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Convergence of the Preconditioned Proximal Point Method and Douglas–Rachford Splitting in the Absence of Monotonicity

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Abstract

The proximal point algorithm (PPA) is the most widely recognized method for solving inclusion problems and serves as the foundation for many numerical algorithms. Despite this popularity, its convergence results have been largely limited to the monotone setting. In this work, we study the convergence of (relaxed) preconditioned PPA for a class of nonmonotone problems that satisfy an oblique weak Minty condition. Additionally, we study the (relaxed) Douglas-Rachford splitting (DRS) method in the nonmonotone setting by establishing a connection between DRS and the preconditioned PPA with a positive semidefinite preconditioner. To better characterize the class of problems covered by our analysis, we introduce the class of semimonotone operators, offering a natural extension to (hypo)monotone and co(hypo)monotone operators, and describe some of their properties. Sufficient conditions for global convergence of DRS involving the sum of two semimonotone operators are provided. Notably, it is shown that DRS converges even when the sum of the involved operators (or of their inverses) is nonmonotone. Various example problems are provided, demonstrating the tightness of our convergence results and highlighting the wide range of applications our theory is able to cover.

Keywords. Convex/nonconvex optimization · monotone/nonmonotone variational inequalities · inclusion problems · preconditioned proximal point algorithm · Douglas–Rachford splitting · semimonotone operators
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1 Introduction

The preconditioned proximal point algorithm (PPPA) is one of the most fundamental methods for solving inclusion problems arising in optimization and variational analysis. It aims to find a solution to the inclusion problem

$$\text{find} \quad x \in \mathbb{R}^n \quad \text{such that} \quad 0 \in Tx, \tag{G-I}$$

where $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued operator which is usually assumed to be monotone. The iterates of the (relaxed) preconditioned proximal point algorithm (PPPA) can be expressed as follows:

$$\begin{cases} \bar{x}^k \in (P+T)^{-1}Px^k \\ x^{k+1} = x^k + \lambda_k(\bar{x}^k - x^k). \end{cases} \tag{PPPA}$$

Here, $P \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix and $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence of strictly positive relaxation parameters. The classic proximal point algorithm (PPA) [46] corresponds to the setting where the preconditioner P is the identity mapping and the relaxation parameter λ_k is one.

Several classical methods are known to be instances of PPPA, each with a specific choice of preconditioner aimed at exploiting structure present in the problem. These include well-known techniques such as Douglas-Rachford splitting (DRS) [23, 14], the alternating direction method of multipliers (ADMM) [20], the (proximal) augmented Lagrangian method (ALM) [60], the primal-dual hybrid gradient (PDHG) method

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[34, 24] (also known as Chambolle-Pock (CP) [12]), iterative refinement [52, §4.1.2], the method of partial inverses [23, §5], progressive hedging [63], and progressive decoupling [62].

The convergence analysis of the methods mentioned above typically relies upon an underlying monotonicity/convexity assumption [60, 61, 62]. See also [68, 45, 22, 11, 26] for related works and generalizations. Departing from this classical setting, we aim to establish convergence of relaxed PPPA for a class of nonmonotone operators that are defined through an *oblique weak Minty condition* (see Definition 2.1). This definition involves a symmetric matrix V that is not required to be positive or negative definite and controls the level of (non)monotonicity. When V is equal to the zero matrix, this condition reduces to the Minty variational inequality (MVI) [48, 27], sometimes also referred to as variational coherence [73]. When V takes the form of $\rho\mathrm{I}$ for negative ρ, one obtains the so-called weak MVI that has been employed in the context of the extragradient method and forward-backward-forward splitting [16, 56, 57, 8]. This condition is closely related to the notion of cohypomonotonicity [6, 41, 71], for which a convergence analysis of classic PPA was already performed in [55, 37, 13]. The key difference between the weak Minty condition and cohypomonotonicity is that the governing inequality is not considered between any two points in the graph of T, but instead only between any point in the graph of T and (a subset of) the zeros of T (see also Example 2.4).

In contrast to the traditional convergence analysis relying on firm nonexpansiveness of the resolvent mapping, our convergence analysis leverages a projective interpretation that was introduced in the monotone case in [67, 66, 38]. We show that the same idea can be applied in the nonmonotone setting for the preconditioned proximal point algorithm provided that a different halfspace is considered. More specifically, each iteration of PPPA is interpreted as a projection onto a certain halfspace, separating the current iterate from a subset of solutions that satisfy the oblique weak Minty assumption. A second key aspect of our convergence analysis is to allow for positive semidefinite preconditioners (as opposed to positive definite ones), which is crucial for establishing a connection with splitting techniques such as (relaxed) Douglas-Rachford splitting (see Section 3.1). We note that in the monotone setting convergence of PPPA with semidefinite preconditioners has been studied in [40, Thm. 3.4], [9, §2.1]. In the nonmonotone setting, the aforementioned halfspace will be constructed using so-called *shadow sequences*, obtained by projecting the sequences $(x^k)_{k\in\mathbb{N}}$ and $(\bar{x}^k)_{k\in\mathbb{N}}$ onto the range of the preconditioner P.

In the second part of this paper, we consider the class of Douglas/Peaceman-Rachford splitting methods and shift our attention towards structured inclusion problems of the form

$$\text{find} \quad x \in \mathbb{R}^n \quad \text{such that} \quad 0 \in T_{\mathrm{P}} x := Ax + Bx, \tag{P-I}$$

where $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are two (possibly nonmonotone) operators. Given stepsize $\gamma > 0$, strictly positive relaxation parameters $\lambda_k > 0$, and a certain initial guess $s^0 \in \mathbb{R}^n$, the (relaxed) Douglas-Rachford splitting method consists of the following iterates:

$$\begin{cases} u^k &\in J_{\gamma A}(s^k) \\ v^k &\in J_{\gamma B}(2u^k - s^k) \\ s^{k+1} &= s^k + \lambda_k(v^k - u^k). \end{cases} \tag{DRS}$$

For $\lambda_k = 1$, this method corresponds to classic Douglas-Rachford splitting [18], while for $\lambda_k = 2$, it is classic Peaceman-Rachford splitting (PRS) [54]. Initially introduced to solve systems of linear equations emerging in heat conduction problems, these methods were later extended in [44] for finding zeros of the sum of two maximally monotone operators, and convergence of the sequence $(u^k)_{k\in\mathbb{N}}$ to a solution was shown [44, 69].

In the optimization setting, global convergence of classic DRS (corresponding to $\lambda_k = 1$) in the nonconvex setting was obtained in [43] assuming that one function is Lipschitz differentiable. Similarly, classic PRS (corresponding to $\lambda_k = 2$) was considered in [42] under the additional requirement that the smooth function is strongly convex. These results were then unified in [70] and tight stepsize ranges for (relaxed) DRS in the nonconvex setting were provided. When restricting to their setting (one function being Lipschitz differentiable), we will show that when the nonsmooth term has a so-called *semimonotone* subdifferential, our theory allows for a larger stepsize range compared to [70] (see Remark 5.3*(iv)*).

A convergence analysis of DRS for minimizing the sum of an α-convex and a β-convex function has been provided in [33] under the assumption that $\alpha + \beta > 0$, i.e. in the convex case. Similar results were obtained for DRS applied to inclusion problems involving the sum of an α- and a β-monotone operator [15, 30] and the sum of an α- and a β-comonotone operator [4], both under the assumption that $\alpha+\beta > 0$. In the former work, it is obvious that the so-called primal inclusion (P-I) is monotone, while in the latter, the so-called dual inclusion

(D-I) is monotone due to [6, Lem. 2.6]. Therefore, the achieved results in this research direction have been limited to the case where either the primal or the dual problem is monotone.

1.1 Contributions

Our main contributions can be summarized as follows:

1. We provide convergence results of PPPA for a class of nonmonotone operators that satisfy the oblique weak Minty condition at one (or more) of its zeros (see Theorem 2.3). The tightness of this result is demonstrated in Example 2.4.
2. Leveraging a primal-dual connection between DRS and PPPA, convergence results of DRS are deduced under an oblique weak Minty condition on the associated primal-dual operator (see Section 3). Additionally, we demonstrate the tightness of this result in Example 6.1*(ii)*.
3. Despite the underlying nonmonotonicity, (local) linear convergence of PPPA is established in Theorem 2.6 under an additional metric subregularity assumption. In turn, this result leads to sufficient conditions for (local) linear convergence of DRS for (possibly) nonmonotone piecewise polyhedral mappings (see Theorem 3.5).
4. We introduce and develop calculus rules for the class of (μ,ρ)-semimonotone operators, which can be viewed as a natural extension of (hypo)monotone and co(hypo)monotone operators (see Remark 4.2). In addition, we provide various examples of well-known (nonconvex) function classes with semimonotone subdifferentials.
5. We provide sufficient conditions for the convergence of DRS, based on the semimonotonicity of the operators A and B (see Theorem 5.2). As discussed in Remark 5.3, the stepsize region obtained through our analysis encompasses and extends existing results on relaxed Douglas/Peaceman-Rachford splitting methods such as [15, 4]. Up to the knowledge of the authors, this is the first work that is able to cover the convergence of DRS in the case where neither the primal nor the dual nor the primal-dual inclusion is monotone. We provide several example problems that demonstrate the applicability of our theory.

1.2 Organisation

The paper is structured as follows. Section 1.3 introduces some notation and recalls some standard definitions. In Section 2 the concept of oblique weak Minty solutions (cf. Definition 2.1) is formally introduced, and the convergence of PPPA is studied for a class of operators that admit at least one such solution. Section 3 establishes a primal-dual connection between PPPA and DRS, which leads to convergence results for DRS under an oblique weak Minty assumption on the associated primal-dual operator. In Section 4 we introduce the class of semimonotone operators, for which we provide several calculus rules and examples. Sufficient conditions for the convergence of DRS, based on the semimonotonicity of the underlying operators are provided in Section 5. In Section 6 various example problems covered by our theory for DRS are presented. Finally, Section 7 concludes the paper. For the sake of readability, several proofs and auxiliary results are deferred to the Appendix.

1.3 Notation

The set of natural numbers including zero is denoted by $\mathbb{N} := \{0, 1, \ldots\}$. The set of real and extended-real numbers are denoted by $\mathbb{R} := (-\infty, \infty)$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, while the positive and strictly positive reals are $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_{++} := (0, \infty)$. We denote the positive part of a real number by $[\cdot]_+ := \max\{0, \cdot\}$ and the negative part by $[\cdot]_- := \min\{0, \cdot\}$. With id we indicate the identity function $x \mapsto x$ defined on a suitable space. The identity matrix is denoted by $\mathrm{I}_n \in \mathbb{R}^{n\times n}$; we write I when no ambiguity occurs. Given a matrix $P \in \mathbb{R}^{n\times n}$, we denote the range of A by $\mathcal{R}(A)$ and the kernel of A by $\mathcal{N}(A)$. We denote by $\mathbb{R}^n$ the standard n-dimensional Euclidean space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. The sets of symmetric, symmetric positive semidefinite and symmetric positive definite n-by-n matrices are denoted by $\mathbb{S}^n$, $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$, respectively. We also write $P \succeq 0$ and $P \succ 0$ for $P \in \mathbb{S}^n_+$ and $P \in \mathbb{S}^n_{++}$, respectively. Given a symmetric matrix $P \in \mathbb{S}^n$ we define the scalar product $\langle x, y\rangle_P = \langle x, Py\rangle$ and the quadratic function $\mathrm{q}_P(x) := \langle x, x\rangle_P$. If $P \in \mathbb{S}^n_{++}$ then its associated induced norm is defined as $\|x\|_P = \sqrt{\mathrm{q}_P(x)}$.

We use the notation $(w^k)_{k\in I}$ to denote a sequence with indices in the set $I \subseteq \mathbb{N}$. When dealing with scalar sequences we use the subscript notation $(\gamma_k)_{k\in I}$.

We say that $(x^k)_{k\in\mathbb{N}}$ converges to a point $x^\star$ (at least) *Q-linearly* with Q-factor given by $c \in (0,1)$ if there exists an index $k_0 \in \mathbb{N}$ such that $\|x^{k+1} - x^\star\| \leq c\|x^k - x^\star\|$ for all $k \geq k_0$. We say that $(x^k)_{k\in\mathbb{N}}$ converges to a point $x^\star$ (at least) *R-linearly* if there exists a sequence of nonnegative scalars $(v^k)_{k\in\mathbb{N}}$ such that $\|x^k - x^\star\| \leq v^k$ and $(v^k)_{k\in\mathbb{N}}$ converges Q-linearly to zero.

An operator or set-valued mapping $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^d$ maps each point $x \in \mathbb{R}^n$ to a subset $A(x)$ of $\mathbb{R}^d$. We will use the notation $A(x)$ and Ax interchangeably. We denote the domain of A by $\operatorname{dom} A := \{x \in \mathbb{R}^n \mid Ax \neq \emptyset\}$, its graph by $\operatorname{gph} A := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^d \mid y \in Ax\}$, and the set of its zeros by $\operatorname{zer} A := \{x \in \mathbb{R}^n \mid 0 \in Ax\}$. The inverse of A is defined through its graph: $\operatorname{gph} A^{-1} := \{(y,x) \mid (x,y) \in \operatorname{gph} A\}$. The *resolvent* of A is defined by $J_A := (\mathrm{id} + A)^{-1}$.

Definition 1.1 (parallel sum of operators)**.** *The parallel sum between operators $A, B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined as $A \,\square\, B := (A^{-1} + B^{-1})^{-1}$.*

Definition 1.2 (parallel sum of extended-real numbers)**.** *Let $a, b \in \overline{\mathbb{R}}$. We say that a and b are parallel summable if either $a = b = 0$ or $a + b \neq 0$ and their parallel sum is defined as*

$$a \,\square\, b := \begin{cases} a, & \text{if } b = \infty, \\ b, & \text{if } a = \infty, \\ 0, & \text{if } a = b = 0, \\ \frac{ab}{a+b}, & \text{otherwise.} \end{cases}$$

Definition 1.3 ((co)monotonicity)**.** *An operator $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be μ-monotone for some $\mu \in \mathbb{R}$ if*

$$\langle x - x', y - y' \rangle \geq \mu\|x - x'\|^2, \qquad \text{for all } (x,y), (x',y') \in \operatorname{gph} A,$$

and it is said to be ρ-comonotone for some $\rho \in \mathbb{R}$ if

$$\langle x - x', y - y' \rangle \geq \rho\|y - y'\|^2, \qquad \text{for all } (x,y), (x',y') \in \operatorname{gph} A.$$

The operator A is said to be maximally (co-)monotone if its graph is not strictly contained in the graph of another (co-)monotone operator.

We say that A is *outer semicontinuous (osc)* at $\bar{x} \in \operatorname{dom} A$ if

$$\limsup_{x\to\bar{x}} Ax := \{y \mid \exists x^k \to \bar{x}, \exists y^k \to y \text{ with } y^k \in Ax^k\} \subseteq A\bar{x}. \tag{1.1}$$

Outer semicontinuity of A everywhere is equivalent to its graph being a closed subset of $\mathbb{R}^n \times \mathbb{R}^d$.

The domain of an extended real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is the set $\operatorname{dom} f := \{x \in \mathbb{R}^n \mid f(x) < \infty\}$. We say that f is proper if $\operatorname{dom} f \neq \emptyset$ and that f is lower semicontinuous (lsc) if the epigraph $\operatorname{epi} f := \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq \alpha\}$ is a closed subset of $\mathbb{R}^{n+1}$. We denote the limiting subdifferential of f by ∂f. We say that f is ℓ-smooth to indicate that f is continuously differentiable and ∇f is Lipschitz continuous with modulus ℓ.

The indicator function of a set $E \subseteq \mathbb{R}^n$ is denoted by δ_E, namely $\delta_E(x) = 0$ if $x \in E$ and ∞ otherwise. We denote the normal cone of E by N_E. The projection onto and the distance from E with respect to $\|\cdot\|_Q$, $Q \in \mathbb{S}^n_{++}$, are denoted by

$$\Pi^Q_E(x) := \operatorname*{arg\,min}_{z\in E}\{\|z - x\|_Q\}, \quad \operatorname{dist}_Q(x,E) := \inf_{z\in E}\{\|z - x\|_Q\},$$

respectively. The absence of super/subscript Q implies the same definitions with respect to the canonical norm.

2 The preconditioned proximal point method

In this section, the convergence of PPPA with positive semidefinite preconditioning is studied for a class of nonmonotone operators, defined through an oblique weak Minty assumption.

We begin by introducing this class of nonmonotone operators and detailing our underlying assumptions. Then, leveraging a projective interpretation of the proximal point algorithm, we provide the corresponding convergence results of PPPA. Subsequently, we demonstrate the tightness of our convergence results through a simple example. Finally, we establish (local) linear convergence under an additional metric subregularity assumption.

2.1 Convergence analysis of PPPA under oblique weak Minty

Consider the following class of operators with oblique weak Minty solutions, which generalizes the class of operators with weak Minty solutions that has been employed in the context of the extragradient method and forward-backward-forward splitting [16, 57, 8].

Definition 2.1 (V-oblique weak Minty solutions)**.** *An operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to have V-oblique weak Minty solutions at (a nonempty set) $\mathcal{S}^\star \subseteq \operatorname{zer} T$ for some symmetric matrix $V \in \mathbb{S}^n$ if*

$$\langle v, u - u^\star \rangle \geq \mathrm{q}_V(v), \qquad \text{for all } u^\star \in \mathcal{S}^\star, (u, v) \in \operatorname{gph} T, \tag{2.1}$$

where the quadratic form $\mathrm{q}_V(v) \coloneqq \langle v, Vv \rangle$. Whenever $V = \rho \mathrm{I}$ for some $\rho \in \mathbb{R}$, we use the notation of ρ-weak Minty solutions.

The generalization of the weak Minty condition to include a general symmetric matrix V instead of a scalar value ρ permits a more detailed characterization of problem classes, which in turn will lead to tight results for the convergence of PPPA and DRS (cf Section 3).

The above definition can be further relaxed by requiring (2.1) to hold instead on $(z, v) \in \operatorname{gph} T \cap (\mathcal{R}((P + T)^{-1}P) \times \mathcal{R}(P))$, where P is the preconditioner from PPPA. Under this weaker condition, the results from Theorems 2.3 and 2.6 are still satisfied since (2.1) is invoked only at points in this restricted set (see (2.6)). We refrain from this relaxation given that it couples the problem class with the preconditioner and does not appear to lead to tighter sufficient conditions in terms of semimonotonicity (cf. Section 5).

As previously mentioned in the introduction, a key aspect of our forthcoming convergence analysis for PPPA is to consider positive semidefinite preconditioners, largely inspired by the convergence analysis performed in [40, Thm. 3.4] and [14, Thm. 3.3]. Unlike the aforementioned works, we do not assume that T is the sum of a maximally monotone and a linear skew-symmetric operator. Instead, we require that it admits at least a single oblique weak Minty solution. More specifically, we work under the following assumptions.

Assumption I. *The operator T in (G-I) and the preconditioner P in (PPPA) satisfy the following properties.*

A1 *$T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous.*

A2 *The preconditioned resolvent $(P + T)^{-1}P$ has full domain.*

A3 *There exists a nonempty set $\mathcal{S}^\star \subseteq \operatorname{zer} T$ and a symmetric, possibly indefinite matrix $V \in \mathbb{S}^n$ such that T has V-oblique weak Minty solutions at $\mathcal{S}^\star$ for V.*

A4 *$P \in \mathbb{S}^n_+$ is a symmetric positive semidefinite matrix such that*

$$\eta_{\min} \coloneqq 1 + \lambda_{\min}(Z^\top V P Z)^{[1]} > 0, \tag{2.2}$$

where Z is any orthonormal basis for the range of P.

Note that the preconditioned resolvent $(P+T)^{-1}P$ can be seen as a particular instance of the warped/nonlinear resolvent [10, 28].

Our convergence analysis relies on a projective interpretation of the preconditioned proximal point algorithm in the monotone setting that dates back to [67, 66, 38]. In particular, it was shown that when the preconditioner is positive definite, each iteration of PPPA can be interpreted as a projection onto a certain halfspace, constructed using the sequences $(x^k)_{k\in\mathbb{N}}$ and $(\bar{x}^k)_{k\in\mathbb{N}}$. We will show that this idea can be extended to our more general setting provided that a different halfspace is considered. More specifically, each iteration of PPPA is interpreted as a projection onto a certain halfspace, separating the current iterate from a subset of solutions that satisfy the oblique weak Minty assumption. This halfspace will be constructed using the so-called shadow sequences $(w^k)_{k\in\mathbb{N}} \coloneqq (\Pi_{\mathcal{R}(P)}\, x^k)_{k\in\mathbb{N}}$ and $(\bar{w}^k)_{k\in\mathbb{N}} \coloneqq (\Pi_{\mathcal{R}(P)}\, \bar{x}^k)_{k\in\mathbb{N}}$ to deal with the fact that in our work the preconditioner is only positive *semidefinite*.

We proceed to present this separating halfspace and its main properties in the following lemma, for which a supporting visualization is provided in Figure 1. Note that the introduction of the matrix $Q \coloneqq P + \Pi_{\mathcal{N}(P)}$ is inspired by the convergence analysis performed in [40, Thm. 3.4] and [14, Thm. 3.3].

[1] As $Z^\top V P Z$ is similar to a symmetric matric, its eigenvalues are real (see (2.9)).

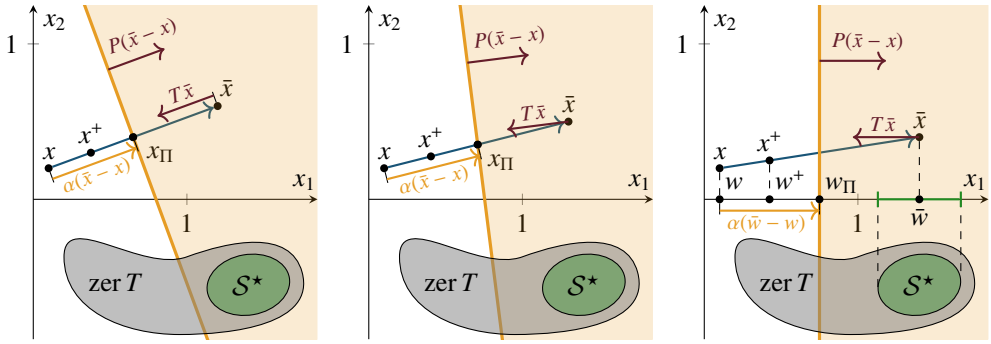

Figure 1: *Visualization of the involved variables in Lemma 2.2 for different types of preconditioners, where* $V = -\frac{3}{2}\mathrm{I}$*,* α *is as in (2.4) and* $\lambda = \alpha/2$*. Here, we use the shorthand notation* $w_\Pi = \Pi^Q_{\mathscr{D}_{w,\bar{w}}}(w)$ *and* $x_\Pi = \Pi^P_{\mathscr{D}_{x,\bar{x}}}(x)$*. In these figures, the vector* $\bar{x}$ *satisfies the update rule (PPPA) (see (2.5)). Note that the precise location of the halfspace is governed by the oblique weak Minty matrix* V*, due to the dependence of* α *on* V*. For example, if* $V = 0$*, then* $\alpha = 1$ *and the corresponding hyperplane would pass through* $\bar{w}$*. (left and middle) Positive definite preconditioner, such that* $\Pi_{\mathcal{R}(P)} = \mathrm{I}$*,* $P = Q$ *and the variables* $x, \bar{x}, x^+, x_\Pi$ *are equal to the variables* $w, \bar{w}, w^+, w_\Pi$*. (left) Multiple of identity preconditioner* $P = \frac{1}{3}\mathrm{I}$*. (middle)* $P = \mathrm{diag}(1/3, 1/6)$*. (right) Positive semidefinite preconditioner* $P = \mathrm{diag}(1/3, 0)$*.*

Lemma 2.2 (hyperplane interpretation)**.** *Suppose that Assumption I holds. Consider the points* $x \in \mathbb{R}^n$*,* $\bar{x} \in (P+T)^{-1}Px$ *and* $x^+ = x + \lambda(\bar{x} - x)$ *satisfying update rule (PPPA), and let* $w := \Pi_{\mathcal{R}(P)}\, x$*,* $\bar{w} := \Pi_{\mathcal{R}(P)}\, \bar{x}$ *and* $w^+ := \Pi_{\mathcal{R}(P)}\, x^+$*. Define the matrix* $Q := P + \Pi_{\mathcal{N}(P)}$ *and the halfspace*

$$\mathscr{D}_{w,\bar{w}} := \{r \in \mathbb{R}^n \;\big|\; \langle w - \bar{w},\, \bar{w} - r\rangle_Q \geq \mathrm{q}_V(Q(w-\bar{w}))\}. \tag{2.3}$$

Then,

(i) The set $\Pi_{\mathcal{R}(P)}\, \mathcal{S}^\star$ *is a subset of* $\mathscr{D}_{w,\bar{w}}$*.*

(ii) If $w \in \mathscr{D}_{w,\bar{w}}$ *then* $\bar{x} \in \operatorname{zer} T$*.*

(iii) If $\bar{x} \notin \operatorname{zer} T$*, then* $\mathrm{q}_P(x - \bar{x}) \neq 0$ *and*

$$\alpha := 1 + \frac{\mathrm{q}_V(P(x-\bar{x}))}{\mathrm{q}_P(x-\bar{x})} \geq \eta_{\min} > 0. \tag{2.4}$$

(iv) If $\lambda \in (0, 2\alpha)$*, then* $w^+ = (1 - \lambda/\alpha)w + \lambda/\alpha\, \Pi^Q_{\mathscr{D}_{w,\bar{w}}}(w)$*.*

Proof.

♠ 2.2*(i)* By update rule (PPPA) it holds that

$$\bar{v} := P(x - \bar{x}) \in T\bar{x}. \tag{2.5}$$

Therefore, owing to the V-oblique weak Minty assumption (Assumption I.A3), for all $x^\star \in \mathcal{S}^\star$

$$\langle P(x-\bar{x}),\, \bar{x} - x^\star\rangle \geq \mathrm{q}_V(P(x-\bar{x})). \tag{2.6}$$

As a result, for all $w^\star := \Pi_{\mathcal{R}(P)}\, x^\star \in \Pi_{\mathcal{R}(P)}\, \mathcal{S}^\star \subseteq \Pi_{\mathcal{R}(P)} \operatorname{zer} T$

$$\begin{aligned}\langle Q(w-\bar{w}),\, \bar{w} - w^\star\rangle &= \langle P(x-\bar{x}),\, \Pi_{\mathcal{R}(P)}(\bar{x} - x^\star)\rangle = \langle P(x-\bar{x}),\, \bar{x} - x^\star\rangle \\ &\overset{(2.6)}{\geq} \mathrm{q}_V(P(x-\bar{x})) = \mathrm{q}_V(Q(w-\bar{w})),\end{aligned}$$

where we used Lemma A.1*(ii)* to relate P and Q in the first and final equality. It is now evident that by construction, $\Pi_{\mathcal{R}(P)}\, \mathcal{S}^\star \subseteq \mathscr{D}_{w,\bar{w}}$.

♠ 2.2*(ii)* Suppose that $w \in \mathscr{D}_{w,\bar{w}}$. Define the shorthand notation $\tilde{w} := w - \bar{w}$, $\tilde{x} := x - \bar{x}$ and let $X := Z^\top QZ$. By Lemma A.1*(iv)* matrix X is positive definite, and we have

$$P = Q\,\Pi_{\mathcal{R}(P)} \;=\; \Pi_{\mathcal{R}(P)}\, Q\, \Pi_{\mathcal{R}(P)} \;= ZZ^\top QZZ^\top = ZXZ^\top, \tag{2.7}$$

due to Lemmas A.1*(i)* and A.1*(ii)*. Consequently, by definition of $\mathscr{D}_{w,\bar{w}}$, it holds that

$$\begin{aligned} 0 \geq \|\tilde{w}\|_Q^2 + \mathrm{q}_V(Q\tilde{w}) &= \langle Q\,\Pi_{\mathcal{R}(P)}\,\tilde{w},\ \tilde{w}\rangle + \langle Q\,\Pi_{\mathcal{R}(P)}\,\tilde{w},\ Q\,\Pi_{\mathcal{R}(P)}\,\tilde{w}\rangle_V = \langle Q\,\Pi_{\mathcal{R}(P)}\,\tilde{w},\ (\mathrm{I} + VQ\,\Pi_{\mathcal{R}(P)})\tilde{w}\rangle \\ &= \langle ZXZ^\top\tilde{w},\ (\mathrm{I} + VZXZ^\top)\tilde{w}\rangle = \langle X^{1/2}Z^\top\tilde{w},\ X^{1/2}Z^\top(\mathrm{I} + VZXZ^\top)\tilde{w}\rangle \\ &= \langle X^{1/2}Z^\top\tilde{w},\ (\mathrm{I} + X^{1/2}Z^\top VZX^{1/2})X^{1/2}Z^\top\tilde{w}\rangle. \end{aligned} \tag{2.8}$$

Furthermore, due to Assumption I.A4 it follows that

$$0 < \eta_{\min} = 1 + \lambda_{\min}(Z^\top VPZ) = 1 + \lambda_{\min}(X^{1/2}Z^\top VZX^{1/2}) \tag{2.9}$$

where the final equality is obtained through similarity transformation by the similarity matrix $X^{1/2}$ [35, Cor. 1.3.4] and (2.7). Hence, $\mathrm{I} + X^{1/2}Z^\top VZX^{1/2}$ is positive definite, which combined with (2.8) and $X \succ 0$ implies that $Z^\top\tilde{w} = 0$. In turn, using Lemmas A.1*(i)* and A.1*(ii)* we have $\bar{v} = P\tilde{x} = P\,\Pi_{\mathcal{R}(P)}\,\tilde{w} = PZZ^\top\tilde{w} = 0$, which by (2.5) implies that $\bar{x} \in \operatorname{zer} T$.

♠ 2.2*(iii)* If $\bar{x} \notin \operatorname{zer} T$, then inclusion (2.5) implies that $P(x - \bar{x}) \neq 0$. Combined with $Q(w - \bar{w}) = P(x - \bar{x})$ (where we used Lemma A.1*(ii)*) and positive definiteness of Q (see Lemma A.1*(iii)*) it follows that $\mathrm{q}_P(x - \bar{x}) = \|w - \bar{w}\|_Q > 0$, ensuring that $\alpha = 1 + \frac{\mathrm{q}_V(Q\tilde{w})}{\|\tilde{w}\|_Q^2}$ is finite-valued. Consequently, using (2.8) and that $\|\tilde{w}\|_Q^2 = \langle Q\,\Pi_{\mathcal{R}(P)}\,\tilde{w},\ \tilde{w}\rangle = \|X^{1/2}Z^\top\tilde{w}\|^2$ by (2.7), α may be written as a Rayleigh quotient [35, Thm. 4.2.2] of $\mathrm{I} + X^{1/2}Z^\top VZX^{1/2}$ and $X^{1/2}Z^\top\tilde{w}$, i.e.,

$$\alpha = 1 + \frac{\mathrm{q}_V(Q\tilde{w})}{\|\tilde{w}\|_Q^2} = \frac{\langle X^{1/2}Z^\top\tilde{w},\,(\mathrm{I}+X^{1/2}Z^\top VZX^{1/2})X^{1/2}Z^\top\tilde{w}\rangle}{\|X^{1/2}Z^\top\tilde{w}\|^2} \geq \lambda_{\min}(\mathrm{I} + X^{1/2}Z^\top VZX^{1/2}) \overset{(2.9)}{=} \eta_{\min}.$$

♠ 2.2*(iv)* For a given vector $u \notin \mathscr{D}_{w,\bar{w}}$ the projection onto the halfspace is given by [5, Ex. 29.20]

$$\Pi^Q_{\mathscr{D}_{w,\bar{w}}}(u) = u + \tfrac{\langle w-\bar{w},u-\bar{w}\rangle_Q + \mathrm{q}_V(Q(w-\bar{w}))}{\|w-\bar{w}\|_Q^2}(\bar{w} - w).$$

Therefore, $\Pi^Q_{\mathscr{D}_{w,\bar{w}}}(w) = w + \alpha(\bar{w} - w)$, which along with $\lambda \in (0, 2\alpha)$ yields

$$(1 - \lambda/\alpha)w + \lambda/\alpha\,\Pi_{\mathscr{D}_{w,\bar{w}}}(w) = w + \lambda(\bar{w} - w) = \Pi_{\mathcal{R}(P)}\left(x + \lambda(\bar{x} - x)\right) = w^+,$$

where $w^+ = \Pi_{\mathcal{R}(P)}\,x^+$ was used in the last equality. □

Most notably, Lemma 2.2*(iv)* establishes that the update rule for the (shadow) sequence generated by PPPA can be interpreted as a relaxed projection onto the halfspace defined in (2.3), while Lemmas 2.2*(i)* and 2.2*(ii)* imply that if at any iteration the current iterate belongs to the halfspace containing the set of projected oblique weak Minty solutions $\Pi_{\mathcal{R}(P)}\,\mathcal{S}^\star$, this implies its optimality. Owing to the non-expansiveness of the projection operator, this leads to the following convergence result for the preconditioned proximal point algorithm.

Theorem 2.3 (convergence of PPPA)**.** *Suppose that Assumption I holds, and consider a sequence* $(x^k, \bar{x}^k)_{k\in\mathbb{N}}$ *generated by PPPA starting from* $x^0 \in \mathbb{R}^n$ *with relaxation parameters* $\lambda_k \in (0, 2\alpha_k)$ *such that* $\liminf_{k\to\infty} \lambda_k(2\alpha_k - \lambda_k) > 0$*, where*

$$\alpha_k = 1 + \frac{\mathrm{q}_V(P(x^k - \bar{x}^k))}{\mathrm{q}_P(x^k - \bar{x}^k)}. \tag{2.10}$$

Then, either a point $\bar{x}^k \in \operatorname{zer} T$ *is reached in a finite number of iterations or the following hold for the sequence* $(x^k, \bar{x}^k)_{k\in\mathbb{N}}$*.*

(i) $\bar{v}^k := P(x^k - \bar{x}^k) \in T\bar{x}^k$ *for all* k *and* $(\bar{v}^k)_{k\in N}$ *converges to zero.*

(ii) Every limit point (if any) of $(\bar{x}^k)_{k\in\mathbb{N}}$ *belongs to* $\operatorname{zer} T$*.*

(iii) The shadow sequences $(\Pi_{\mathcal{R}(P)}\,x^k)_{k\in\mathbb{N}}$, $(\Pi_{\mathcal{R}(P)}\,\bar{x}^k)_{k\in\mathbb{N}}$ *are bounded.*

(iv) If $\lambda_k(2\alpha_k - \lambda_k) \geq \kappa > 0$ *uniformly for all* k *then, for all* $x^\star \in \mathcal{S}^\star$ *it holds that*

$$\min_{k=0,1,\ldots,N} \|\bar{v}^k\|^2 \leq \frac{1}{N+1}\frac{\|P\|}{\kappa}\,\mathrm{q}_P(x^0 - x^\star).$$

Moreover, if $(P+T)^{-1}P$ is (single-valued) continuous then,

(v) The limit points of $(\Pi_{\mathcal{R}(P)}\, x^k)_{k\in\mathbb{N}}$, $(\Pi_{\mathcal{R}(P)}\, \bar{x}^k)_{k\in\mathbb{N}}$ are in $\Pi_{\mathcal{R}(P)} \operatorname{zer} T$.

(vi) If in Assumption I.A3, $\mathcal{S}^\star = \operatorname{zer} T$, then $(\Pi_{\mathcal{R}(P)}\, x^k)_{k\in\mathbb{N}}$, $(\Pi_{\mathcal{R}(P)}\, \bar{x}^k)_{k\in\mathbb{N}}$ both converge to some element of $\Pi_{\mathcal{R}(P)} \operatorname{zer} T$ and $(\bar{x}^k)_{k\in\mathbb{N}}$ converges to some element of $\operatorname{zer} T$. Finally, if λ_k is uniformly bounded in the interval $(0, 2)$, then $(x^k)_{k\in\mathbb{N}}$ also converges to some element of $\operatorname{zer} T$.

Proof. Note that as long as $\bar{x}^k \notin \operatorname{zer} T$ it follows from Lemma 2.2*(iii)* that $\alpha_k \geq \eta_{\min} > 0$. Consider the shadow sequences $(w^k)_{k\in\mathbb{N}} := (\Pi_{\mathcal{R}(P)}\, x^k)_{k\in\mathbb{N}}$ and $(\bar{w}^k)_{k\in\mathbb{N}} := (\Pi_{\mathcal{R}(P)}\, \bar{x}^k)_{k\in\mathbb{N}}$ with corresponding update rule

$$w^{k+1} = w^k + \lambda_k(\bar{w}^k - w^k). \tag{2.11}$$

If $w^k \in \mathscr{D}_{w^k,\bar{w}^k}$ for some $k \in \mathbb{N}$ then by Lemma 2.2*(ii)* the algorithm has reached a point $\bar{x}^k \in \operatorname{zer} T$ in a finite number of iterations. Therefore, we will consider the case when $w^k \notin \mathscr{D}_{w^k,\bar{w}^k}$ in the remainder of the proof.

♠ 2.3*(i)*: From the update rule (PPPA), it directly follows that $\bar{v}^k := P(x^k - \bar{x}^k) \in T\bar{x}^k$ for all k. It remains to show that $(\bar{v}^k)_{k\in\mathbb{N}}$ converges to zero. By Lemma 2.2*(iv)*, update step (2.11) is equivalent to the relaxed projection $(1-\zeta_k)\mathrm{id} + \zeta_k \Pi_{\mathscr{D}_{w^k,\bar{w}^k}}$ where $\zeta_k := \lambda_k/\alpha_k$, which is $\zeta_k/2$-averaged [5, Prop. 4.40] in the space with inner product $\langle\cdot,\cdot\rangle_Q$. Note that for any $w^\star \in \Pi_{\mathcal{R}(P)}\, \mathcal{S}^\star$ it holds by Lemma 2.2*(i)* that $w^\star \in \mathscr{D}_{w^k,\bar{w}^k}$. Therefore, it follows from [5, Prop. 4.35(iii)] for any $w^\star \in \Pi_{\mathcal{R}(P)}\, \mathcal{S}^\star$ that

$$\begin{aligned}
\|w^{k+1} - w^\star\|_Q^2 &\leq \|w^k - w^\star\|_Q^2 - \tfrac{1-\zeta_k/2}{\zeta_k/2}\|w^{k+1} - w^k\|_Q^2 \\
&= \|w^k - w^\star\|_Q^2 - \tfrac{2\alpha_k - \lambda_k}{\lambda_k}\,\|\lambda_k(w^k - \bar{w}^k)\|_Q^2 \\
&= \|w^k - w^\star\|_Q^2 - \lambda_k(2\alpha_k - \lambda_k)\,\|w^k - \bar{w}^k\|_Q^2,
\end{aligned} \tag{2.12}$$

establishing that $(w^k)_{k\in\mathbb{N}}$ is Fejér monotone with respect to $\Pi_{\mathcal{R}(P)}\, \mathcal{S}^\star$ [5, Def. 5.1]. By telescoping (2.12), and since $\liminf_{k\to\infty} \lambda_k(2\alpha_k - \lambda_k) > 0$, it follows that $\sum_{k=0}^{\infty} \|w^k - \bar{w}^k\|_Q^2 < +\infty$, which implies that

$$\|w^k - \bar{w}^k\|_Q \to 0. \tag{2.13}$$

The convergence of $(\bar{v}^k)_{k\in N}$ to zero is established by noting that

$$\|\bar{v}^k\|^2 = \|P(w^k - \bar{w}^k)\|^2 \leq \|P^{1/2}\|^2\, \|P^{1/2}(w^k - \bar{w}^k)\|^2 = \|P\|\, \|w^k - \bar{w}^k\|_Q^2. \tag{2.14}$$

♠ 2.3*(ii)*: Since $(\bar{v}^k)_{k\in\mathbb{N}}$ converges to zero and $\bar{v}^k \in T\bar{x}^k$, it follows from outer semicontinuity of T (due to Assumption I.A1) that any limit point of $(\bar{x}^k)_{k\in\mathbb{N}}$ belongs to $\operatorname{zer} T$.

♠ 2.3*(iii)*: It follows from (2.12) that $(\|w^k - w^\star\|_Q)_{k\in\mathbb{N}}$ converges, and in particular that $(w^k)_{k\in\mathbb{N}}$ is bounded. In turn, using (2.13) and the triangle inequality $\|\bar{w}^k\|_Q \leq \|\bar{w}^k - w^k\|_Q + \|w^k\|_Q$, it follows that $(\bar{w}^k)_{k\in\mathbb{N}}$ is bounded as claimed.

♠ 2.3*(iv)*: By telescoping (2.12) and from the uniform bound $\forall k \in \mathbb{N} : \lambda_k(2\alpha_k - \lambda_k) \geq \kappa > 0$, it follows that

$$\min_{0\leq k\leq N} \|w^k - \bar{w}^k\|_Q^2 \leq \tfrac{1}{(N+1)\kappa}\|w^0 - w^\star\|_Q^2. \tag{2.15}$$

The claimed upper bound on $\min_{0\leq k\leq N} \|\bar{v}^k\|^2$ can thus be established by noting that $\|w^0 - w^\star\|_Q^2 = \mathrm{q}_P(x^0 - x^\star)$ and using (2.14).

♠ 2.3*(v)*: To show that limit points of $(w^k)_{k\in\mathbb{N}}$ and $(\bar{w}^k)_{k\in\mathbb{N}}$ are in $\Pi_{\mathcal{R}(P)} \operatorname{zer} T$, consider $G = (P+T)^{-1}P$ and $G' = \Pi_{\mathcal{R}(P)}\, G$. Then, both G and G' are (single-valued) continuous as G' is the composition of a linear mapping and a continuous one. Next, observe that $\bar{w}^k = G'(w^k)$ and take a subsequence $(w^k)_{k\in K}$ converging to some limit point w^∞. Since $(\bar{w}^k - w^k)_{k\in\mathbb{N}}$ converges to zero by (2.13), we have that $(\bar{w}^k)_{k\in K}$ also converges to the same limit point w^∞. Since G' is continuous, we obtain $w^\infty = G'(w^\infty)$, i.e., $w^\infty \in \operatorname{fix} G'$. We will now show that such a fixed point of G' belongs to $\Pi_{\mathcal{R}(P)} \operatorname{zer} T$.

It holds that $Pw^\infty = PG'(w^\infty) = P\,\Pi_{\mathcal{R}(P)}(P+T)^{-1}Pw^\infty = P(P+T)^{-1}Pw^\infty$. Therefore, there exists a $\bar{w}^\infty = (P+T)^{-1}Pw^\infty$ such that $Pw^\infty = P\bar{w}^\infty$. Furthermore, it holds that $P(w^\infty - \bar{w}^\infty) = T\bar{w}^\infty$, implying that $\bar{w}^\infty \in \operatorname{zer} T$. Hence, $w^\infty = G'(w^\infty) = \Pi_{\mathcal{R}(P)}(P+T)^{-1}Pw^\infty = \Pi_{\mathcal{R}(P)}\, \bar{w}^\infty \in \Pi_{\mathcal{R}(P)} \operatorname{zer} T$, as claimed in Theorem 2.3*(v)*.

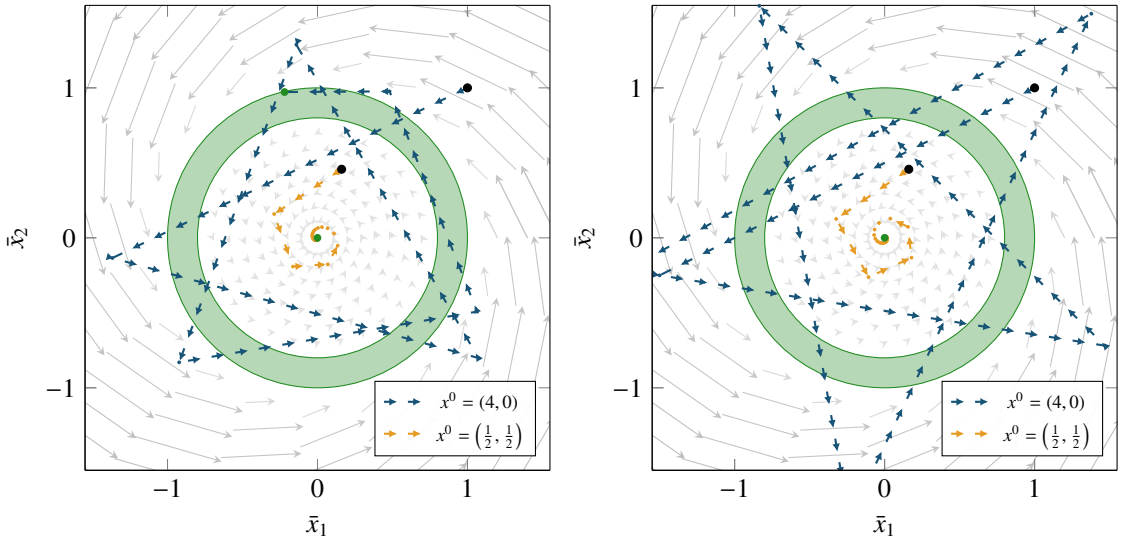

Figure 2: *Visualization of the sequences $(\bar{x}^k)_{k\in\mathbb{N}}$ for the relaxed proximal point algorithm applied to Example 2.4, where $a = 2$ and $b = 1$. The zeros of T are marked in green and its gradient flow $-T$ is indicated using gray arrows. (left) Sequences generated using $\lambda = 2.3 < \bar{\lambda}$, where $\bar{\lambda}$ is the upper bound on λ_k from (2.16). The orange sequence converges to the single (b/a^2+b^2)-weak Minty solution $(0, 0)$ while the blue sequence converges to a zero of T which is not the weak Minty solution (cf. Theorem 2.3(ii)). (right) Sequences generated using $\lambda = 2.5 > \bar{\lambda}$. In this setting, it is no longer guaranteed by Theorem 2.3 that the sequence $(\bar{x}^k)_{k\in\mathbb{N}}$ converges, as can be seen from the (diverging) blue sequence.*

♠ 2.3*(vi)*: Finally, if $\mathcal{S}^\star = \operatorname{zer} T$, then the convergence of $(\Pi_{\mathcal{R}(P)}\, x^k)_{k\in\mathbb{N}}$, $(\Pi_{\mathcal{R}(P)}\, \bar{x}^k)_{k\in\mathbb{N}}$ to some element of $\Pi_{\mathcal{R}(P)} \operatorname{zer} T$ follows by [5, Theorem 5.5]. Using $G\,\Pi_{\mathcal{R}(P)} = G$, it follows that $\bar{x}^k = G(x^k) = G(w^k)$. Hence, by continuity of G, the sequence $(\bar{x}^k)_{k\in\mathbb{N}}$ converges to $\bar{x}^\infty = Gw^\infty$, and by Theorem 2.3*(ii)* it holds that $\bar{x}^\infty \in \operatorname{zer} T$.

It remains to show that $(x^k)_{k\in\mathbb{N}}$ also converges to some element of $\operatorname{zer} T$ if λ_k is uniformly bounded in the interval $(0, 2)$. Note that

$$\|x^{k+1} - \bar{x}^\infty\| \leq \lambda_k \|\bar{x}^k - \bar{x}^\infty\| + |1 - \lambda_k| \|x^k - \bar{x}^\infty\|.$$

Consequently, by uniform boundedness of λ_k, i.e., $(\lambda_k)_{k\in\mathbb{N}} \subseteq [\epsilon, 2 - \epsilon]$ for some $\epsilon > 0$, and by [58, Lemma 3] it follows that $\lim_{k\to\infty} \|x^{k+1} - \bar{x}^\infty\| = 0$, i.e. $(x^k)_{k\in\mathbb{N}}$ also converges to $\bar{x}^\infty \in \operatorname{zer} T$. □

It is worth reiterating that in the preceding theorem, all results except the final one apply even when only a subset of $\operatorname{zer} T$ are oblique weak Minty solutions. Most notably, Theorem 2.3*(ii)* states that in our setting the limit points of $(\bar{x}^k)_{k\in\mathbb{N}}$ are zeros of T, which do not have to be oblique weak Minty solutions. The significance of this result is highlighted by the first part of the following example, where this convergence behavior is observed in practice (see also Figure 2). Additionally, the second part of this example (where $f \equiv 1$) demonstrates the tightness of our results.

Example 2.4 (toy example)**.** Consider the operator

$$T(x) := f\big(\|x\|\big) \begin{bmatrix} b & a \\ -a & b \end{bmatrix} x,$$

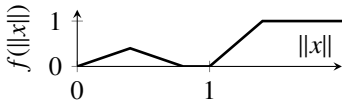

Figure 3: *Definition of $f : \mathbb{R}_+ \to [0, 1]$ in Example 2.4.*

where $a \in \mathbb{R}$, $b \in \mathbb{R}_{++}$ and $f : \mathbb{R}_+ \to [0, 1]$. If f is defined as in Figure 3, then the following hold.

(i) T has a single (b/a^2+b^2)-weak Minty solution at the point $\mathcal{S}^\star = (0, 0) \subset \operatorname{zer} T$ while the other zeros of T are not ρ-weak Minty solutions for any $\rho \in \mathbb{R}$.

(ii) By Theorem 2.3*(ii)*, the limit points of the sequence $(\bar{x}^k)_{k\in\mathbb{N}}$ generated by the proximal point algorithm with fixed relaxation parameter λ belong to $\operatorname{zer} T$ (not necessarily to $\mathcal{S}^\star$) if λ is selected according to (2.16). This result has been verified numerically in Figure 2.

If $f \equiv 1$, then the following hold.

(iii) T has a single (b/a^2+b^2)-weak Minty solution at $\mathcal{S}^\star = \operatorname{zer} T = (0,0)$.

(iv) By Theorem 2.3*(vi)*, the sequence $(\bar{x}^k)_{k\in\mathbb{N}}$ generated by the proximal point algorithm with fixed relaxation parameter λ converges to $\operatorname{zer} T$ if λ lies in the interval

$$0 < \lambda < \bar{\lambda} := 2\Big(1 + \tfrac{b}{a^2+b^2}\Big). \tag{2.16}$$

By examining the spectral radius of the algorithmic operator, it can be seen that this result is tight.

The proof is deferred to Appendix C. □

2.2 Linear convergence of PPPA under metric subregularity

Metric subregularity is a well-known concept used to describe the behavior of a set-valued mapping in a neighborhood around a given point and can be seen as a one-point version of metric regularity. Many algorithms used in optimization, such as splitting methods, have been shown to have good convergence properties when applied to operators that are metrically subregular, see e.g. [72, 19, 40, 39, 28]. Metric subregularity is defined as follows.

Definition 2.5 (metric subregularity)**.** *A set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^d$ is* metrically subregular *at $\bar{x}$ for $\bar{y}$ if $(\bar{x},\bar{y}) \in \operatorname{gph} T$ and there exists a positive constant η together with a* neighborhood of subregularity *$\mathcal{U}$ of $\bar{x}$ such that*

$$\operatorname{dist}(x, T^{-1}\bar{y}) \le \eta \operatorname{dist}(\bar{y}, Tx), \qquad \text{for all } x \in \mathcal{U}. \tag{2.17}$$

There exist many prominent examples of operator classes that are metrically subregular. For instance, all piecewise polyhedral mappings are metrically subregular at all points in their graph [17, §3]. Another important example is the subdifferential operator ∂f of a proper lsc convex function satisfying quadratic growth conditions (cf. [1]). We refer the interested reader to [17, §3] and [64, §9] for further discussion.

The main result of this subsection is Theorem 2.6, where it is shown that (local) linear convergence of PPPA can be attained under an additional metric subregularity assumption. This result will in turn lead to sufficient conditions for (local) linear convergence of DRS for (possibly) nonmonotone piecewise polyhedral mappings (cf. Theorem 3.5).

Theorem 2.6 (linear convergence of PPPA under metric subregularity)**.** *In addition to Assumption I, suppose that $\operatorname{zer} T$ is nonempty, closed and convex, that Assumption I.A3 holds with $\mathcal{S}^\star = \operatorname{zer} T$, that $(P+T)^{-1}P$ is continuous, and that T is metrically subregular at all $x^\star \in \operatorname{zer} T$ for 0. Consider a sequence $(x^k,\bar{x}^k)_{k\in\mathbb{N}}$ generated by PPPA starting from $x^0 \in \mathbb{R}^n$ with relaxation parameters $\lambda_k \in (0, 2\alpha_k)$ such that $\liminf_{k\to\infty} \lambda_k(2\alpha_k - \lambda_k) > 0$, where α_k is defined as in (2.10). Then, $(\|\Pi_{\mathcal{R}(P)} x^k - \Pi_{\mathcal{R}(P)} \bar{x}^k\|)_{k\in\mathbb{N}}$ and $(\Pi_{\mathcal{R}(P)} x^k)_{k\in\mathbb{N}}$ converge R-linearly to zero.*

Proof. By Theorem 2.3*(vi)* the sequence $(\bar{x}^k)_{k\in\mathbb{N}}$ converges to a point $x^\star \in \operatorname{zer} T$. Up to possibly discarding initial iterates, it holds that $\bar{x}^k \in \mathcal{U}_{x^\star}$, where $\mathcal{U}_{x^\star}$ denotes the neighborhood of subregularity associated with $x^\star$. Metric subregularity of T at $x^\star$ for 0 along with the fact that $P(x^k - \bar{x}^k) \in T\bar{x}^k$ (cf. Theorem 2.3*(i)*) implies

$$\operatorname{dist}(\bar{x}^k, \operatorname{zer} T) \le \eta \|P(x^k - \bar{x}^k)\|, \tag{2.18}$$

for some $\eta \in (0,\infty)$. The distance of the shadow sequence from the set $\Pi_{\mathcal{R}(P)} \operatorname{zer} T$ may then be bounded as

$$\begin{aligned}\operatorname{dist}(\bar{w}^k, \Pi_{\mathcal{R}(P)} \operatorname{zer} T) = \inf_{v\in\Pi_{\mathcal{R}(P)} \operatorname{zer} T}\{\|v - \bar{w}^k\|\} = \inf_{v\in\operatorname{zer} T}\{\|\Pi_{\mathcal{R}(P)}(v - \bar{x}^k)\|\} \le \|\Pi_{\mathcal{R}(P)}\| \operatorname{dist}(\bar{x}^k, \operatorname{zer} T)\\ \overset{(2.18)}{\le} \eta\|P(x^k - \bar{x}^k)\|\end{aligned} \tag{2.19}$$

where we used that $\|\Pi_{\mathcal{R}(P)}\| = 1$. Note that since the set $\mathcal{S}^\star = \operatorname{zer} T$ is convex, so is $\Pi_{\mathcal{R}(P)} \operatorname{zer} T$, implying that $\operatorname{dist}(\bar{w}^k, \Pi_{\mathcal{R}(P)} \operatorname{zer} T) = \|\bar{w}^k - \Pi_{\Pi_{\mathcal{R}(P)} \operatorname{zer} T}(\bar{w}^k)\|$ is attained. By the triangle inequality

$$\begin{aligned}\|w^k - \Pi_{\Pi_{\mathcal{R}(P)} \operatorname{zer} T}(\bar{w}^k)\| \le \|\bar{w}^k - \Pi_{\Pi_{\mathcal{R}(P)} \operatorname{zer} T}(\bar{w}^k)\| + \|w^k - \bar{w}^k\| = \operatorname{dist}(\bar{w}^k, \Pi_{\mathcal{R}(P)} \operatorname{zer} T) + \|w^k - \bar{w}^k\|\\ \overset{(2.19)}{\le} \eta\|P(x^k - \bar{x}^k)\| + \|w^k - \bar{w}^k\|\end{aligned}$$

$$\overset{(2.14)}{\leq} \xi \|w^k - \bar{w}^k\|_Q,$$

where $\xi = \eta\|P\|^{1/2} + \|P\|^{-1/2}$. It then follows that

$$\operatorname{dist}_Q^2(w^k, \Pi_{\mathcal{R}(P)} \operatorname{zer} T) = \inf_{v \in \Pi_{\mathcal{R}(P)} \operatorname{zer} T} \{\|w^k - v\|_Q^2\} \leq \|w^k - \Pi_{\Pi_{\mathcal{R}(P)} \operatorname{zer} T}(\bar{w}^k)\|_Q^2 \leq \xi^2 \|P\| \|w^k - \bar{w}^k\|_Q^2. \tag{2.20}$$

Since $\liminf_{k\to\infty} \lambda_k(2\alpha_k - \lambda_k) > 0$, there exists some $k_0 \geq 0$ and some $\kappa > 0$ such that $\lambda_k(2\alpha_k - \lambda_k) \geq \kappa$ for all $k \geq k_0$. Consequently, due to (2.12) and (2.20) we obtain for all k large enough that

$$\begin{aligned}
\operatorname{dist}_Q^2(w^{k+1}, \Pi_{\mathcal{R}(P)} \operatorname{zer} T) &\leq \|w^{k+1} - \Pi^Q_{\Pi_{\mathcal{R}(P)} \operatorname{zer} T}(w^k)\|_Q^2 \\
&\overset{(2.12)}{\leq} \|w^k - \Pi^Q_{\Pi_{\mathcal{R}(P)} \operatorname{zer} T}(w^k)\|_Q^2 - \lambda_k(2\alpha_k - \lambda_k)\|w^k - \bar{w}^k\|_Q^2 \\
&= \operatorname{dist}_Q^2(w^k, \Pi_{\mathcal{R}(P)} \operatorname{zer} T) - \lambda_k(2\alpha_k - \lambda_k)\|w^k - \bar{w}^k\|_Q^2 \\
&\overset{(2.20)}{\leq} \left(1 - \tfrac{\kappa}{\xi^2\|P\|}\right) \operatorname{dist}_Q^2(w^k, \Pi_{\mathcal{R}(P)} \operatorname{zer} T),
\end{aligned} \tag{2.21}$$

establishing that the sequence $(\operatorname{dist}_Q^2(w^k, \Pi_{\mathcal{R}(P)} \operatorname{zer} T))_{k\in\mathbb{N}}$ converges Q-linearly to zero. Therefore, it follows from (2.21) that $(\|w^k - \bar{w}^k\|_Q)_{k\in\mathbb{N}}$ converges R-linearly to zero since

$$\operatorname{dist}_Q^2(w^k, \Pi_{\mathcal{R}(P)} \operatorname{zer} T) \geq \lambda_k(2\alpha_k - \lambda_k)\|w^k - \bar{w}^k\|_Q^2 \geq \kappa \|w^k - \bar{w}^k\|_Q^2.$$

Consequently, by (2.11) it follows that

$$\|w^{k+1} - w^k\|_Q^2 = \lambda_k^2 \|\bar{w}^k - w^k\|_Q^2 \leq \tfrac{\lambda_k^2}{\kappa} \operatorname{dist}_Q^2(w^k, \Pi_{\mathcal{R}(P)} \operatorname{zer} T).$$

Combined with the fact that $\lambda_k < 2\alpha_k < +\infty$ is bounded (the latter inequality follows directly from (2.10)), and since the distance on the right-hand side converges Q-linearly to zero, $(\|w^{k+1} - w^k\|_Q)_{k\in\mathbb{N}}$ converges R-linearly to zero. This implies that $(\|w^k - w^\star\|_Q)_{k\in\mathbb{N}}$ (with $w^\star \in \Pi_{\mathcal{R}(P)} \operatorname{zer} T$ being the point where the sequence converges owing to Theorem 2.3*(vi)*) also converges R-linearly to zero. Noting that Q is positive definite completes the proof. □

3 Douglas-Rachford splitting

In the previous section, convergence results of the preconditioned proximal point method have been provided for a class of nonmonotone operators. In this section, we will leverage these results to obtain convergence results of DRS in the nonmonotone setting.

First, we establish the equivalence between DRS and PPPA through the general duality framework of [2]. Subsequently, we provide the convergence results for DRS under an oblique weak Minty assumption on the primal-dual operator. Finally, we show (local) linear convergence for piecewise polyhedral mappings.

3.1 Equivalence between DRS and PPPA

In the monotone setting, the convergence of DRS has been studied in [14] by establishing its connection with the preconditioned proximal point method with a positive semidefinite preconditioner. In the nonmonotone setting, this equivalence heavily relies upon the abstract duality framework introduced in [2], [3, Section 6.9]. In this framework, the inclusion problem (P-I) is referred to as the *primal inclusion*. Associated with the primal inclusion is the *dual inclusion*, given by

$$\text{find} \quad y \in \mathbb{R}^n \quad \text{such that} \quad 0 \in T_{\mathrm{D}} y := -A^{-1}(-y) + B^{-1}(y). \tag{D-I}$$

A key property of the dual inclusion is that the primal inclusion is solvable if and only if the dual inclusion is solvable [3, Proposition 6.9.1]. Finally, the *primal-dual inclusion* is given by

$$\text{find} \quad z = (x, y) \in \mathbb{R}^{2n} \quad \text{such that} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \overbrace{\begin{bmatrix} Ax \\ B^{-1}y \end{bmatrix} + \begin{bmatrix} y \\ -x \end{bmatrix}}^{T_{\mathrm{PD}} z}. \tag{PD-I}$$

For the primal-dual inclusion, it holds by [3, Proposition 6.9.2] that

$$(x^\star, y^\star) \in \operatorname{zer} T_{\mathrm{PD}} \iff (x^\star, -y^\star) \in \operatorname{gph} A,\ (x^\star, y^\star) \in \operatorname{gph} B,$$

where $\operatorname{zer} T_{\mathrm{P}} = \{x^\star \mid \exists y^\star : (x^\star, y^\star) \in \operatorname{zer} T_{\mathrm{PD}}\}$ and $\operatorname{zer} T_{\mathrm{D}} = \{y^\star \mid \exists x^\star : (x^\star, y^\star) \in \operatorname{zer} T_{\mathrm{PD}}\}$. Therefore, the primal-dual inclusion encompasses the solutions of both the primal and the dual inclusion, in the sense that $(x^\star, y^\star)$ is a solution of (PD-I) if and only if $x^\star$ is a solution of (P-I) and $y^\star$ is a solution of (D-I) [3, Proposition 6.9.2]. Hence, to obtain a zero of T_{P}, one may apply PPPA to the primal-dual operator T_{PD} starting from an initial point $z^0 = (x^0, y^0)$ with iterates $z^k = (x^k, y^k)$ and $\bar{z}^k = (\bar{x}^k, \bar{y}^k)$. Most notably, PPPA applied to the primal-dual operator T_{PD} reduces to DRS when the preconditioner P is selected to be equal to

$$P = \begin{bmatrix} \gamma^{-1}\mathrm{I}_n & -\mathrm{I}_n \\ -\mathrm{I}_n & \gamma \mathrm{I}_n \end{bmatrix}, \tag{3.1}$$

where $\gamma \in \mathbb{R}_{++}$ is a strictly positive stepsize parameter. In the following lemma, the equivalence between PPPA and DRS is formally summarized.

Lemma 3.1 (equivalence of DRS and PPPA)**.** *Let $z^0 = (x^0, y^0) \in \mathbb{R}^{2n}$ and set $s^0 = x^0 - \gamma y^0$. Then, to any sequence $(u^k, v^k, s^k)_{k\in\mathbb{N}}$ generated by DRS (initialized with s^0) for solving (P-I), there correspond sequences $(z^k)_{k\in\mathbb{N}} = (x^k, y^k)_{k\in\mathbb{N}}$, $(\bar{z}^k)_{k\in\mathbb{N}} = (\bar{x}^k, \bar{y}^k)_{k\in\mathbb{N}}$ generated by PPPA (initialized with z^0) for solving (PD-I) with preconditioner (3.1) (and vice versa), and the correspondence is as follows.*

(i) $s^k = x^k - \gamma y^k$.

(ii) $\bar{x}^k = u^k$.

(iii) $\bar{y}^k = {}^1\!/\!_\gamma(2u^k - s^k - v^k)$.

Proof. See Appendix B.1. □

3.2 Convergence analysis of DRS under oblique weak Minty

Despite the fact that DRS is a primal algorithm, Lemma 3.1 shows that it can be interpreted as applying PPPA to the associated primal-dual operator T_{PD}. Based on this equivalence, we proceed to derive convergence results for DRS, leveraging the results obtained in Section 2. First, the underlying assumptions for PPPA provided in Assumption I are translated to conditions on the individual operators A and B as follows.

Assumption II. *In problem (P-I), the following hold.*

A1 *Operators A and B are outer semicontinuous.*

A2 *For the selected stepsizes, the resolvents have full domain, i.e.,* $\operatorname{dom} J_{\gamma A} = \operatorname{dom} J_{\gamma B} = \mathbb{R}^n$.

A3 *There exists a nonempty set $\mathcal{S}^\star \subseteq \operatorname{zer} T_{\mathrm{PD}}$ such that the primal-dual operator (PD-I) has V-oblique weak Minty solutions at $\mathcal{S}^\star$, where*

$$V := \operatorname{blkdiag}(\beta_{\mathrm{P}}\mathrm{I}_n, \beta_{\mathrm{D}}\mathrm{I}_n) \tag{3.2}$$

and $\beta_{\mathrm{P}}, \beta_{\mathrm{D}}, \in \mathbb{R}$ satisfy $[\beta_{\mathrm{D}}]_-[\beta_{\mathrm{P}}]_- < \frac{1}{4}$.

Note that these assumptions are not very restrictive. For instance, only the existence of a single oblique weak Minty solution is required for the primal-dual inclusion. Furthermore, it can be seen that the range of β_{P} and β_{D} in Assumption II.A3 covers cases where neither the primal inclusion (P-I) nor the dual inclusion (D-I) satisfies the Minty variational inequality (MVI), which is an immediate consequence of the following lemma (see also Table 1).

Lemma 3.2 (weak Minty for primal and dual operator)**.** *Suppose that Assumption II.A3 holds and let*

$$\mathcal{S}_P^\star := \{x^\star \mid \exists y^\star : (x^\star, y^\star) \in \mathcal{S}^\star\} \quad \text{and} \quad \mathcal{S}_D^\star := \{y^\star \mid \exists x^\star : (x^\star, y^\star) \in \mathcal{S}^\star\}.$$

Then, the primal operator T_{P} has β_{P}-weak Minty solutions at $\mathcal{S}_P^\star$ and the dual operator T_{D} has β_{D}-weak Minty solutions at $\mathcal{S}_D^\star$.

	$\beta_D \geq 0$	$\beta_D < 0$
$\beta_P \geq 0$	T_P, T_D, T_{PD}	T_P
$\beta_P < 0$	T_D	-

Table 1: *Table summarizing which operators from the abstract duality framework satisfy the Minty variational inequality (MVI), depending on the sign of β_P and β_D.*

Proof. See Appendix B.1. □

The following theorem follows as a consequence of Theorem 2.3 and characterizes the convergence of DRS in the nonmonotone setting, assuming the corresponding primal-dual operator has V-oblique weak Minty solutions. To prove this result, we rely upon Lemma A.4, which shows that when Assumption II holds for DRS, Assumptions I.A1 to I.A3 are satisfied for the corresponding preconditioned proximal point algorithm. Note that Assumption I.A4 is not covered by this lemma. To ensure satisfaction of this final assumption, the stepsize γ of DRS needs to be selected appropriately (see condition (3.3)).

Theorem 3.3 (convergence of DRS)**.** *Suppose that Assumption II holds and let*

$$\gamma \in \left(\frac{2[-\beta_P]_+}{1+\sqrt{1-4\beta_D\beta_P}}, \frac{1+\sqrt{1-4\beta_D\beta_P}}{2[-\beta_D]_+} \right). \tag{3.3}$$

Consider a sequence $(u^k, v^k, s^k)_{k\in\mathbb{N}}$ generated by DRS starting from $s^0 \in \mathbb{R}^n$ with stepsize γ and relaxation parameters $\lambda_k \in (0, 2\alpha)$ such that $\liminf_{k\to\infty} \lambda_k(2\alpha - \lambda_k) > 0$, where $\alpha := 1 + \frac{1}{\gamma}\beta_P + \gamma\beta_D > 0$. Then, either a point $(u^k, 1/\gamma(2u^k - s^k - v^k)) \in \operatorname{zer} T_{PD}$ is reached in a finite number of iterations or the following hold for the sequence $(u^k, v^k, s^k)_{k\in\mathbb{N}}$.

(i) $\gamma^{-1}(u^k - v^k) \in Au^k + Bv^k$ for all k, and $(u^k - v^k)_{k\in\mathbb{N}}$ converges to zero.

(ii) Every limit point (if any) of $(u^k, 1/\gamma(2u^k - s^k - v^k))_{k\in\mathbb{N}}$ belongs to $\operatorname{zer} T_{PD}$.

(iii) The sequence $(s^k)_{k\in\mathbb{N}}$ is bounded.

(iv) If $\lambda_k(2\alpha - \lambda_k) \geq \kappa > 0$ for all k, then, for all $z^\star = (x^\star, y^\star) \in \mathcal{S}^\star$ letting $s^\star = x^\star - \gamma y^\star$ we have

$$\min_{k=0,1,\dots,N} \|u^k - v^k\|^2 \leq \frac{\|s^0 - s^\star\|^2}{(N+1)\kappa}.$$

Moreover, if $J_{\gamma A}$ and $J_{\gamma B}$ are (single-valued) continuous, the following hold.

(v) The limit points of $(s^k)_{k\in\mathbb{N}}$ are in $\{(x^\star - \gamma y^\star) \mid (x^\star, y^\star) \in \operatorname{zer} T_{PD}\}$.

(vi) If in Assumption II.A3 $\mathcal{S}^\star = \operatorname{zer} T_{PD}$, then $(s^k)_{k\in\mathbb{N}}$ converges to some element of $\{(x^\star - \gamma y^\star) \mid (x^\star, y^\star) \in \operatorname{zer} T_{PD}\}$ and $(u^k, 1/\gamma(2u^k - s^k - v^k))_{k\in\mathbb{N}}$ converges to some element of $\operatorname{zer} T_{PD}$.

Proof. Recall that DRS is cast as an instance of PPPA, as summarized in Lemma 3.1. Therefore, all the assertions follow from those of Theorem 2.3 applied to the primal-dual inclusion (PD-I) with P as defined in (3.1). Moreover, by Lemma A.4 it follows immediately that Assumptions I.A1 to I.A3 hold so that it only remains to show that Assumption I.A4 holds whenever γ satisfies (3.3).

Throughout the proof, T_{PD} is as in (PD-I). The preconditioning matrix P has two distinct eigenvalues, namely zero and $\gamma+\gamma^{-1}$, each with a multiplicity of n. Therefore, $P = (\gamma+\gamma^{-1})ZZ^\top$, where Z is an orthonormal basis for the range of P as defined in Assumption I.A4. In particular, we have

$$Z = \frac{1}{\sqrt{\gamma^2+1}}\begin{bmatrix} \mathrm{I} \\ -\gamma\mathrm{I} \end{bmatrix} \quad \text{and} \quad \Pi_{\mathcal{R}(P)} = ZZ^\top = \frac{1}{\gamma^2+1}\begin{bmatrix} \mathrm{I} & -\gamma\mathrm{I} \\ -\gamma\mathrm{I} & \gamma^2\mathrm{I} \end{bmatrix}.$$

Therefore, substituting $V := \operatorname{blkdiag}(\beta_P \mathrm{I}_n, \beta_D \mathrm{I}_n)$ into (2.2), it follows that $\eta_{\min} = 1 + \frac{1}{\gamma}\beta_P + \gamma\beta_D$. Owing to Fact A.2*(ii)*, $\eta_{\min} > 0$ iff γ satisfies (3.3), and therefore Assumption I.A4 holds.

Having established that the underlying assumptions of Theorem 2.3 are satisfied, it remains to translate the sequences therein in terms of the variables of DRS. First, leveraging the equivalences established in Lemma 3.1, it follows that

$$P(z^k - \bar{z}^k) = (-\tfrac{1}{\gamma}(u^k - v^k), u^k - v^k) \tag{3.4}$$

$$\mathrm{q}_P(z^k - \bar{z}^k) = \tfrac{1}{\gamma}\|u^k - v^k\|^2 \tag{3.5}$$

$$\mathrm{q}_V(P(z^k - \bar{z}^k)) = (\tfrac{1}{\gamma^2}\beta_{\mathrm{P}} + \beta_{\mathrm{D}})\|u^k - v^k\|^2, \tag{3.6}$$

which shows that α_k defined in (2.10) coincides with α as defined in this theorem.

♠ 3.3*(i)*: By Theorem 2.3*(i)* and (3.4), the sequence $(u^k - v^k)_{k\in\mathbb{N}}$ converges to zero and

$$\tfrac{1}{\gamma}(u^k - v^k) \in A\bar{x}^k + \bar{y}^k, \quad -(u^k - v^k) \in B^{-1}\bar{y}^k - \bar{x}^k,$$

implying the claimed inclusion.

♠ 3.3*(ii)*: It follows from Theorem 2.3*(ii)* that every limit point (if any) of $(\bar{x}^k, \bar{y}^k)_{k\in\mathbb{N}}$ belongs to $\operatorname{zer} T_{\mathrm{PD}}$.

♠ 3.3*(iii)* and 3.3*(v)*: The claims for s^k follow directly from Theorems 2.3*(iii)* and 2.3*(v)* after noting that $\Pi_{\mathcal{R}(P)}\, z^k = \frac{1}{1+\gamma^2}(s^k, -\gamma s^k)$.

♠ 3.3*(iv)*: The claimed rate follows from Theorem 2.3*(iv)* since $\|P(z^k - \bar{z}^k)\|^2 = (1 + \frac{1}{\gamma^2})\|u^k - v^k\|^2$, $\mathrm{q}_P(z^0 - \bar{z}^\star) = \frac{1}{\gamma}\|s^0 - s^\star\|^2$ and $\|P\| = \gamma + \frac{1}{\gamma}$.

♠ 3.3*(v)*: By Lemma 3.1 we know that $\bar{z}^k \in (P + T_{\mathrm{PD}})^{-1}Pz^k$ iff $\bar{x}^k \in J_{\gamma A}(x^k - \gamma y^k)$ and $\bar{y}^k \in {}^1\!/\!_\gamma(2J_{\gamma A}(x^k - \gamma y^k) - x^k + \gamma y^k - J_{\gamma B}(2J_{\gamma A}(x^k - \gamma y^k) - x^k + \gamma y^k))$. Consequently, since $J_{\gamma A}$ and $J_{\gamma B}$ are assumed to be (single-valued) continuous, $(P + T_{\mathrm{PD}})^{-1}P$ is (single-valued) continuous as well. The claimed assertion for s^k then follows immediately from Theorem 2.3*(v)*.

♠ 3.3*(vi)*: The assertion for $(s^k)_{k\in\mathbb{N}}$ is of immediate verification based on the assertion for $(\Pi_{\mathcal{R}(P)}\, z^k)_{k\in\mathbb{N}}$ in Theorem 2.3*(vi)*. The last claim holds since $(\bar{z}^k)_{k\in\mathbb{N}} = ((\bar{x}^k, \bar{y}^k))_{k\in\mathbb{N}}$ converges to an element of $\operatorname{zer} T_{\mathrm{PD}}$. □

Note that in Theorem 3.3*(vi)* we show that under suitable conditions $(u^k, {}^1\!/\!_\gamma(2u^k - s^k - v^k))_{k\in\mathbb{N}}$ converges to some element of $\operatorname{zer} T_{\mathrm{PD}}$. By [3, Proposition 6.9.2], this immediately implies that $(u^k)_{k\in\mathbb{N}}$ converges to some element of $\operatorname{zer} T_{\mathrm{P}}$ and $({}^1\!/\!_\gamma(2u^k - s^k - v^k))_{k\in\mathbb{N}}$ converges to some element of $\operatorname{zer} T_{\mathrm{D}}$. An analogous argument holds for the limit point result of Theorem 3.3*(ii)*.

3.3 Linear convergence of DRS for piecewise polyhedral mappings

In this subsection, we present the convergence rate of DRS for structured inclusion problems involving piecewise polyhedral mappings, which are defined as follows.

Definition 3.4 (piecewise polyhedral mappings)**.** *A set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be piecewise polyhedral if its graph can be expressed as the union of finitely many polyhedral sets.*

This class trivially includes piecewise linear mappings and the subgradient of all piecewise linear quadratic convex functions such as normal cones of polyhedral sets [64, §12].

The following theorem shows that DRS achieves (local) linear convergence for (possibly) nonmonotone piecewise polyhedral mappings. The proof is similar to [39, Lem. IV.4] and relies on the fact that piecewise polyhedral mappings are metrically subregular at all points in their graph [17, §3] and on previous results from Theorem 2.6 and Lemma 3.1.

Theorem 3.5 (linear convergence of DRS for piecewise polyhedral mappings)**.** *Suppose that A and B are piecewise polyhedral mappings, that $\operatorname{zer} T_{\mathrm{PD}}$ is nonempty closed convex, and that Assumption II.A3 holds with $\mathcal{S}^\star = \operatorname{zer} T_{\mathrm{PD}}$. Set the stepsize γ according to (3.3) and suppose that the resolvents $J_{\gamma A}$, $J_{\gamma B}$ have full domain and are continuous. Consider a sequence $(u^k, v^k, s^k)_{k\in\mathbb{N}}$ generated by DRS starting from $s^0 \in \mathbb{R}^n$ with stepsize γ and relaxation parameters $\lambda_k \in (0, 2\alpha)$ such that $\liminf_{k\to\infty} \lambda_k(2\alpha - \lambda_k) > 0$, where α is defined as in Theorem 3.3. Then, $(s^k)_{k\in\mathbb{N}}$ and $(\|u^k - v^k\|)_{k\in\mathbb{N}}$ converge R-linearly to zero.*

Proof. See Appendix B.1. □

4 Semimonotone operators

In this section, we introduce the class of semimonotone operators and provide several examples of well-known function classes with semimonotone subdifferentials. Sufficient conditions for the convergence of DRS applied to the sum of two semimonotone operators will be provided in Section 5.

We begin by providing the definition of (μ,ρ)-semimonotone operators, which can be seen as a natural extension of (hypo)monotone and co(hypo)monotone operators. The semimonotone terminology adapted here is inspired by the notion of semimonotonicity introduced in [51, Def. 2] (which corresponds to the case $\mu = \rho$).

Definition 4.1 (semimonotonicity)**.** *Let $\mu,\rho \in \mathbb{R}$. An operator $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be (μ,ρ)-semimonotone at $(x',y') \in \operatorname{gph} A$ if*

$$\langle x - x', y - y' \rangle \geq \mu\|x - x'\|^2 + \rho\|y - y'\|^2, \qquad \text{for all } (x,y) \in \operatorname{gph} A. \tag{4.1}$$

An operator A is said to be (μ,ρ)-semimonotone if it is (μ,ρ)-semimonotone at all $(x',y') \in \operatorname{gph} A$. It is said to be maximally (μ,ρ)-semimonotone if its graph is not strictly contained in the graph of another (μ,ρ)-semimonotone operator.

Due to the additional degree of freedom this operator class enjoys, it is able to cover a wide array of other commonly used operator classes available in literature. We list the most prominent connections below.

Remark 4.2 (relationship with other types of operators)**.**

(i) $(0,0)$-semimonotonicity is equivalent to the monotonicity of A (and of its inverse A^{-1}).

(ii) $(\mu,0)$-semimonotonicity is equivalent to μ-monotonicity, which is also known as $|\mu|$-*hypo*monotonicity [64, Example 12.28] when $\mu < 0$ and as *strong* monotonicity when $\mu > 0$.

(iii) Similarly, $(0,\rho)$-semimonotonicity is equivalent to ρ-comonotonicity, which is also referred to as $|\rho|$-co*hypo*monotonicity when $\rho < 0$ and as ρ-cocoercivity when $\rho > 0$.

(iv) $(0,\rho)$-semimonotonicity of an operator T at a point $(x^\star,0) \in \operatorname{gph} T$ is equivalent to T having ρ-weak Minty solutions at $\bar{x}^\star \in \operatorname{zer} T$, sometimes also referred to as ρ-star-cocoercivity around $x^\star$ [31, Def. 2.1].

(v) When $\mu = \rho < 0$, our definition of (μ,ρ)-semimonotonicity is equivalent to the definition of $|\rho|$-semimonotonicity proposed in [51, Def. 2].

(vi) The definition of α-averagedness of an operator is equivalent to $(1 - {}^1/_{(2-2\alpha)}, {}^1/_{(2-2\alpha)})$-semimonotonicity [5, Prop. 4.35(iv)]. Additionally, an operator is firmly nonexpansive if and only if it is ${}^1/_2$-averaged [5, Remark 4.34(iii)], which holds for any $(0,\rho)$-semimonotone operator with $\rho \geq 1$.

(vii) When $\mu > 0$ and $\rho < 0$, then the definition of (μ,ρ)-semimonotonicity coincides with the definition of $(-\rho,\mu)$-relaxed-cocoercivity [36, Def. 2.3]. □

Figure 4 provides the Scaled Relative Graph (SRG) [65] of different operator classes and a graphical representation of the connections between them. In Figure 4a, the green (resp. red) hatched area represents the region where all operators (resp. no operators) satisfy (μ,ρ)-semimonotonicity, which is a consequence of the following proposition.

Proposition 4.3 (existence of semimonotone operators)**.** *Let $\mu,\rho \in \mathbb{R}$. Then, the following holds.*

(i) If $[\mu]_-[\rho]_- \geq \frac{1}{4}$, then all operators $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfy the definition of (μ,ρ)-semimonotonicity.

(ii) If $[\mu]_+[\rho]_+ > \frac{1}{4}$, then there does not exist any operator $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ that is (μ,ρ)-semimonotone.

(iii) If $[\mu]_+[\rho]_+ = \frac{1}{4}$, then operator $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$

1. *is (μ,ρ)-semimonotone iff for some $c \in \mathbb{R}^n$ it holds that $\operatorname{gph} A \subseteq \operatorname{gph} T$, where $T : x \mapsto 2\mu x + c$.*
2. *is maximally (μ,ρ)-semimonotone iff for some $c \in \mathbb{R}^n$ it holds that $A : x \mapsto 2\mu x + c$.*

Proof. See Appendix B.2. □

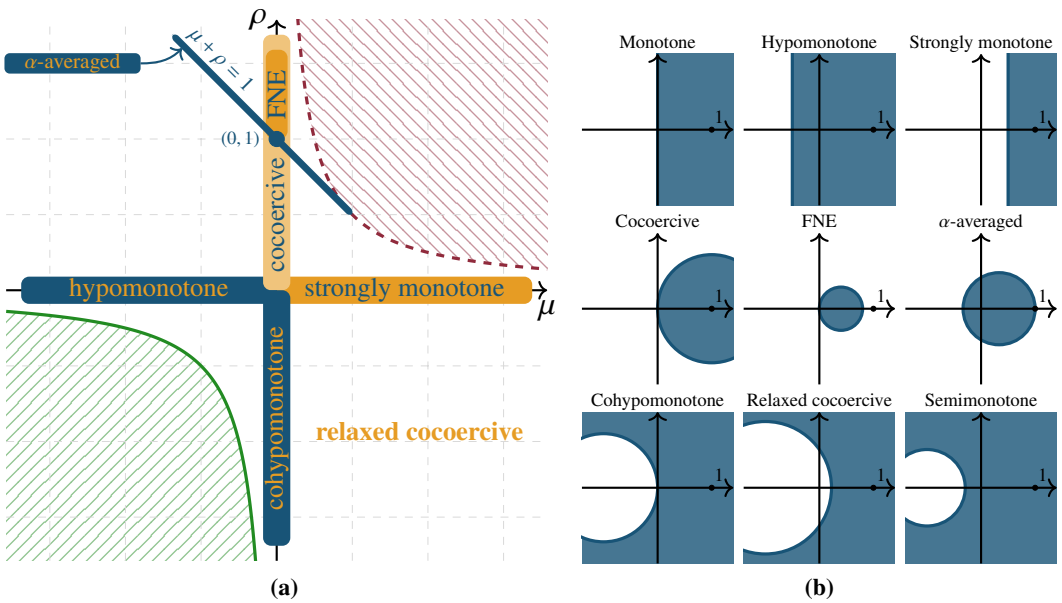

Figure 4: *(left) Relationship between semimonotonicity and other types of operators. The green hatched area represents the region where all operators satisfy (μ,ρ)-semimonotonicity and the red hatched area represents the region for which there do not exist any (μ,ρ)-semimonotone operators (see Proposition 4.3). (right) Scaled Relative Graph (SRG) for different operator classes. For instance, in the top row, the SRG of a μ-monotone operator is visualized [65, Prop. 3.3 and Thm. 3.5]. In the bottom right figure a (μ,ρ)-semimonotone operator with negative μ and ρ is visualized. For more details on the SRG and similar geometric illustrations, we refer the interested reader to [21, 29, 65].*

We proceed to examine basic properties and calculus rules for the class of (maximally) semimonotone operators. For instance, the following proposition states that their inverses belong to the same class of operators, with the roles of μ and ρ reversed.

Proposition 4.4 (inverse operator)**.** *An operator $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is (maximally) (μ,ρ)-semimonotone [at $(x',y') \in \operatorname{gph} A$] if and only if the inverse operator A^{-1} is (maximally) (ρ,μ)-semimonotone [at $(y',x') \in \operatorname{gph} A^{-1}$].*

Linear operators are a particular instance of semimonotone operators. For these operators, there exists an interplay between μ and ρ, which is summarized in the following proposition. This result generalizes [6, Prop. 5.1] for μ-monotone and ρ-comonotone operators. Additionally, a subsequent corollary for multiple of identity operators is provided, including its maximality.

Proposition 4.5 (linear operator)**.** *Let $A \in \mathbb{R}^{n\times n}$ and let $\mu,\rho \in \mathbb{R}$. Then, A is (μ,ρ)-semimonotone if and only if $\frac{1}{2}(A + A^\top) - \rho A^\top A \succeq \mu \mathrm{I}$.*

Proof. Consider $x \in \mathbb{R}^n$. Then, it holds by (μ,ρ)-semimonotonicity that $\langle x, Ax\rangle \geq \mu\|x\|^2 + \rho\|Ax\|^2 \Leftrightarrow \langle x, (A - \mu\mathrm{I} - \rho A^\top A)x\rangle \geq 0 \Leftrightarrow \left\langle x, \left(\frac{1}{2}(A + A^\top) - \mu\mathrm{I} - \rho A^\top A\right)x\right\rangle \geq 0 \Leftrightarrow \frac{1}{2}(A + A^\top) - \mu\mathrm{I} - \rho A^\top A \succeq 0.$ □

Corollary 4.6 (multiple of identity)**.** *If $A = \alpha \mathrm{I}_n$ for some $\alpha \in \mathbb{R}$, then A is $(\alpha(1 - c\alpha), c)$-semimonotone for any $c \in \mathbb{R}$. Furthermore, A is maximal except when $\alpha < 0$ and $c = 1/2\alpha$.*

Proof. See Appendix B.2. □

The upcoming proposition discusses the sum and the parallel sum of two semimonotone operators, A and B. Notice that the semimonotonicity parameters of the resulting operator always involve both a sum and a parallel sum of A and B's semimonotonicity parameters. As expected, the commutative property holds as swapping A and B does not affect the result.

Proposition 4.7 (sum and parallel sum)**.** *Let operator $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be (μ_A,ρ_A)-semimonotone [at $(x'_A,y'_A) \in \operatorname{gph} A$] and operator $B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be (μ_B,ρ_B)-semimonotone [at $(x'_B,y'_B) \in \operatorname{gph} B$]. Then, it holds that*

(i) [if $x' = x'_A = x'_B$, then] $A + B$ is $(\mu_A + \mu_B, \rho_A \boxdot \rho_B)$-semimonotone [at $(x', y'_A + y'_B)$] if either $\rho_A + \rho_B > 0$ or $\rho_A = \rho_B = 0$.

(ii) [if $y' = y'_A = y'_B$, then] $A \boxdot B$ is $(\mu_A \boxdot \mu_B, \rho_A + \rho_B)$-semimonotone [at $(x'_A + x'_B, y')$] if either $\mu_A + \mu_B > 0$ or $\mu_A = \mu_B = 0$.

Proof. See Appendix B.2. □

Combining the previous proposition for the sum with the characterization of multiple of identity operators from Corollary 4.6, one may derive the semimonotonicity of the sum of a semimonotone and a multiple of identity operator. This strategy leads precisely to the statement of Proposition 4.8*(ii)*. However, under a particular condition, a much stronger equivalence relationship can be obtained by exploiting the multiple of identity structure present. This result is provided in Proposition 4.8*(i)* and generalizes [6, Lem. 2.6 & 2.8].

Proposition 4.8 (sum with identity)**.** *Let $\mu_A, \rho_A \in \mathbb{R}$. Consider the operator $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and define $T := A + \alpha \mathrm{I}$ where $\alpha \in \mathbb{R}$. Then, the following hold.*

(i) If $1 + 2\rho_A\alpha > 0$, then T is (maximally) $\left(\frac{\mu_A + \alpha(1+\rho_A\alpha)}{1+2\rho_A\alpha}, \frac{\rho_A}{1+2\rho_A\alpha}\right)$-semimonotone [at $(x', y' + \alpha x') \in \operatorname{gph} T$] if and only if A is (maximally) (μ_A, ρ_A)-semimonotone [at $(x', y') \in \operatorname{gph} A$].

(ii) If $1 + 2\rho_A\alpha \leq 0$, then T is $(\mu_A + \alpha(1 - c\alpha), \rho_A \boxdot c)$-semimonotone [at $(x', y' + \alpha x') \in \operatorname{gph} T$] for all $c > -\rho_A$ if A is (μ_A, ρ_A)-semimonotone [at $(x', y') \in \operatorname{gph} A$].

Proof. See Appendix B.2. □

Leveraging previous results, we now formalize the relationship between the semimonotonicity of an operator T defined in function of another (maximally) monotone operator A. In practical applications, this result can be leveraged to check (maximal) semimonotonicity of a given operator or to generate (maximally) semimonotone operators from a given monotone operator.

Corollary 4.9 (connection to monotone operators)**.** *Let $\xi, \nu \in \mathbb{R}$. Consider the operator $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and define $T := (A + \xi \mathrm{id})^{-1} + \nu \mathrm{id}$. Then, the following hold.*

(i) If $1 + 2\xi\nu > 0$, then T is (maximally) $\left(\frac{\nu(1+\xi\nu)}{1+2\xi\nu}, \frac{\xi}{1+2\xi\nu}\right)$-semimonotone if and only if A is (maximally) monotone.

(ii) If $1 + 2\xi\nu \leq 0$, then T is $\big(\nu(1 - c\nu), \xi \boxdot c\big)$-semimonotone for all $c > -\xi$ if A is monotone.

Proof. By applying Propositions 4.4 and 4.8 it follows that A is (maximally) monotone if and only if $A + \xi \mathrm{id}$ is (maximally) $(\xi, 0)$-semimonotone if and only if $(A + \xi \mathrm{id})^{-1}$ is (maximally) $(0, \xi)$-semimonotone. The claimed result then follows immediately by using Proposition 4.8 again. □

The following corollary is a direct consequence of Corollary 4.9*(i)*, noting that for the choice of ξ and ν from (4.2) it holds that $1 + 2\xi\nu = 1/\sqrt{1-4\rho\mu} > 0$. This result generalizes [51, Thm. 2], which only covers the case $\mu = \rho$.

Corollary 4.10. *A set-valued operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is (maximally) (μ, ρ)-semimonotone with $\rho\mu < \frac{1}{4}$ if and only if $A := (T - \nu \mathrm{id})^{-1} - \xi \mathrm{id}$ is a (maximally) monotone operator where*

$$\xi = \frac{\rho}{\sqrt{1-4\mu\rho}} \quad \text{and} \quad \nu = \frac{2\mu}{1+\sqrt{1-4\mu\rho}}. \tag{4.2}$$

In light of the above connection with maximally monotone operators, a (μ, ρ)-semimonotone operator A with $\rho\mu < \frac{1}{4}$ is maximal if $(A - \nu \mathrm{id})^{-1}$ is continuous (since any continuous monotone mapping is maximally monotone).

The final proposition discusses the domain and the Lipschitz continuity of the resolvent of a semimonotone operator. This result generalizes that of [6, Prop. 2.17] for comonotone operators and [51, Thm. 3], which only covers the case $\mu = \rho$.

Proposition 4.11 (resolvent of a semimonotone operator)**.** *Let $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a (μ, ρ)-semimonotone operator with either $\rho\mu < \frac{1}{4}$ or $[\mu]_+[\rho]_+ = \frac{1}{4}$. Let*

$$\gamma \in \left(\frac{2[-\rho]_+}{1+\sqrt{1-4\rho\mu}}, \frac{1+\sqrt{1-4\rho\mu}}{2[-\mu]_+} \right). \tag{4.3}$$

Then, the following hold.

(i) The resolvent $J_{\gamma A}$ has full domain if and only if A is maximal.

(ii) The resolvent $J_{\gamma A}$ is Lipschitz continuous with Lipschitz constant $L(\gamma) = \frac{|1+2\rho\gamma^{-1}|+\sqrt{1-4\mu\rho}}{2(1+\mu\gamma+\rho\gamma^{-1})}$.

Proof. See Appendix B.2. □

4.1 Examples of functions with semimonotone subdifferentials

In what follows, we present various function classes of which the (limiting) subdifferentials belong to the class of semimonotone operators. We start by studying the class of functions with upper and/or lower curvature, which are defined as follows.

Definition 4.12 (upper and lower curvature)**.** *Let $\sigma, \ell \in \mathbb{R}$ and consider a proper lsc function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Then,*

(i) (lower curvature) f is said to have lower curvature with modulus σ (denoted by $f \in \mathscr{F}_\sigma(\mathbb{R}^n)$) if $\varphi_\sigma := f - \frac{\sigma}{2}\|\cdot\|^2$ is convex.

(ii) (upper curvature) f is said to have upper curvature with modulus ℓ (denoted by $f \in \mathscr{F}^\ell(\mathbb{R}^n)$) if f is finite-valued and $\varphi^\ell := \frac{\ell}{2}\|\cdot\|^2 - f$ is convex.

A function with both upper and lower curvatures is denoted by $\mathscr{F}^\ell_\sigma(\mathbb{R}^n)$.

Many traditional function classes fit within this curvature framework. For instance, functions with positive (resp. negative) lower curvatures are widely known as strongly (resp. hypo) convex functions. Another notable example is that of ℓ-smooth functions, which have upper curvature ℓ and lower curvature $-\ell$ [70, Lem. 2.1].

The next proposition presents the semimonotonicity properties of the subdifferential of functions according to their curvature. Additionally, a corollary is provided for the pointwise minimum of functions with upper curvatures.

Proposition 4.13 (functions with upper and lower curvature)**.** *Let $\sigma, \ell \in \mathbb{R}$. Then, the following hold.*

(i) If $f \in \mathscr{F}_\sigma(\mathbb{R}^n)$, i.e., f is σ-convex, then, ∂f is $(\sigma, 0)$-semimonotone.

(ii) If f is continuous and $f \in \mathscr{F}^\ell(\mathbb{R}^n)$ with $\ell < 0$, then, ∂f is $(0, \frac{1}{\ell})$-semimonotone.

(iii) If f is continuous and $f \in \mathscr{F}^\ell(\mathbb{R}^n)$ with $\ell \geq 0$, then, ∂f is $(\alpha(1-c\alpha), c \,\square\, \frac{1}{\ell})$-semimonotone for all $\alpha > \ell$ and $c > \frac{1}{\alpha-\ell}$.

If $f \in \mathscr{F}^\ell_\sigma(\mathbb{R}^n)$, then, $\sigma \leq \ell$, f is continuously differentiable and the following hold.

(iv) If $\ell + \sigma > 0$, then ∇f is $(\sigma \,\square\, \ell, \frac{1}{\ell+\sigma})$-semimonotone.

(v) If $\ell + \sigma \leq 0$, ∇f is $(\sigma(1-c\sigma), c \,\square\, \frac{1}{\ell-\sigma})$-semimonotone for any constant c such that $1 + c(\ell - \sigma) > 0$.

Proof. See Appendix B.3. □

Corollary 4.14 (pointwise minimum)**.** *Let f be the pointwise min of finite-valued and continuous functions $f_i \in \mathscr{F}^{\ell_{f_i}}(\mathbb{R}^n)$ where $\ell_{f_i} \in \mathbb{R}$, i.e., $f(x) = \min\{f_1(x), \ldots, f_N(x)\}$. Let $\ell_{\max} = \max_i \ell_{f_i}$. Then, the following hold.*

(i) If $\ell_{\max} < 0$, then ∂f is $\left(0, \frac{1}{\ell_{\max}}\right)$-semimonotone.

(ii) If $\ell_{\max} \geq 0$, then ∂f is $\left(\alpha(1-c\alpha), c \,\square\, \frac{1}{\ell_{\max}}\right)$-semimonotone for all $\alpha > \ell_{\max}$ and $c > \frac{1}{\alpha-\ell_{\max}}$.

Proof. See Appendix B.3. □

One of the most fundamental operations for functions is the infimal convolution, which is the subject of the upcoming proposition. To obtain this result, we first show that the subdifferential of the infimal convolution of two semimonotone operators belongs to the parallel sum of the subdifferentials of these two functions (see Proposition 4.15*(i)*), and then we use Proposition 4.7*(ii)*.

Proposition 4.15 (infimal convolution)**.** *For $i = 1, 2$, let $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper lsc functions whose subdifferentials are (μ_i, ρ_i)-semimonotone with $\mu_i, \rho_i \in \mathbb{R}$ such that $\mu_1 + \mu_2 > 0$ or $\mu_1 = \mu_2 = 0$. Denote their infimal convolution by*

$$\varphi : s \mapsto \min_{w \in \mathbb{R}^n} \{f_1(w) + f_2(s - w)\}.$$

Suppose that for each $r_1 \geq 0$ and $\bar{s} \in \mathbb{R}^n$ there exists $r_2 > 0$ such that the set

$$\left\{(w_1, w_2) \in \mathbb{R}^n \;\middle|\; \textstyle\sum_{i=1}^2 f_i(w_i) \leq r_1,\ \left\|\sum_{i=1}^2 w_i - \bar{s}\right\| \leq r_2\right\} \tag{4.4}$$

is bounded. Then, the following hold.

(i) $\partial\varphi(\bar{s}) \subseteq (\partial f_1 \,\square\, \partial f_2)(\bar{s})$.

(ii) $\partial\varphi$ *is* $(\mu_1 \,\square\, \mu_2, \rho_1 + \rho_2)$*-semimonotone.*

Proof. See Appendix B.3. □

As a final example, we show that the associated saddle operator of a minimax optimization problem is semimonotone under the so-called interaction dominance condition [32].

Proposition 4.16 (saddle operator)**.** *Suppose that $\varphi(x, y) : \mathbb{R}^n \times \mathbb{R}^m$ is twice continuously differentiable and σ-convex-concave for some $\sigma \in \mathbb{R}$, i.e., $\nabla^2_{xx}\varphi(z) \succeq \sigma \mathrm{I}$ and $-\nabla^2_{yy}\varphi(z) \succeq \sigma \mathrm{I}$ for all $z \in \mathbb{R}^{n+m}$. Moreover, the α-interaction dominance condition [32, Def. 1] holds for some $\alpha > -\frac{1}{\delta}$ and $\delta \in (0, 1/[-\sigma]_+)$, i.e., for all $z \in \mathbb{R}^{n+m}$*

$$\nabla^2_{xx}\varphi(z) + \nabla^2_{xy}\varphi(z)\Big(\tfrac{1}{\delta}\mathrm{I} - \nabla^2_{yy}\varphi(z)\Big)^{-1}\nabla^2_{yx}\varphi(z) \succeq \alpha \mathrm{I},$$
$$-\nabla^2_{yy}\varphi(z) + \nabla^2_{yx}\varphi(z)\Big(\tfrac{1}{\delta}\mathrm{I} + \nabla^2_{xx}\varphi(z)\Big)^{-1}\nabla^2_{xy}\varphi(z) \succeq \alpha \mathrm{I}.$$

Then, the following hold for the associated saddle operator $T_\varphi(z) \coloneqq (\nabla_x\varphi(z), -\nabla_y\varphi(z))$.

(i) If $\alpha < \frac{1}{\delta}$, then T_φ is $(\frac{\alpha}{1-\alpha\delta}, -\frac{\delta}{1-\alpha\delta})$-semimonotone.

(ii) If $\alpha \geq 0$, then T_φ is $(0, -\delta)$-semimonotone.

Proof. See Appendix B.3. □

5 Douglas-Rachford splitting for semimonotone operators

In practical applications, it might be difficult to determine whether the conditions of Assumption II for DRS are satisfied. To address this issue, we provide a set of sufficient conditions for the convergence of DRS involving the sum of two semimonotone operators. More specifically, we work under the following assumptions on the underlying operators A and B from problem (P-I).

Assumption III. *In problem (P-I), there exist parameters $\mu_A, \rho_A, \mu_B, \rho_B \in \mathbb{R}$ and a nonempty set $\mathcal{S}^\star \subseteq \operatorname{zer} T_{\mathrm{PD}}$ such that for every $(x^\star, y^\star) \in \mathcal{S}^\star$, the following hold.*

A1 *Either $\mu_A + \mu_B > 0$ or $\mu_A = \mu_B = 0$, either $\rho_A + \rho_B > 0$ or $\rho_A = \rho_B = 0$ and $[\mu_A \,\square\, \mu_B]_-[\rho_A \,\square\, \rho_B]_- < \frac{1}{4}$.*

A2 *Operator A is (μ_A, ρ_A)-semimonotone at $(x^\star, -y^\star) \in \operatorname{gph} A$.*

A3 *Operator B is (μ_B, ρ_B)-semimonotone at $(x^\star, y^\star) \in \operatorname{gph} B$.*

Under these assumptions, it can be shown that Assumption II.A3 is satisfied, i.e., that the primal-dual operator (PD-I) has oblique weak Minty solutions. This key result is stated in the following lemma.

Lemma 5.1 (primal-dual operator)**.** *Suppose that Assumption III holds. Then, the primal-dual operator* (PD-I) *has V-oblique weak Minty solutions at $\mathcal{S}^\star$, where V is given by* (3.2) *and*

$$\beta_{\mathrm{P}} = \rho_A \,\square\, \rho_B, \quad \text{and} \quad \beta_{\mathrm{D}} = \mu_A \,\square\, \mu_B. \tag{5.1}$$

Proof. See Appendix B.4. □

Note that not only Assumption II.A3 but also Assumption II.A2 can also be verified through the semimonotonicity properties of the operators A and B. To this end, we consider the following assumption, which includes an additional maximality assumption compared to Assumption III. Under this additional assumption, the resolvents that arise in DRS have full domain by construction (as outlined in Lemma A.5).

Assumption IV. *In problem* (P-I)*, Assumption III holds for maximally semimonotone A and B.*

We are now in a position to state the main result of this section: sufficient conditions for the convergence of DRS, through the semimonotonicity properties of the operators A and B.

Theorem 5.2 (convergence of DRS under semimonotonicity)**.** *Suppose that*

(i) either Assumption II.A1, Assumption II.A2 and Assumption III hold,

(ii) or Assumption II.A1 and Assumption IV hold.

Let γ be selected according to Table 2, where

$$\gamma_- := -\frac{2(\rho_A \square \rho_B)}{1+\sqrt{1-4(\mu_A \square \mu_B)(\rho_A \square \rho_B)}} \quad \text{and} \quad \gamma_+ := -\frac{1+\sqrt{1-4(\mu_A \square \mu_B)(\rho_A \square \rho_B)}}{2(\mu_A \square \mu_B)}. \tag{5.2}$$

Then, $\alpha := 1 + \frac{1}{\gamma}(\rho_A \square \rho_B) + \gamma(\mu_A \square \mu_B) > 0$. Consider a sequence $(u^k, v^k, s^k)_{k\in\mathbb{N}}$ generated by DRS starting from $s^0 \in \mathbb{R}^n$ with stepsize γ and relaxation parameters $\lambda_k \in (0, 2\alpha)$ such that $\liminf_{k\to\infty} \lambda_k(2\alpha - \lambda_k) > 0$. Then, all the claims of Theorem 3.3 hold.

Proof. See Appendix B.4. □

<table>
<tr><td colspan="2" rowspan="2"></td><td rowspan="2">$\mu_A = \mu_B = 0$</td><td colspan="2">$\mu_A + \mu_B > 0$</td></tr>
<tr><td>$\mu_A\mu_B \geq 0$</td><td>$\mu_A\mu_B < 0$</td></tr>
<tr><td colspan="2">$\rho_A = \rho_B = 0$</td><td colspan="2" rowspan="2">$\gamma \in (0, +\infty)$</td><td>$\gamma \in \left(0, -\frac{1}{\mu_A \square \mu_B}\right)$</td></tr>
<tr><td rowspan="2">$\rho_A + \rho_B > 0$</td><td>$\rho_A\rho_B \geq 0$</td><td>$\gamma \in (0, \gamma_+)$</td></tr>
<tr><td>$\rho_A\rho_B < 0$</td><td>$\gamma \in \left(-(\rho_A \square \rho_B), +\infty\right)$</td><td>$\gamma \in (\gamma_-, +\infty)$</td><td>$\gamma \in (\gamma_-, \gamma_+)$</td></tr>
</table>

Table 2: *Range of γ for DRS for semimonotone operators.*

Theorem 5.2 provides a general framework for the convergence of DRS, unifying and extending existing convergence results on relaxed Douglas/Peaceman Rachford splitting methods. We provide several examples.

Remark 5.3 (comparison with existing theory)**.**

(i) First of all, when A and B are maximally $(0, 0)$-semimonotone, i.e., maximally monotone, the convergence result of classic DRS is recovered, see e.g. [44, 23].

(ii) In the strongly monotone case, i.e. when A and B are both μ-strongly monotone, we recover the convergence result from [49], showing that in this case, DRS converges for $\lambda \in (0, 2 + \gamma\mu)$ instead of the traditional $\lambda \in (0, 2)$.

(iii) In [15], DRS is analyzed for the sum of a μ_A- and a μ_B-monotone operator when either $\mu_A + \mu_B > 0$ or $\mu_A = \mu_B = 0$, and their stepsize range matches the range provided by Table 2. Similarly, the results of [4] for the sum of a ρ_A- and a ρ_B-comonotone operator are also covered by our analysis.

(iv) In the optimization setting, global convergence of DRS has recently been shown in [43, 70] for structured nonsmooth, nonconvex optimization problems where the cost is split as the sum of one Lipschitz differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ and one proper lsc function $g : \mathbb{R}^n \to \mathbb{R}$, i.e., $A = \nabla f$ and $B = \partial g$. These results rely on showing descent of the iterates on the so-called Douglas-Rachford envelope [53], for which only properties of the Lipschitz differentiable function f are exploited to determine the range of admissible stepsizes. However, we will show that when the semimonotonicity properties of g are also known, a larger stepsize range for DRS can be used than what was previously known in this setting.

For instance, consider the setting where f is ℓ-smooth and σ-convex, where $\ell > 0$ and $\sigma \in [-\ell, 0)$. We know from Proposition 4.13*(iv)* that in this setting A is (μ_A, ρ_A)-semimonotone for some $\mu_A < 0$ and $\rho_A \geq 0$. When $\arg\min_x f(x) + g(x)$ is nonempty, it is shown in [70] that DRS converges globally for $\lambda \in (0, 2)$ and $\gamma < \min\{\frac{\lambda-2}{2\sigma}, \frac{1}{\ell}\} < -\frac{1}{\sigma}$.

Now, suppose that ∂g is (μ_B, ρ_B)-semimonotone at $(x^\star, y^\star) \in \operatorname{gph} \partial g$ where $\mu_A + \mu_B > 0$ and $\rho_B \geq 0$. Then, by Theorem 5.2 we know that DRS converges for $\gamma \in (0, \gamma_+)$. As it is of immediate verification that $\gamma_+ > -\frac{1}{\sigma}$, our theory allows to choose a larger range of stepsizes γ when information about the semimonotonicity of the nonsmooth term is provided.

□

6 Example problems

In this section, we provide several example problems for DRS, highlighting the wide range of applications our theory is able to cover. Proof of the assertions provided in this section can be found in Appendix C.

We commence by applying DRS to a simple linear inclusion problem. This setting is of particular interest for two specific reasons. Firstly, this example demonstrates the tightness of the stepsize range provided in Theorem 3.3 (see Example 6.1*(ii)*). Secondly, this example highlights that our theory is able to cover cases where neither the primal nor the dual nor the primal-dual inclusion is monotone (see Example 6.1*(iii)*).

Example 6.1 (saddle point problem)**.** Consider the problem of finding a stationary point of a linear inclusion, with a particular structure that emerges naturally in saddle point problems [25, §1.4.1]

$$0 \in T_{\mathrm{P}} x = \begin{bmatrix} b & a \\ -a & b \end{bmatrix} x = \underbrace{\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}}_{B} x, \tag{6.1}$$

where $a, b \in \mathbb{R} \setminus \{0\}$. In particular, any solution to the inclusion problem $0 \in T_{\mathrm{P}} x$ is a minimax solution of $f(x_1, x_2) := a x_1 x_2 + \frac{b}{2}(x_1^2 - x_2^2)$ when $b > 0$ and a maximin solution when $b < 0$. For this problem, the following assertions hold.

(i) By examining the spectral radius of the algorithmic operator, it can be seen that the sequence $(u^k, v^k, s^k)_{k \in \mathbb{N}}$ generated by DRS with fixed relaxation parameter λ converges iff λ lies in the interval

$$0 < \lambda < \bar{\lambda} := 2\frac{(a^2\gamma + b)(1 + \gamma b)}{\gamma(a^2 + b^2)}, \tag{6.2}$$

which is nonempty iff either $b > 0$ or

$$b < 0,\ a^2 \neq b^2 \text{ and } \gamma \in \left(\min\{-\tfrac{1}{b}, -\tfrac{b}{a^2}\}, \max\{-\tfrac{1}{b}, -\tfrac{b}{a^2}\}\right).$$

A numerical experiment verifying this convergence result is provided in Figure 5.

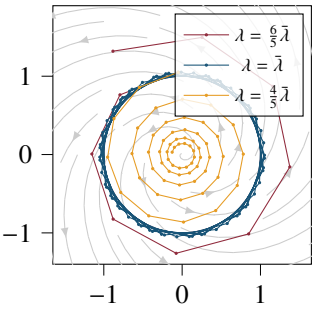

Figure 5: *Sequences u^k generated by applying DRS on Example 6.1 for $a = 2$, $b = -1$ and stepsize $\gamma = \frac{1}{2}$.*

(ii) The results from Theorem 3.3 are tight in the sense that the entire range of relaxation parameters λ from (6.2) is covered, when in (2.1) the vector v is restricted to $\mathcal{R}(P)$ (see the remark below Definition 2.1).

(iii) The range of a and b from Example 6.1*(i)* for which DRS converges covers cases where neither the primal, nor the dual, nor the primal-dual problem is monotone.

Additionally, an analysis of the tightness of Theorem 5.2 is provided in Appendix C. In this analysis, the semimonotonicity parameters $\mu_A, \mu_B, \rho_A, \rho_B$ are selected according to Proposition 4.5 such that the relaxation parameter in Theorem 5.2 is maximized. Notice that although A is obviously $(0,0)$-semimonotone, the maximal relaxation parameter is in general attained for other values of μ_A and ρ_A. □

Next, we turn our attention to a nonsmooth optimization problem. This example highlights the generality of our underlying assumptions, as the convergence of DRS for this problem is not covered by any previous works on DRS (to the best of the authors' knowledge).

Example 6.2 (nonsmooth optimization)**.** Consider the separable nonsmooth optimization problem

$$\underset{x}{\text{minimize}} \quad f_A(x) + f_B(x)$$

which is visualized in Figure 6, where

$$f_A(x) := \begin{cases} 3x^2 - 3x + 8, & \text{if } x \in [-1, 1], \\ -\frac{1}{2}x^2 - 3x + \frac{23}{2}, & \text{otherwise,} \end{cases}$$

$$f_B(x) := \begin{cases} 2x^2 - x, & \text{if } x < -1, \\ x^2 + 3x + 5, & \text{if } x \in [-1, 2], \\ x^2 + 15x - 19, & \text{if } x > 2. \end{cases}$$

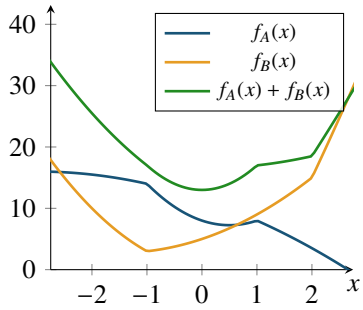

Figure 6: *Visualization of the (nonsmooth) functions involved in Example 6.2.*

The following hold for the inclusion $0 \in T_{\mathrm{P}} x = \partial f_A(x) + \partial f_B(x)$.

(i) Assumption II.A1 and Assumption II.A2 hold when $\gamma \neq 1$.

(ii) Assumption III holds for $\mathcal{S}^\star = \{(0, 3)\}$ and $\mu_A = -1.2, \rho_A = 0.2, \mu_B = 1.6$ and $\rho_B = 0.1$.

(iii) Theorems 3.3*(i)* to 3.3*(iv)* hold for the sequence $(u^k, v^k, s^k)_{k\in\mathbb{N}}$ generated by DRS with stepsize $\gamma \in (0, 0.2614)$ and relaxation parameter $\lambda \in (0, 2 + 2/15\gamma - 48\gamma/5)$. □

Remark 6.3 (comparison with previous works)**.**

(i) Since both f_A and f_B are nonsmooth, convergence results for DRS in the optimization setting such as [43, 70] can not be applied here.

(ii) Since in this example $\rho_A > 0$ and $\rho_B > 0$, it holds by definition that ∂f_A is μ_A-monotone at $(x^\star, -y^\star)$ and ∂f_B is μ_B-monotone at $(x^\star, y^\star)$. However, as the monotonicity of A and B does not hold globally, the results from [15] are not applicable here. □

Finally, consider the following example, where we apply DRS to the subdifferential of a separable nonsmooth function. For this example, we show that our analysis of DRS covers its convergence behavior towards the stationary points, and provide numerical results in Figure 7. Notice that in the iterates are attracted by the local minimum, while in contrast e.g. descent methods would quickly diverge.

Example 6.4 (stationary point)**.** Let $\mu_A = -0.3, \rho_A = -0.1, \mu_B = 0.4$ and $\rho_B = 0.4$. Consider the problem of finding a stationary point of $f_A(x) + f_B(x)$, visualized in Figure 7, where

$$f_A(x) := \begin{cases} \frac{1-\sqrt{1-4\mu_A\rho_A}}{4\rho_A}x^2 - x - 3, & \text{if } x \in [-3, 3], \\ \frac{1+\sqrt{1-4\mu_A\rho_A}}{4\rho_A}x^2 - x - \frac{9\sqrt{1-4\mu_A\rho_A}}{2\rho_A} - 3, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_B(x) := \begin{cases} x^2 + x + 2, & \text{if } x \in [-1, 1], \\ \frac{1}{4}x^2 + x + \frac{11}{4}, & \text{otherwise.} \end{cases}$$

The following hold for the inclusion $0 \in T_{\mathrm{P}} x = \partial f_A(x) + \partial f_B(x)$.

(i) Assumption II.A1 and Assumption II.A2 hold for all $\gamma \in \left(0, \frac{-2\rho_A}{1+\sqrt{1-4\mu_A\rho_A}}\right) \cup \left(\frac{-2\rho_A}{1+\sqrt{1-4\mu_A\rho_A}}, 1/5\right)$.

(ii) Assumption III holds for $\mathcal{S}^\star = \{(0, -1)\}$ and the given $\mu_A, \rho_A, \mu_B, \rho_B$.

(iii) Theorems 3.3*(i)* to 3.3*(iv)* hold for the sequence $(u^k, v^k, s^k)_{k\in\mathbb{N}}$ generated by DRS with stepsize $\gamma \in (1/6, 1/5)$ and fixed relaxation parameter $\lambda \in (0, 2 - 4/15\gamma - 12\gamma/5)$ (see Figure 7 for numerical results). □

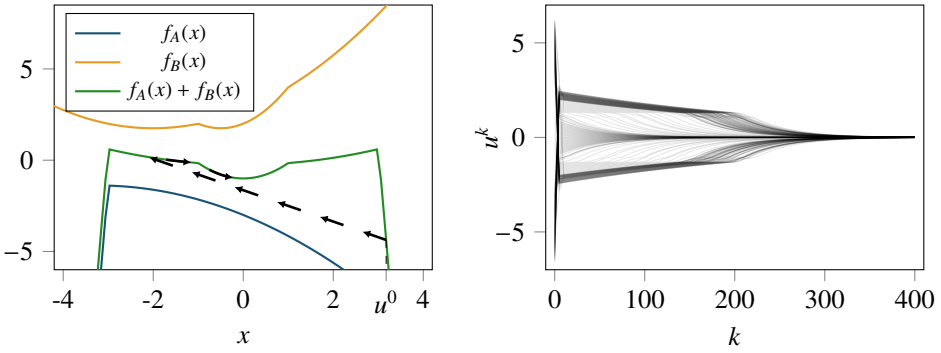

Figure 7: *(left) Visualization of the functions involved in Example 6.4. Additionally, the sequence $(u^k)_{k=0}^{400}$ is indicated using black arrows, which is obtained by applying DRS to $0 \in \partial f_A(x) + \partial f_B(x)$ with stepsize $\gamma = {}^{11}/_{60} \in ({}^1/_6, {}^1/_5)$ and relaxation $\lambda = {}^9/_{10}(2 - {}^4/_{15\gamma} - {}^{12\gamma}/_5) \approx 0.095$. When evaluating the (multi-valued) resolvents $J_{\gamma A}$ and $J_{\gamma B}$, each element is sampled with equal probability. (right) Visualization of the sequence $(u^k)_{k=0}^{400}$ for 200 such experiments, of which the initializations s^0 are evenly spaced within the interval $[-5, 5]$.*

7 Conclusion

This paper presented a comprehensive study of the (relaxed) preconditioned proximal point algorithm (PPPA) and the (relaxed) Douglas-Rachford splitting (DRS) method for a class of nonmonotone problems that satisfy an oblique weak Minty condition. To make the achieved results for DRS readily applicable in practice, the class of semimonotone operators was introduced, and sufficient conditions were developed for the convergence of DRS for the sum of two semimonotone operators. Various relevant example problems were provided thoughout the paper, showing tightness of the achieved convergence results.

Future research directions include investigating inexact variants of PPPA, analyzing other splitting methods in nonmonotone settings, and studying applications in noncooperative game theory and nonconvex distributed optimization.

A Auxiliary lemmas

Lemma A.1. *Let $P \in \mathbb{S}_+^n$ be a symmetric positive semidefinite matrix , Z be an orthonormal basis for $\mathcal{R}(P)$ and define $Q := P + \Pi_{\mathcal{N}(P)}$. Then, the following properties hold.*

(i) The projection onto the range of P is given by $\Pi_{\mathcal{R}(P)} = ZZ^\top$.

(ii) $P = P\,\Pi_{\mathcal{R}(P)} = \Pi_{\mathcal{R}(P)}\,P = Q\,\Pi_{\mathcal{R}(P)}$.

(iii) $Q \succ 0$.

(iv) $Z^\top PZ = Z^\top QZ \succ 0$.

Proof. Statement A.1*(i)* follows from [47, §5.13] and A.1*(ii)* follows directly by definition of P and Q.

♠ A.1*(iii)*: Let $x \in \mathbb{R}^n \backslash \{0\}$ and consider decomposing it as $x = x_1 + x_2$ where $x_1 \in \mathcal{R}(P)$ and $x_2 \in \mathcal{N}(P)$. By definition of Q we have

$$\langle x, Qx\rangle = \langle x, Px\rangle + \langle x, \Pi_{\mathcal{N}(P)}\,x\rangle = \langle x_1, Px\rangle + \langle x_2, \Pi_{\mathcal{N}(P)}\,x\rangle = \langle x_1, Px_1\rangle + \|x_2\|^2 > 0$$

where the inequality follows from Lemma A.1*(iv)* after noting that $x_1 = Zw$ for some w.

♠ A.1*(iv)*: The equivalence $Z^\top PZ = Z^\top QZ$ follows from Lemmas A.1*(i)* and A.1*(ii)* using $Z^\top Z = \mathrm{I}$. Positive definiteness follows from Lemma A.1*(iii)* and the fact that Z has full column rank [7, Prop. 8.1.2(xiii)]. □

Fact A.2 (Solution of quadratic inequality)**.** *Let $a, b, c \in \mathbb{R}$ and suppose that $b > -2\sqrt{ac}$ when $ac \geq 0$. Then,*

(i) There exists a $\gamma > 0$ satisfying $a\gamma^2 + b\gamma + c > 0$ if and only if it holds that

$$b > 2\sqrt{ac} \quad \text{when} \quad a < 0 \text{ and } c < 0. \tag{A.1}$$

<table>
<tr><td colspan="2" rowspan="2"></td><td rowspan="2">$\mu_A = \mu_B = 0$</td><td colspan="2">$\mu_A + \mu_B > 0$</td></tr>
<tr><td>$\mu_A\mu_B \geq 0$</td><td>$\mu_A\mu_B < 0$</td></tr>
<tr><td colspan="2">$\rho_A = \rho_B = 0$</td><td colspan="2" rowspan="2">$\gamma \in (0, +\infty)$</td><td>$\gamma \in \left(0, -\frac{1}{\min\{\mu_A,\mu_B\}}\right)$</td></tr>
<tr><td rowspan="2">$\rho_A + \rho_B > 0$</td><td>$\rho_A\rho_B \geq 0$</td><td>$\gamma \in (0, \bar{\gamma}_+)$</td></tr>
<tr><td>$\rho_A\rho_B < 0$</td><td>$\gamma \in (-\min\{\rho_A,\rho_B\}, +\infty)$</td><td>$\gamma \in (\bar{\gamma}_-, +\infty)$</td><td>$\gamma \in (\bar{\gamma}_-, \bar{\gamma}_+)$</td></tr>
</table>

Table 3: *Ranges of γ for which the resolvents $J_{\gamma A}$ and $J_{\gamma B}$ have full domain.*

(ii) If (A.1) *holds, then $\gamma > 0$ satisfies $a\gamma^2 + b\gamma + c > 0$ if and only if $\gamma \in \left(\frac{2[-c]_+}{b+\sqrt{b^2-4ac}}, \frac{b+\sqrt{b^2-4ac}}{2[-a]_+}\right)$.*

Fact A.3. *Let $a, b \in \mathbb{R}$ such that either $a + b > 0$ or $a = b = 0$. Then, $a \,\square\, b < \min\{a, b\}$.*

Lemma A.4 (implications of Assumption II)**.** *If Assumption II holds, then Assumptions I.A1 to I.A3 hold for operator T_{PD} from* (PD-I) *and preconditioner P as in* (3.1)*.*

Proof. First, outer semicontinuity of T_{PD} follows from that of A and B [64, Theorem 5.7(a)], showing I.A1. Second, I.A2 holds since $J_{\gamma A}$ and $J_{\gamma B}$ having full domain implies that the preconditioned resolvent $(P+T_{\mathrm{PD}})^{-1}P$ has full domain, owing to [64, Lemma 12.14]. Finally, statement I.A3 is immediate, completing the proof. □

Lemma A.5 (implications of Assumption IV)**.** *Suppose that Assumption IV holds and that γ is selected according to Table 2, where γ_- and γ_+ are defined as in (5.2). Then, Assumption II.A2 holds and the resolvents $J_{\gamma A}$ and $J_{\gamma B}$ are Lipschitz continuous.*

Proof. First, observe that $[\mu_A \,\square\, \mu_B]_-[\rho_A \,\square\, \rho_B]_- < \frac{1}{4}$, which by Fact A.3 and Proposition 4.3*(ii)* implies that $[\mu_A]_-[\rho_A]_- < \frac{1}{4}$ and $[\mu_B]_-[\rho_B]_- < \frac{1}{4}$. Therefore, owing to Proposition 4.11*(i)*, it follows that the resolvents $J_{\gamma A}$ and $J_{\gamma B}$ are Lipschitz continuous and have full domain if γ complies with Table 3, where

$$\bar{\gamma}_- = \begin{cases} \frac{-2\rho_A}{1+\sqrt{1-4\mu_A\rho_A}}, & \text{if } \rho_A < 0, \\ \frac{-2\rho_B}{1+\sqrt{1-4\mu_B\rho_B}}, & \text{otherwise,} \end{cases} \quad \text{and} \quad \bar{\gamma}_+ = \begin{cases} -\frac{1+\sqrt{1-4\mu_A\rho_A}}{2\mu_A}, & \text{if } \mu_A < 0, \\ -\frac{1+\sqrt{1-4\mu_B\rho_B}}{2\mu_B}, & \text{otherwise.} \end{cases} \tag{A.2}$$

As γ is selected according to Table 2, it simply remains to prove that these ranges are a subset of the ranges for γ provided in Table 3, which can be verified directy using Fact A.3. □

B Omitted proofs

B.1 Douglas-Rachford splitting

Proof of Lemma 3.1 *(equivalence of DRS and PPPA).* The update rule (PPPA) applied to (PD-I) with preconditioner (3.1) can be equivalently written as

$$\bar{x}^k \in J_{\gamma A}(x^k - \gamma y^k) \tag{B.1a}$$

$$\bar{y}^k \in (\gamma \mathrm{id} + B^{-1})^{-1}(2\bar{x}^k - x^k + \gamma y^k) \overset{[64,\ \text{Lem. } 12.14]}{\iff} 2\bar{x}^k - x^k + \gamma y^k - \gamma\bar{y}^k \in J_{\gamma B}(2\bar{x}^k - x^k + \gamma y^k) \tag{B.1b}$$

$$x^{k+1} = x^k + \lambda_k(\bar{x}^k - x^k) \tag{B.1c}$$

$$y^{k+1} = y^k + \lambda_k(\bar{y}^k - y^k). \tag{B.1d}$$

We proceed by induction. The base case $k = 0$ is vacuously satisfied through initialization $s^0 = x^0 - \gamma y^0$, and by using (B.1a) and (B.1b), noting that for any $\bar{x}^0$, $\bar{y}^0$ there exists $u^0 = \bar{x}^0 \in J_{\gamma A}(s^0)$, $v^0 = 2u^0 - s^0 - \gamma\bar{y}^0 \in J_{\gamma B}(2u^0 - s^0)$ (and vice versa). Suppose now that the claim holds for some $k \geq 0$. In particular, $s^k = x^k - \gamma y^k$, $u^k = \bar{x}^k$ and $v^k = 2u^k - s^k - \gamma\bar{y}^k$. According to update rule (DRS), it holds that

$$s^{k+1} = s^k + \lambda_k(v^k - u^k) = s^k + \lambda_k(\bar{x}^k - s^k - \gamma\bar{y}^k) = x^k - \gamma y^k + \lambda_k(\bar{x}^k - \gamma\bar{y}^k - x^k + \gamma y^k) = x^{k+1} - \gamma y^{k+1},$$

where the induction hypothesis was used in the first two equalities, and (B.1c) and (B.1d) were used in the last equality. Having established $s^{k+1} = x^{k+1} - \gamma y^{k+1}$, the claim for the correspondences between $\bar{x}^{k+1}, \bar{y}^{k+1}$ and u^{k+1}, v^{k+1} follow from (B.1a) and (B.1b) as argued in the base case. □

Proof of Lemma 3.2 *(weak Minty for primal and dual operator).* Let $(x_A, y_A) \in \operatorname{gph} A$ and $(x_B, y_B) \in \operatorname{gph} B$. Then, by definition of the primal-dual operator and using that $(y_B, x_B) \in \operatorname{gph} B^{-1}$ it holds that $(z, v) := \big((x_A, y_B), (y_A + y_B, x_B - x_A)\big) \in \operatorname{gph} T_{\mathrm{PD}}$. Therefore, by Assumption II.A3 it holds for all $z^\star = (x^\star, y^\star) \in \mathcal{S}^\star$ that $\langle v, z - z^\star \rangle \geq \mathrm{q}_V(v)$, i.e.,

$$\left\langle y_A + y_B, x_A - x^\star \right\rangle + \left\langle x_B - x_A, y_B - y^\star \right\rangle \geq \beta_{\mathrm{P}} \|y_A + y_B\|^2 + \beta_{\mathrm{D}} \|x_B - x_A\|^2. \tag{B.2}$$

Let $x_A = x_B =: x_P$. Then, from (B.2) it follows that $\langle y_A + y_B, x_P - x^\star \rangle \geq \beta_{\mathrm{P}} \|y_A + y_B\|^2$ for all $x^\star \in \mathcal{S}_P^\star$. Since $(x_P, y_A + y_B) \in \operatorname{gph} T_{\mathrm{P}}$ it is of immediate verification that T_{P} has β_{P}-weak Minty solutions at $\mathcal{S}_P^\star$.

Similarly, let $-y_A = y_B =: y_D$. Then, due to (B.2) $\langle x_B - x_A, y_D - y^\star \rangle \geq \beta_{\mathrm{D}} \|x_B - x_A\|^2$ for all $x^\star \in \mathcal{S}_D^\star$ and the claim for T_{D} follows analogously from the fact that $(y_D, x_B - x_A) \in \operatorname{gph} T_{\mathrm{D}}$, completing the proof. □

Proof of Theorem 3.5 *(linear convergence of DRS for piecewise polyhedral mappings).*
Since operator B is piecewise polyhedral so is its inverse, and in turn so is the mapping $\hat{T} = A \times B^{-1}$. Therefore, T_{PD} being the sum of $\hat{T}$ and a linear mapping is also piecewise polyhedral. Since the inverse of a piecewise polyhedral mapping is piecewise polyhedral, it follows from [17, 3H.1 and 3H.3] that T_{PD} is metrically subregular at any z for any v with $(z, v) \in \operatorname{gph} T_{\mathrm{PD}}$. The claimed rates for DRS are an immediate consequence of Theorem 2.6, owing to the relations $U_1^\top z^k = s^k$ and (3.5). □

B.2 Semimonotone operators

Proof of Proposition 4.3 *(existence of semimonotone operators).* Consider $(x, y), (x', y') \in \operatorname{gph} A$ and let $\epsilon > 0$. Then, by the Fenchel-Young inequality, it holds that

$$-\epsilon \|x - x'\|^2 - \epsilon^{-1} \|y - y'\|^2 \leq 2\langle x - x', y - y' \rangle \leq \epsilon \|x - x'\|^2 + \epsilon^{-1} \|y - y'\|^2, \tag{B.3}$$

♠ 4.3*(i)*: The first inequality of (B.3) implies that (4.1) is always satisfied for $\mu \leq -\frac{\epsilon}{2}$ and $\rho \leq -\frac{1}{2\epsilon}$, i.e., for $[\mu]_- [\rho]_- \geq \frac{1}{4}$, which completes the proof.

♠ 4.3*(ii)*: The second inequality of (B.3) implies that (4.1) can never be satisfied when $\mu > \frac{\epsilon}{2}$ and $\rho > \frac{1}{2\epsilon}$, i.e., when $[\mu]_+ [\rho]_+ > \frac{1}{4}$.

♠ 4.3*(iii)*: The second inequality of (B.3) implies that (4.1) can only be satisfied with equality when $\mu = \frac{\epsilon}{2} > 0$ and $\rho = \frac{1}{2\epsilon} > 0$, i.e.,

$$\langle x - x', y - y' \rangle = \mu \|x - x'\| + \tfrac{1}{4\mu} \|y - y'\|^2, \qquad \text{for all } (x, y), (x', y') \in \operatorname{gph} A,$$

which is equivalent to

$$\mu \left\| x - x' - \tfrac{1}{2\mu}(y - y') \right\|^2 = 0, \qquad \text{for all } (x, y), (x', y') \in \operatorname{gph} A.$$

This may only hold if $\operatorname{gph} A \subseteq \operatorname{gph} T$ for some $c \in \mathbb{R}^n$. Since $\operatorname{gph} T$ cannot be enlarged by another point $(\hat{x}, \hat{y})$ while maintaining (μ, ρ)-semimonotonicity, T is maximally (μ, ρ)-semimonotone, and the proof is complete. □

***Proof of Corollary 4.6** (multiple of identity).* From Proposition 4.5 and using that $A = \alpha \mathrm{I}_n$, it follows that A is (μ, ρ)-semimonotone if and only if $\alpha \geq \mu + \rho\alpha^2$, which is trivially satisfied when $(\mu, \rho) = (\alpha(1 - c\alpha), c)$ for any $c \in \mathbb{R}$. Now, let $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^n$ and suppose that

$$\langle x - \hat{x}, \alpha x - \hat{y} \rangle \geq \alpha(1 - c\alpha)\|x - \hat{x}\|^2 + c\|\alpha x - \hat{y}\|^2, \qquad \forall x \in \mathbb{R}^n. \tag{B.4}$$

To show when A is maximally $(\alpha(1 - c\alpha), c)$-semimonotone, we must show when it is guaranteed that $(\hat{x}, \hat{y}) \in \operatorname{gph} A$, i.e., that $\hat{y} = \alpha\hat{x}$. Through elementary algebra, it can be shown that (B.4) is equivalent to

$$0 \geq (1 - 2c\alpha)\langle x, \hat{y} - \alpha\hat{x} \rangle + \alpha(1 - c\alpha)\|\hat{x}\|^2 - \langle \hat{x}, \hat{y} \rangle + c\|\hat{y}\|^2, \qquad \forall x \in \mathbb{R}^n. \tag{B.5}$$

Due to the dependence of the first term on x, necessarily either $\hat{y} = \alpha\hat{x}$ or $c = \frac{1}{2\alpha}$. Plugging in $\hat{y} = \alpha\hat{x}$ into (B.5), we get $0 \geq 0$ (as expected). Plugging in $c = \frac{1}{2\alpha}$ into (B.5), we get

$$0 \geq \tfrac{1}{2}\alpha\|\hat{x}\|^2 - \langle \hat{x}, \hat{y} \rangle + \tfrac{1}{2\alpha}\|\hat{y}\|^2 = \tfrac{1}{2\alpha}\|\alpha\hat{x} - \hat{y}\|^2, \qquad \forall x \in \mathbb{R}^n.$$

and thus (B.5) holds if and only if either (i) $\hat{y} = \alpha\hat{x}$ or (ii) $c = \frac{1}{2\alpha}$ and $\alpha < 0$, completing the proof. □

***Proof of Proposition 4.7** (sum and parallel sum).*

♠ 4.7*(i)*: First, consider the case where A and B are only semimonotone at a single point. Let $(x, y_A) \in \operatorname{gph} A$ and $(x, y_B) \in \operatorname{gph} B$. Then, it holds that

$$\begin{aligned}\langle x - x', y_A + y_B - (y'_A + y'_B) \rangle &= \langle x - x', y_A - y'_A \rangle + \langle x - x', y_B - y'_B \rangle \\ &\geq (\mu_A + \mu_B)\|x - x'\|^2 + \rho_A\|y_A - y'_A\|^2 + \rho_B\|y_B - y'_B\|^2.\end{aligned}$$

Consequently, by definition $A + B$ is $(\mu_A + \mu_B, \rho)$-semimonotone at $(x', y'_A + y'_B)$ if and only if

$$\rho\|y_A + y_B - (y'_A + y'_B)\|^2 = \rho\|y_A - y'_A\|^2 + 2\rho\langle y_A - y'_A, y_B - y'_B \rangle + \rho\|y_B - y'_B\|^2 \leq \rho_A\|y_A - y'_A\|^2 + \rho_B\|y_B - y'_B\|^2,$$

which is equivalent to the LMI $\left(\begin{smallmatrix} \rho_A - \rho & -|\rho| \\ -|\rho| & \rho_B - \rho \end{smallmatrix}\right) \succeq 0$. This is satisfied if and only if the determinant and trace are nonnegative, i.e.,

$$\rho_A\rho_B - \rho(\rho_A + \rho_B) \geq 0 \ \text{ and } \ \rho_A + \rho_B - 2\rho \geq 0 \quad \Leftrightarrow \quad \rho \leq \begin{cases} \frac{\rho_A\rho_B}{\rho_A + \rho_B}, & \text{if } \rho_A + \rho_B > 0, \\ 0, & \text{if } \rho_A = \rho_B = 0. \end{cases} \tag{B.6}$$

Since by definition the largest possible value for ρ satisfying (B.6) is given by $\rho_A \,\square\, \rho_B$ (if either $\rho_A + \rho_B > 0$ or $\rho_A = \rho_B = 0$), the first part of the proof is completed.

If A and B are semimonotone then by definition A is (μ_A, ρ_A)-semimonotone at all $(x'_A, y'_A) \in \operatorname{gph} A$ and B is (μ_B, ρ_B)-semimonotone at all $(x'_B, y'_B) \in \operatorname{gph} B$. Hence, using the previous result we know that $A + B$ is $(\mu_A + \mu_B, \rho_A \,\square\, \rho_B)$-semimonotone at all points in the set

$$\{(x', v'_A + v'_B) \mid x' \in \operatorname{dom} A \cap \operatorname{dom} B,\ v'_A \in Ax',\ v'_B \in Bx'\}.$$

Since this set is equal to $\operatorname{gph}(A + B)$, the proof is completed.

♠ 4.7*(ii)*: This is an immediate consequence of the definition of the parallel sum, owing to Proposition 4.4 and Proposition 4.7*(i)*. □

***Proof of Proposition 4.8** (sum with identity).*

♠ 4.8*(i)*: First, consider the assertion where the semimonotonicity holds only at a single point. Operator A is (μ_A, ρ_A)-semimonotone at $(x', y') \in \operatorname{gph} A$ if and only if $\langle x - x', y - y' \rangle \geq \mu_A\|x - x'\|^2 + \rho_A\|y - y'\|^2$ for all $(x, y) \in \operatorname{gph} A$. Defining $p := y + \alpha x$ and $p' := y' + \alpha x'$, so that $(x, p), (x', p') \in \operatorname{gph} T$, this is equivalent to

$$(1 + 2\rho_A\alpha)\langle x - x', p - p' \rangle \geq (\mu_A + \alpha(1 + \rho_A\alpha))\|x - x'\|^2 + \rho_A\|p - p'\|^2, \qquad \forall (x, p) \in \operatorname{gph} T.$$

Dividing both sides by $1 + 2\rho_A\alpha > 0$, we obtain

$$\langle x - x', p - p' \rangle \geq \tfrac{\mu_A + \alpha(1+\rho_A\alpha)}{1+2\rho_A\alpha}\|x - x'\|^2 + \tfrac{\rho_A}{1+2\rho_A\alpha}\|p - p'\|^2, \qquad \forall (x, p) \in \operatorname{gph} T. \tag{B.7}$$

establishing the claimed equivalence between (μ_A, ρ_A)-semimonotonicity of A at (x', y') and (μ_T, ρ_T)-semimonotonicity of T at $(x', y' + \alpha x')$. The assertion for (global) semimonotonicity follows analogously by considering all points $(x', y') \in \operatorname{gph} A$.

Finally, the assertion for maximality holds since $\operatorname{gph} A$ can be enlarged with a point $(\hat{x}, \hat{y})$ without breaking (μ_A, ρ_A)-semimonotonicity if and only if $\operatorname{gph} T$ can be enlarged with $(\hat{x}, \hat{y} + \alpha\hat{x})$ without violating (B.7).

♠ 4.8*(ii)*: This result immediately follows from Proposition 4.7*(i)* and Corollary 4.6. □

***Proof of Proposition 4.11** (resolvent of a semimonotone operator).*

♠ 4.11*(i)*: First assume that $\mu\rho < \frac{1}{4}$. Define $\theta := \sqrt{1 - 4\rho\mu}$ and let $\nu := \frac{2\mu}{1+\theta}$, $\xi := \frac{\rho}{\theta}$ and $\omega := \frac{\gamma}{1+\gamma\nu}$. We have

$$1 + \gamma\nu = 1 + \gamma\tfrac{2\mu}{1+\theta} \geq 1 - \tfrac{2\gamma[-\mu]_+}{1+\theta} > 0,$$

where the upper bound $\gamma < \frac{1+\theta}{2[-\mu]_+}$ from (4.3) was used. Consequently, $\omega + \xi > 0$ since

$$\tfrac{\gamma}{1+\gamma\nu} + \tfrac{\rho}{\theta} > 0 \overset{1+\gamma\nu>0}{\iff} \gamma + \left(1 + \gamma\tfrac{2\mu}{1+\theta}\right)\tfrac{\rho}{\theta} > 0 \overset{\theta>0}{\iff} \left(2\theta + \tfrac{4\mu\rho}{1+\theta}\right)\gamma + 2\rho > 0 \iff (1+\theta)\gamma + 2\rho > 0$$

which is satisfied by the lower bound $\gamma > \frac{2[-\rho]_+}{1+\theta}$ from (4.3). Denote $\hat{A} := (A - \nu\mathrm{id})^{-1} + \omega\mathrm{id}$. Owing to $\omega + \xi > 0$ and Minty's theorem [5, Thm. 21.1], the operator $\frac{1}{\omega+\xi}\hat{A} = \frac{1}{\omega+\xi}\left((A - \nu\mathrm{id})^{-1} - \xi\mathrm{id}\right) + \mathrm{id}$ has full range iff $\frac{1}{\omega+\xi}((A - \nu\mathrm{id})^{-1} - \xi\mathrm{id})$ is maximally monotone. Since $\omega + \xi > 0$, this implies that $\hat{A}$ has full range iff the operator $\tilde{A} := (A - \nu\mathrm{id})^{-1} - \xi\mathrm{id}$ is maximally monotone.

On the other hand, by definition $1 + 2\xi\nu = 1 + \frac{1-\theta^2}{\theta(1+\theta)} = \frac{1}{\theta} > 0$. Hence, using Corollary 4.9*(i)*, the operator $\tilde{A} := (A - \nu\mathrm{id})^{-1} - \xi\mathrm{id}$ is maximally monotone if and only if A is maximally $\left(\frac{\nu(1+\xi\nu)}{1+2\xi\nu}, \frac{\xi}{1+2\xi\nu}\right) = (\mu, \rho)$ semimonotone, where the equality is the result of substituting ξ and ν. Therefore, it suffices to show $\hat{A}$ has full range iff $\mathcal{R}(\mathrm{id} + \gamma A) = \mathbb{R}^n$, i.e., that the resolvent has full domain. First suppose that $\hat{A}$ has full range. Fix $y \in \mathbb{R}^n$. Then, by the full range assumption for $\hat{A}$, there exists some $x \in \mathbb{R}^n$ such that $\frac{\omega}{\gamma}y \in (A - \nu\mathrm{id})^{-1}x + \omega x$. Letting $z = \omega(\frac{1}{\gamma}y - x)$ we have

$$\tfrac{1}{\gamma}y - \tfrac{1}{\omega}z \in (A - \nu\mathrm{id})z \iff \tfrac{1}{\gamma}y \in (A + (\tfrac{1}{\omega} - \nu)\mathrm{id})z \iff y \in (\gamma A + \mathrm{id})z, \tag{B.8}$$

where in the last equivalence we used the fact that $\gamma = \frac{\omega}{1-\omega\nu}$. Since the choice of y was arbitrary, $J_{\gamma A}$ has full domain. The converse follows similarly. Assume that $\mathcal{R}(\mathrm{I} + \gamma A) = \mathbb{R}^n$. For any fixed $y \in \mathbb{R}^n$ there exists some $z \in \mathbb{R}^n$ such that $y \in \gamma Az + z$. Therefore, by (B.8) and letting $x = \frac{1}{\gamma}y - \frac{1}{\omega}z$ we obtain $\frac{\omega}{\gamma}y \in \hat{A}x$, thus establishing that $\mathcal{R}(\hat{A}) = \mathbb{R}^n$.

Finally, consider the case $[\mu]_+[\rho]_+ = \frac{1}{4}$. The range for the stepsize in (4.3) then reduces to $\gamma \in (0, +\infty)$. First, suppose that $J_{\gamma A}$ has full domain, i.e., $\mathcal{R}(\mathrm{id} + \gamma A) = \mathbb{R}^n$. Fix $v \in \mathbb{R}^n$. Then, there exists $(x, y) \in \operatorname{gph} A$ such that $v = x + \gamma y$. By Proposition 4.3*(iii)*-1., $y = 2\mu x + c$ for some $c \in \mathbb{R}$ and some $x \in \operatorname{dom} A$, and therefore, $v = (1 + 2\gamma\mu)x + c\gamma$. Since $1 + 2\gamma\mu > 0$ and the choice of v was arbitrary, necessarily $\operatorname{dom} A = \mathbb{R}^n$, and as such A is maximally (μ, ρ)-semimonotone, see Proposition 4.3*(iii)*-2.. Now, suppose that A is maximally (μ, ρ)-semimonotone. Then, by Proposition 4.3*(iii)*-2. it holds that $A : x \mapsto 2\mu x + c$ for some $c \in \mathbb{R}^n$, which implies that A is maximally monotone. Therefore by Minty's theorem [5, Thm. 21.1] the resolvent $J_{\gamma A}$ has full domain.

♠ 4.11*(ii)*: Take $(u, v), (u', v') \in \operatorname{gph} J_{\gamma A}$. By semimonotonicity of A and using the fact that $\frac{1}{\gamma}(u - v) \in Av$, $\frac{1}{\gamma}(u' - v') \in Av'$, we have

$$\mu\|v - v'\|^2 + \rho\gamma^{-2}\|(u - v) - (u' - v')\|^2 \leq \gamma^{-1}\langle (u - v) - (u' - v'), v - v' \rangle.$$

Expanding the last term of the left-hand side and rearranging, we get

$$\begin{aligned}\left(1+\mu\gamma+\rho\gamma^{-1}\right)\|v-v'\|^2+\rho\gamma^{-1}\|u-u'\|^2 &\leq \left(1+2\rho\gamma^{-1}\right)\langle u-u', v-v'\rangle \\ &\leq \left|1+2\rho\gamma^{-1}\right|\|u-u'\|\|v-v'\|.\end{aligned} \tag{B.9}$$

By Fact A.2*(ii)* it follows that (4.3) provides all strictly positive solutions for γ of $1+\mu\gamma+\rho\gamma^{-1}>0$. Hence, if $u=u'$ then necessarily $v=v'$, and the Lipschitz inequality holds trivially. Otherwise, divide both sides by $\|u-u'\|^2$ to obtain

$$\left(1+\mu\gamma+\rho\gamma^{-1}\right)\omega^2-\left|1+2\rho\gamma^{-1}\right|\omega+\rho\gamma^{-1}\leq 0.$$

where $\omega \coloneqq \frac{\|x-y\|}{\|u-v\|}$. Solving this quadratic inequality for ω yields the claimed Lipschitz modulus. □

B.3 Examples of functions with semimonotone subdifferentials

Proof of Proposition 4.13 *(functions with upper and lower curvature).*

♠ 4.13*(i)*: Follows by definition.

♠ 4.13*(ii)*: By [64, Prop. 12.60] and strong convexity of $-f$, the conjugate $(-f)^*$ is $-\frac{1}{\ell}$-smooth, i.e.

$$\langle \nabla(-f)^*(x)-\nabla(-f)^*(x'), x-x'\rangle \leq -\tfrac{1}{\ell}\|x-x'\|^2, \quad \forall x,x'\in\mathbb{R}^n. \tag{B.10}$$

Moreover, since $-f$ is continuous and convex, we have by [64, Prop. 11.3] that $\nabla(-f)^* = (\partial(-f))^{-1}$, so that (B.10) is equivalent to

$$\langle y-y', x-x'\rangle \leq -\tfrac{1}{\ell}\|y-y'\|^2, \quad \forall (x,y),(x',y')\in \operatorname{gph}\partial(-f). \tag{B.11}$$

Finally, since $-f$ is finite and convex, it follows from [64, Cor. 9.21] that $\partial f(\bar{x})\subseteq -\partial(-f)(\bar{x})$, $\forall \bar{x}\in\mathbb{R}^n$. Consequently,

$$\langle y-y', x-x'\rangle \geq \tfrac{1}{\ell}\|y-y'\|^2, \quad \forall (x,y),(x',y')\in \operatorname{gph}\partial f \subseteq -\operatorname{gph}\partial(-f),$$

which completes the proof.

♠ 4.13*(iii)*: Consider the function $\varphi = f-\frac{\alpha}{2}\|\cdot\|^2$. Then, by definition $\varphi\in\mathscr{F}^{\ell-\alpha}(\mathbb{R}^n)$, implying that $\partial\varphi$ is $(0,\frac{1}{\ell-\alpha})$-semimonotone owing to Proposition 4.13*(ii)* since $\alpha>\ell$. Since $\partial f = \partial\varphi+\alpha\mathrm{id}$, [59, Ex. 8.8(c)], it follows from Proposition 4.8 that for $c>\frac{1}{\alpha-\ell}$ the claimed semimonotonicity holds.

♠ 4.13*(iv)*, 4.13*(v)*: Suppose that $\ell>\sigma$, since if $\ell=\sigma$, then, $\nabla f=\ell\mathrm{id}$ and the claims follow trivially. Consider the function $\varphi = f-\frac{\sigma}{2}\|\cdot\|^2\in\mathscr{F}_0^{\ell-\sigma}(\mathbb{R}^n)$. Since φ is finite and convex its subdifferential $\partial\varphi(x)$ is nonempty for all $x\in\mathbb{R}^n$ [59, Thm. 23.4]. Consequently, φ is continuously differentiable due to [70, Lem. 2.1] and $\nabla\varphi$ is $(0,\frac{1}{\ell-\sigma})$-semimonotone due to [50, Thm. 2.1.10]. Therefore, by Proposition 4.4, there exists a monotone mapping $\tilde{A}$ for which $(\nabla\varphi)^{-1}=\tilde{A}+\frac{1}{\ell-\sigma}$ and thus, owing to $\nabla f=\nabla\varphi+\sigma\mathrm{id}$ [64, Ex. 8.8(c)] it holds that

$$\nabla f = \nabla\varphi+\sigma\mathrm{id} = (\tilde{A}+\tfrac{1}{\ell-\sigma}\mathrm{id})^{-1}+\sigma\mathrm{id}.$$

The claimed results from Propositions 4.13*(iv)* and 4.13*(v)* then follow from Corollary 4.9, where we let $\xi=\frac{1}{\ell-\sigma}$ and $\nu=\sigma$. This is because $\ell-\sigma>0$ and therefore $1+2\xi\nu>0$ (resp. ≤ 0) if and only if $\sigma+\ell>0$ (resp. ≤ 0). □

Proof of Corollary 4.14 *(pointwise minimum).* Since $f_i\in\mathscr{F}^{\ell_{f_i}}(\mathbb{R}^n)$, it follows that $\frac{\ell_{\max}}{2}\|\cdot\|^2-f_i$ is convex. Therefore, $\frac{\ell_{\max}}{2}\|x\|^2-f(x) = \max\left\{\frac{\ell_{\max}}{2}\|x\|^2-f_1(x),\dots,\frac{\ell_{\max}}{2}\|x\|^2-f_N(x)\right\}$ is convex being the pointwise max of convex functions and thus $f\in\mathscr{F}^{\ell_{\max}}(\mathbb{R}^n)$. The claimed result then follows immediately from Propositions 4.13*(ii)* and 4.13*(iii)*. □

Proof of Proposition 4.15 *(infimal convolution).*

♠ 4.15*(i)*: Let $f(w) = f_1(w_1) + f_2(w_2)$, and L be the linear mapping $Lw = w_1 + w_2$. Then, the infimal convolution may be viewed as the parametric minimization of $\tilde{f}(w, s) = f(w) + \delta_{\{0\}}(Lw - s)$ with respect to the variable w, i.e., $\varphi(s) = \min_w \tilde{f}(w, s)$. Let $M(s) := \arg\min_{(w_1,w_2)} \{f_1(w_1) + f_2(w_2) \mid w_1 + w_2 = s\}$. Then, owing to (4.4) it holds by [64, Thm. 10.13] for all $\bar{s} \in \mathbb{R}^n$ that

$$\partial\varphi(\bar{s}) \subseteq \bigcup_{\bar{x}\in M(\bar{s})} \big\{y \in R^n \mid (0, y) \in \partial\tilde{f}(\bar{x}, \bar{s})\big\}. \tag{B.12}$$

Consider $\bar{f}(w, u) = f(w) + \delta_{\{0\}}(u)$ and the linear transformation $K : (w, s) \mapsto (w, Lw - s)$. Then, $\tilde{f} = \bar{f} \circ K$. Since K has full rank, by [64, Ex. 10.7] for any pair $(\bar{x}, \bar{s})$ such that $\bar{x} \in M(\bar{s})$ we have

$$\partial\tilde{f}(\bar{x}, \bar{s}) = (K^\top \circ \partial\bar{f} \circ K)(\bar{x}, \bar{s}) \quad \text{where} \quad \partial\bar{f}(w, u) = (\partial f(w), N_{\{0\}}(u)).$$

Using this in (B.12) yields

$$\partial\varphi(\bar{s}) \subseteq \bigcup_{\bar{x}\in M(\bar{s})} \big\{y \in R^n \mid (0, y) \in \big(\partial f(\bar{x}) + L^\top N_{\{0\}}(L\bar{x} - \bar{s}), -N_{\{0\}}(L\bar{x} - \bar{s})\big)\big\} = \bigcup_{(\bar{x}_1,\bar{x}_2)\in M(\bar{s})} \partial f_1(\bar{x}_1) \cap \partial f_2(\bar{x}_2).$$

Hence, the claimed result follows from [5, Prop. 25.30(i)].

♠ 4.15*(ii)*: Follows immediately from Proposition 4.15*(i)* and Proposition 4.7*(ii)*. □

Proof of Proposition 4.16 *(saddle operator).* Consider the saddle envelope

$$\varphi_\delta(x, y) := \min_{u\in\mathbb{R}^n} \max_{v\in\mathbb{R}^m} \{\varphi(u, v) + \tfrac{1}{2\delta}\|u - x\|^2 - \tfrac{1}{2\delta}\|v - y\|^2\},$$

along with the associated resolvent $J_{\delta T_\varphi} := (\mathrm{id} + \delta T_\varphi)^{-1}$. For any $(z, w) \in \operatorname{gph} J_{\delta T_\varphi}$ with the decomposition $z = (x, y)$ and $w = (u, v)$ we have by construction that

$$0 = \nabla_x\varphi(w) + \tfrac{1}{\delta}(u - x) = \nabla_x\varphi(w) - \nabla_x\varphi_\delta(z) \quad \text{and} \quad 0 = \nabla_y\varphi(w) - \tfrac{1}{\delta}(v - y) = \nabla_y\varphi(w) - \nabla_y\varphi_\delta(z).$$

Hence,

$$T_\varphi(w) = (\nabla_x\varphi(w), -\nabla_y\varphi(w)) = (\nabla_x\varphi_\delta(z), -\nabla_y\varphi_\delta(z)) =: T_{\varphi_\delta}(z). \tag{B.13}$$

On the other hand, it follows from [32, Prop. 1] that

$$\tfrac{1}{\delta}\mathrm{I} \succeq \nabla_{xx}\varphi_\delta(z) \succeq \tfrac{\alpha}{1+\delta\alpha}\mathrm{I}, \quad \tfrac{1}{\delta}\mathrm{I} \succeq -\nabla_{yy}\varphi_\delta(z) \succeq \tfrac{\alpha}{1+\delta\alpha}\mathrm{I}, \quad \forall z, \tag{B.14}$$

thus implying that the operator T_{φ_δ} is $\frac{\alpha}{1+\delta\alpha}$-monotone. Let $w_1, w_2 \in \mathbb{R}^n$ and $z_i = w_i + \delta T_\varphi w_i$ for $i = 1, 2$. Then, $(z_i, w_i) \in \operatorname{gph} J_{\delta T_\varphi}$ for $i = 1, 2$ and we have that

$$\begin{aligned}
\langle T_\varphi w_1 - T_\varphi w_2, w_1 - w_2\rangle &= \langle T_\varphi w_1 - T_\varphi w_2, z_1 - z_2\rangle - \delta\|T_\varphi w_1 - T_\varphi w_2\|^2 \\
&\overset{\text{(B.13)}}{=} \langle T_{\varphi_\delta} z_1 - T_{\varphi_\delta} z_2, z_1 - z_2\rangle - \delta\|T_\varphi w_1 - T_\varphi w_2\|^2 \\
&\overset{\text{(B.14)}}{\geq} \tfrac{\alpha}{1+\alpha\delta}\|z_1 - z_2\|^2 - \delta\|T_\varphi w_1 - T_\varphi w_2\|^2.
\end{aligned} \tag{B.15}$$

The second assertion follows from the above inequality noting that $1 + \alpha\delta > 0$. Moreover, plugging $z_i = w_i + \delta T_\varphi w_i$ in (B.15) and multiplying by $(1 + \alpha\delta)$ yields

$$(1 - \alpha\delta)\langle T_\varphi w_1 - T_\varphi w_2, w_1 - w_2\rangle \geq \alpha\|w_1 - w_2\|^2 - \delta\|T_\varphi w_1 - T_\varphi w_2\|^2, \tag{B.16}$$

establishing the claim. □

B.4 Douglas-Rachford splitting for semimonotone operators

***Proof of Lemma 5.1** (primal-dual operator).* Let $(x_A, y_A) \in \operatorname{gph} A$ and $(x_B, y_B) \in \operatorname{gph} B$, so that $(y_B, x_B) \in \operatorname{gph} B^{-1}$. Then, by definition of the primal-dual operator from (PD-I), it holds that

$$(z, v) \coloneqq \big((x_A, y_B), (y_A + y_B, x_B - x_A)\big) \in \operatorname{gph} T_{\mathrm{PD}}.$$

Therefore, by Assumption III and Proposition 4.4, it holds for all $z^\star = (x^\star, y^\star) \in \mathcal{S}^\star$ that

$$\begin{aligned}\big\langle v, z - z^\star \big\rangle &= \big\langle y_A + y_B, x_A - x^\star \big\rangle + \big\langle x_B - x_A, y_B - y^\star \big\rangle \\ &= \big\langle y_A + y^\star, x_A - x^\star \big\rangle + \big\langle x_B - x^\star, y_B - y^\star \big\rangle \\ &\geq \mu_A \|x_A - x^\star\|^2 + \rho_A \|y_A + y^\star\|^2 + \rho_B \|y_B - y^\star\|^2 + \mu_B \|x_B - x^\star\|^2. \end{aligned} \tag{B.17}$$

On the other hand, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}\mathrm{q}_V(v) &= \beta_{\mathrm{P}} \|y_A + y_B\|^2 + \beta_{\mathrm{D}} \|x_B - x_A\|^2 \\ &= \beta_{\mathrm{P}} \|y_A + y^\star\|^2 + \beta_{\mathrm{P}} \|y_B - y^\star\|^2 + 2\beta_{\mathrm{P}} \langle y_A + y^\star, y_B - y^\star \rangle \\ &\quad + \beta_{\mathrm{D}} \|x_B - x^\star\|^2 + \beta_{\mathrm{D}} \|x_A - x^\star\|^2 + 2\beta_{\mathrm{D}} \langle x_B - x^\star, x_A - x^\star \rangle \\ &\leq \beta_{\mathrm{P}} \|y_A + y^\star\|^2 + \beta_{\mathrm{P}} \|y_B - y^\star\|^2 + 2|\beta_{\mathrm{P}}| \|y_A + y^\star\| \|y_B - y^\star\| \\ &\quad + \beta_{\mathrm{D}} \|x_B - x^\star\|^2 + \beta_{\mathrm{D}} \|x_A - x^\star\|^2 + 2|\beta_{\mathrm{D}}| \|x_B - x^\star\| \|x_A - x^\star\|. \end{aligned} \tag{B.18}$$

Then, combining (B.17) and (B.18) it follows that T_{PD} has V-oblique weak Minty solutions at $\mathcal{S}^\star$ provided that $X \coloneqq \left(\begin{smallmatrix} \mu_A - \beta_{\mathrm{D}} & -|\beta_{\mathrm{D}}| \\ -|\beta_{\mathrm{D}}| & \mu_B - \beta_{\mathrm{D}} \end{smallmatrix}\right) \geq 0$ and $Y \coloneqq \left(\begin{smallmatrix} \rho_B - \beta_{\mathrm{P}} & -|\beta_{\mathrm{P}}| \\ -\beta_{\mathrm{P}} & \rho_A - |\beta_{\mathrm{P}}| \end{smallmatrix}\right) \geq 0$. As previously discussed in the proof of Proposition 4.7*(i)*, the largest possible value for β_{D} satisfying $X \geq 0$ is given by $\mu_A \,\square\, \mu_B$ if either $\mu_A + \mu_B > 0$ or $\mu_A = \mu_B = 0$ (and analogously the largest value for β_{P} satisfying $Y \geq 0$ is given by $\rho_A \,\square\, \rho_B$ if either $\rho_A + \rho_B > 0$ or $\rho_A = \rho_B = 0$), which completes the proof. □

***Proof of Theorem 5.2** (convergence of DRS under semimonotonicity).* First, we will show that all conditions from Assumption II are satisfied.

♠ For 5.2*(i)*, it follows from Assumption III and Lemma 5.1 that Assumption II.A3 holds.

♠ For 5.2*(ii)*, it follows from Assumption IV and Lemma A.5 that Assumption II.A2 holds and from Assumption IV and Lemma 5.1 that Assumption II.A3 holds.

Therefore, all conditions from Assumption II are satisfied. It remains to show that γ satisfies (3.3), which follows immediately by observing that the provided bounds on γ from Table 2 are obtained by plugging in (5.1) into (3.3). Hence, all the claims of Theorem 3.3 hold, concluding the proof. □

C Examples

***Proof of Example 2.4** (toy example).* Recall that for a given $\rho \in \mathbb{R}$, the ρ-weak Minty solutions of T are given by the set $\mathcal{S}^\star \subseteq \operatorname{zer} T$ for which it holds that

$$\langle T(x), x - x^\star \rangle \geq \rho \|T(x)\|, \quad \forall x \in \mathbb{R}^2, x^\star \in \mathcal{S}^\star. \tag{C.1}$$

♠ 2.4*(i)*: The analytical expression for f as defined in Figure 3 is given by

$$f\big(\|x\|\big) \coloneqq \begin{cases} \|x\|, & \text{if } \|x\| \leq 2/5, \\ 4/5 - \|x\|, & \text{if } \|x\| \in (2/5, 4/5), \\ 0, & \text{if } \|x\| \in [4/5, 1], \\ 5/2(\|x\| - 1), & \text{if } \|x\| \in (1, 7/5), \\ 1, & \text{if } \|x\| \geq 7/5. \end{cases} \tag{C.2}$$

Therefore, by definition of T, it holds that $\operatorname{zer} T = \left\{x \in \mathbb{R}^2 \mid \|x\| \in \{0\} \cup [4/5, 1]\right\}$ and that (C.1) corresponds to

$$f\Big(\|x\|\Big)\Big(b\|x\|^2 - b(x_1x_1^\star + x_2x_2^\star) + a(x_1x_2^\star + x_1^\star x_2)\Big) \geq \rho(a^2+b^2)f\Big(\|x\|\Big)^2\|x\|^2, \qquad \forall x \in \mathbb{R}^2, x^\star \in \mathcal{S}^\star. \quad \text{(C.3)}$$

This obviously holds for all $x \in \operatorname{zer} T$. For $x \notin \operatorname{zer} T$, we distinguish between the following two cases.

- $\diamond$ $\|x^\star\| = 0$: Then, (C.3) holds iff for all $x \notin \operatorname{zer} T$ it holds that $\rho \leq \frac{b}{(a^2+b^2)f(\|x\|)}$. Since $f(\|x\|) \in (0, 1]$, this condition is satisfied for $\rho = b/a^2+b^2$.
- $\diamond$ $\|x^\star\| \in [4/5, 1]$: Let $x = \epsilon x^\star$ where $\epsilon \in (0, 2/5)$. Then, $f(\|x\|) = \epsilon\|x^\star\|$ and (C.3) holds iff $\rho \leq \frac{b(1-\epsilon^{-1})}{\epsilon(a^2+b^2)\|x^\star\|}$ Taking the limit for ϵ to zero from above, we get that $\lim_{\epsilon\to 0+} \frac{b(1-\epsilon^{-1})}{\epsilon(a^2+b^2)\|x^\star\|} = -\infty$ and thus (C.3) can not be satisfied for any $\rho \in \mathbb{R}$.

♠ 2.4*(iii)*: By definition of T and using that $f \equiv 1$, it is straightforward to verify that $\operatorname{zer} T = (0, 0)$ and that (C.1) corresponds to $\rho \leq b/a^2+b^2$.

♠ 2.4*(iv)*: To show that the result from Theorem 2.3*(vi)* is tight, define

$$H := \mathrm{I}_2 + \lambda\Big((\mathrm{I}_2 + T)^{-1} - \mathrm{I}_2\Big) = \mathrm{I}_2 + \lambda\left(\begin{bmatrix} 1+b & a \\ -a & 1+b \end{bmatrix}^{-1} - \mathrm{I}_2\right) = \mathrm{I}_2 + \lambda\left(\tfrac{1}{(1+b)^2+a^2}\begin{bmatrix} 1+b & -a \\ a & 1+b \end{bmatrix} - \mathrm{I}_2\right)$$

such that the iteration of the proximal point algorithm with fixed relaxation parameter λ can be expressed as the linear dynamical system $x^{k+1} = Hx^k$. This system is globally asymptotically stable if and only if the spectral radius of H is strictly less than one, which through elementary algebra can be shown to hold iff (2.16) is satisfied, completing the proof. □

***Proof of Example 6.1** (saddle point problem).*

♠ 6.1*(i)*: By defining $\overline{H} := (\mathrm{I} + \gamma A)^{-1}$, $\widehat{H} := (\mathrm{I} + \gamma B)^{-1}(2\overline{H} - \mathrm{I})$ and $H := \mathrm{I} + \lambda(\widehat{H} - \overline{H})$, DRS can be expressed as the linear dynamical system $s^{k+1} = Hs^k$. This system is globally asymptotically stable if and only if the spectral radius of H is strictly less than one, which through elementary algebra can be shown to hold iff (6.2) is satisfied.

♠ 6.1*(ii)*: The primal-dual operator and its inverse are given by

$$T_{\mathrm{PD}} = \begin{bmatrix} A & \mathrm{I}_2 \\ -\mathrm{I}_2 & B^{-1} \end{bmatrix} \quad \text{and} \quad T_{\mathrm{PD}}^{-1} = \begin{bmatrix} (A+B)^{-1} & -(A+B)^{-1}B \\ B(A+B)^{-1} & B - B(A+B)^{-1}B \end{bmatrix}$$

where we used the Schur complement lemma. Consequently, when in (2.1) the vector v is restricted to $\mathcal{R}(P)$ we may equivalently state Assumption II.A3 as

$$z^\top\left(\tfrac{T_{\mathrm{PD}}+T_{\mathrm{PD}}^\top}{2} - T_{\mathrm{PD}}^\top V T_{\mathrm{PD}}\right)z \geq 0, \qquad \text{for all } z \in \mathbb{R}^n : z \in T_{\mathrm{PD}}^{-1}\mathcal{R}(P), \quad \text{(C.4)}$$

where V is defined as in (3.2). Using that $Z = \left[\begin{smallmatrix} \mathrm{I}_2 \\ -\gamma\mathrm{I}_2 \end{smallmatrix}\right]$ is an orthogonal basis for $\mathcal{R}(P)$, it follows that (C.4) is satisfied iff

$$(T_{\mathrm{PD}}^{-1}Z)^\top\left(\tfrac{T_{\mathrm{PD}}+T_{\mathrm{PD}}^\top}{2} - T_{\mathrm{PD}}^\top V T_{\mathrm{PD}}\right)T_{\mathrm{PD}}^{-1}Z \geq 0 \quad \iff \quad \tfrac{1}{\gamma}\beta_{\mathrm{P}} + \gamma\beta_{\mathrm{D}} \leq \tfrac{b(a^2\gamma+1)}{\gamma(a^2+b^2)}.$$

Consequently, for the upper bound on λ implied by Theorem 3.3, denoted by $\lambda_{\max}$, it holds that

$$\lambda_{\max} := 2\Big(1 + \tfrac{1}{\gamma}\beta_{\mathrm{P}} + \gamma\beta_{\mathrm{D}}\Big) \leq 2 + 2\tfrac{b(a^2\gamma+1)}{\gamma(a^2+b^2)}.$$

As this bound matches the upper bound on λ from (6.2), the proof is completed.

♠ 6.1*(iii)*: For example, consider $a = 2, b = -1$, for which the trace of T_{P},T_{D} and T_{PD} are all equal to -2.

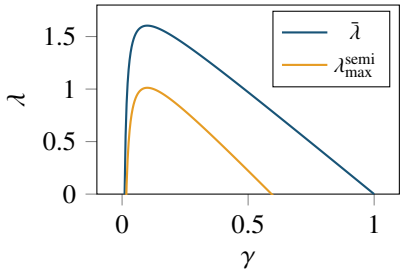

Figure 8: *Comparison between the upper bounds on the relaxation parameter λ in function of the stepsize γ for Example 6.1 with $a = 10$, $b = -1$. $\bar{\lambda}$ is the bound given by (6.2) and $\lambda_{\max}^{\text{semi}}$ is the bound implied by Theorem 5.2.*

♠ Finally, we will discuss the range of relaxation parameters λ covered by Theorem 5.2. To this end, the semimonotonicity parameters $\mu_A, \rho_A, \mu_B, \rho_B$ need to be selected such that the relaxation parameter is maximized, i.e.

$$\begin{aligned} \lambda_{\max}^{\text{semi}} \coloneqq \max_{\mu_A,\rho_A,\mu_B,\rho_B} \quad & 2\Big(1 + \tfrac{1}{\gamma}(\rho_A \,\square\, \rho_B) + \gamma(\mu_A \,\square\, \mu_B)\Big) \\ \text{subject to} \quad & \mu_A + a^2\rho_A \le 0 \quad \text{and} \quad \mu_B + b^2\rho_B - b \le 0, \end{aligned}$$

where the constraints are obtained using Proposition 4.5 by enforcing (μ_A, ρ_A)- and (μ_B, ρ_B)-semimonotonicity of operators A and B, respectively. Note that this optimization problem can be encoded as a convex SDP. Solving this problem numerically for $a = 10$, $b = -1$, the result visualized in Figure 8 is obtained. Hence, when characterizing the nonmonotonicity of the problem through the semimonotonicity properties of A and B separately rather than through the (oblique) weak Minty condition for the combined primal-dual operator, some looseness is inevitably introduced. □

***Proof of Example 6.2** (nonsmooth optimization).* The corresponding limiting subdifferentials are given by

$$\partial f_A(x) = \begin{cases} -x - 3, & \text{if } x < -1, \\ \{-2, -9\}, & \text{if } x = -1, \\ 6x - 3, & \text{if } x \in (-1, 1), \\ \{-4, 3\}, & \text{if } x = 1, \\ -x - 3, & \text{if } x > 1, \end{cases} \qquad \partial f_B(x) = \begin{cases} 4x - 1, & \text{if } x < -1, \\ [-5, 1] & \text{if } x = -1, \\ 2x + 3, & \text{if } x \in (-1, 2), \\ [7, 19], & \text{if } x = 2, \\ 2x + 15, & \text{if } x > 2. \end{cases} \tag{C.5}$$

♠ Assumption II.A1: Follows by definition of the limiting subdifferential [59, Def. 8.3(b)].

♠ Assumption II.A2: It is easy to verify that for $\gamma \neq 1$ it holds that $\mathcal{R}(1 + \gamma\partial f_A) = \mathcal{R}(1 + \gamma\partial f_B) = \mathbb{R}$. Hence, the resolvents $J_{\gamma\partial f_A}$ and $J_{\gamma\partial f_B}$ have full domain.

♠ Assumption III: Let $(x^\star, y^\star) = (0, 3)$. Then, it is of immediate verification that the subdifferentials ∂f_A and ∂f_B are semimonotone at respectively $(x^\star, -y^\star)$ and $(x^\star, y^\star)$ with $\mu_A = -1.2$, $\rho_A = 0.2$, $\mu_B = 1.6$ and $\rho_B = 0.1$.

♠ Follows directly by applying Theorem 5.2 and using that $\mu_A \,\square\, \mu_B = -24/5$ and $\rho_A \,\square\, \rho_B = 2/30$. □

***Proof of Example 6.4** (stationary point).* The limiting subdifferentials of f_A and f_B are given by

$$\partial f_A(x) = \begin{cases} \left\{ \frac{1-\sqrt{1-4\mu_A\rho_A}}{2\rho_A} x, \frac{1+\sqrt{1-4\mu_A\rho_A}}{2\rho_A} x \right\} - 1, & \text{if } x \in \{-3, 3\}, \\ \frac{1-\sqrt{1-4\mu_A\rho_A}}{2\rho_A} x - 1, & \text{if } x \in (-3, 3), \\ \frac{1+\sqrt{1-4\mu_A\rho_A}}{2\rho_A} x - 1, & \text{otherwise}, \end{cases} \qquad \partial f_B(x) = \begin{cases} \left\{2x, \tfrac{1}{2}x\right\} + 1, & \text{if } x \in \{-1, 1\}, \\ 2x + 1, & \text{if } x \in (-1, 1), \\ \tfrac{1}{2}x + 1, & \text{otherwise}. \end{cases}$$

♠ Assumption II.A1: Follows by definition of the limiting subdifferential [59, Def. 8.3(b)].

♠ Assumption II.A2: The resolvents $J_{\gamma\partial f_A}$ and $J_{\gamma\partial f_B}$ have full domain iff $\mathcal{R}(1+\gamma\partial f_A) = \mathcal{R}(1+\gamma\partial f_B) = \mathbb{R}$. It is easy to verify that $\mathcal{R}(1+\gamma\partial f_B) = \mathbb{R}$ for all $\gamma \in \mathbb{R}_+$. On the other hand, we have that

$$(1+\gamma\partial f_A)(x) = \begin{cases} x - \gamma + \left\{\frac{1-\sqrt{1-4\mu_A\rho_A}}{2\rho_A}\gamma x, \frac{1+\sqrt{1-4\mu_A\rho_A}}{2\rho_A}\gamma x\right\}, & \text{if } x \in \{-3,3\}, \\ x - \gamma + \frac{1-\sqrt{1-4\mu_A\rho_A}}{2\rho_A}\gamma x, & \text{if } x \in (-3,3), \\ x - \gamma + \frac{1+\sqrt{1-4\mu_A\rho_A}}{2\rho_A}\gamma x, & \text{otherwise} \end{cases}$$

from which follows that $\mathcal{R}(1+\gamma\partial f_A) = \mathbb{R}$ for all $\gamma \in \mathbb{R}_+$ iff γ is selected according to the provided range.

♠ Assumption III: Let $(x^\star, y^\star) = (0, -1)$. Then, it is of immediate verification that the subdifferentials ∂f_A and ∂f_B are semimonotone at respectively $(x^\star, -y^\star)$ and $(x^\star, y^\star)$ for the given parameters $\mu_A, \rho_A, \mu_B, \rho_B$. Furthermore, these parameters satisfy Assumption III.A1.

♠ Follows directly by applying Theorem 5.2 and using that $\mu_A \square \mu_B = -6/5$ and $\rho_A \square \rho_B = -4/30$. □

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