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# BEHAVIORS DEFINED BY RATIONAL FUNCTIONS

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***Abstract*— In this article behaviors defined by 'differential equations' involving matrices of rational functions are introduced. Conditions in terms of controllability and stabilizability for the existence of rational representations that are prime over various rings are derived.**

***Index Terms*— Behaviors, rational functions, controllability, stabilizability, observability.**

## I. INTRODUCTION

The behavioral theory of linear time-invariant differential systems has been dominated by polynomial matrix representations, but representations using rational functions have hardly been considered. Only recently the idea of how to define formally a behavior in terms of rational functions has been introduced in [6] for discrete-time systems (here we deal with continuous-time systems). In this conference write-up, we only give the basic notions and an outline of results. Details and proofs will appear in [7].

We first explain some background notions and notation. $\mathbb{R}[\xi]$ denotes the polynomials with real coefficients in the indeterminate $\xi$, and $\mathbb{R}(\xi)$ the real rational functions in the indeterminate $\xi$. $\mathbb{R}[\xi]$ is a ring, and $\mathbb{R}[\xi]^{\mathtt{n}}$ a module over $\mathbb{R}[\xi]$. $\mathbb{R}(\xi)$ is a field, and $\mathbb{R}(\xi)^{\mathtt{n}}$ is an $\mathtt{n}$ dimensional vector over $\mathbb{R}(\xi)$.

The following three rings each have $\mathbb{R}[\xi]$ as their field of fractions:

1) the ring $\mathbb{R}[\xi]$ of polynomials,
2) the ring $\mathbb{R}(\xi)_{\mathscr{P}}$ of proper rational functions, and
3) the ring $\mathbb{R}(\xi)_{\mathscr{S}}$ of stable proper rational functions.

(i) means all poles at $\infty$, (ii) no poles at $\infty$, (iii) only finite stable poles, meaning no poles in the closed right half of the complex plane. The poles and zeros of $M \in \mathbb{R}(\xi)^{\mathtt{n}_1 \times \mathtt{n}_2}$ are defined in the usual way using the *Smith-McMillan form*. When there are no common poles and zeros, the zeros are simply the complex numbers where $M(\lambda), \lambda \in \mathbb{C}$, drops rank, and the poles where it 'goes to infinity'.

We now discuss unimodularity and prime factorizations over each of these rings. Of course, one can do this once and for all in an abstract algebraic setting (see [4]), but we consider each of these rings for the sake of concreteness.

An element $U \in \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$ is said to be *unimodular over* $\mathbb{R}[\xi]$ if it has an inverse in $\mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$. This is the case iff $\det(U)$ is a non-zero constant. We denote the $\mathbb{R}[\xi]$-unimodular elements of $\mathbb{R}[\xi]^{\bullet\times\bullet}$ by $\mathscr{U}_{\mathbb{R}[\xi]}$. $M \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_2}$ is said to be *left prime over* $\mathbb{R}[\xi]$ if for every factorization $M = FM'$ with $F \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_1}$, $M' \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_2}$, $F$ is unimodular over $\mathbb{R}[\xi]$. $M_1, M_2, \ldots, M_{\mathtt{n}} \in \mathbb{R}[\xi]^{\mathtt{n}\times\bullet}$ are said to be *left co-prime* over $\mathbb{R}[\xi]$ if $M = \operatorname{row}(M_1, M_2, \ldots, M_{\mathtt{n}})$ is left prime over $\mathbb{R}[\xi]$.

The pair $(P,Q)$ is said to be a *left factorization over* $\mathbb{R}[\xi]$ of $M \in \mathbb{R}(\xi)^{\mathtt{n}_1\times\mathtt{n}_2}$ if

(i) $P \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{p}}$ and $Q \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{m}}$,
(ii) $\det(P) \neq 0$, and
(iii) $M = P^{-1}Q$.

It is said to be a *left co-prime factorization of M over* $\mathbb{R}[\xi]$ if, in addition,

(iv) $P$ and $Q$ are left co-prime over $\mathbb{R}[\xi]$.

The existence, for every $M \in \mathbb{R}(\xi)^{\mathtt{n}_1\times\mathtt{n}_2}$, of a left co-prime factorization over $\mathbb{R}[\xi]$ is readily deduced from the Smith-McMillan form. It is unique up to left multiplication of $P$ and $Q$ by a unimodular element $U \in \mathscr{U}_{\mathbb{R}[\xi]}$.

The *relative degree* of $f \in \mathbb{R}(\xi), f = \frac{n}{d}, n, d \in \mathbb{R}[\xi]$, is defined as the degree of the denominator $d$ minus the degree of the numerator $n$. The rational function $f \in \mathbb{R}(\xi)$ is said to be *proper* if the relative degree is $\geq 0$, *strictly proper* if it is $> 0$, and *bi-proper* if it is 0. Denote

$$\mathbb{R}(\xi)_{\mathscr{P}} := \left\{ f \in \mathbb{R}(\xi) \,\middle|\, f \text{ is proper} \right\}.$$

$\mathbb{R}(\xi)_{\mathscr{P}}$ is a ring, in fact, a proper Euclidean domain. $M \in \mathbb{R}(\xi)^{\mathtt{n}_1\times\mathtt{n}_2}$ is said to be *proper* if each of its elements is proper. $M \in \mathbb{R}(\xi)^{\mathtt{n}\times\mathtt{n}}$ is said to be *bi-proper* if $\det(M) \neq 0$ and $M, M^{-1}$ are both proper. $U \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}\times\mathtt{n}}$ is said to be *unimodular over* $\mathbb{R}(\xi)_{\mathscr{P}}$ if it has an inverse in $\mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}\times\mathtt{n}}$. Clearly $[\![ U \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}\times\mathtt{n}}$ is unimodular over $\mathbb{R}(\xi)_{\mathscr{P}} ]\!] \Leftrightarrow [\![$it is bi-proper$]\!] \Leftrightarrow [\![ \det(U)$ is bi-proper$]\!]$. Denote the unimodular elements of $\mathbb{R}(\xi)_{\mathscr{P}}^{\bullet\times\bullet}$ by $\mathscr{U}_{\mathbb{R}(\xi)_{\mathscr{P}}}$.

$M \in \mathbb{R}(\xi)^{\mathtt{n}_1\times\mathtt{n}_2}$ is said to be *left prime* over $\mathbb{R}(\xi)_{\mathscr{P}}$ if every factorization $M = FM'$ with $F \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1\times\mathtt{n}_1}, M' \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1\times\mathtt{n}_2}$ is such that $F$ is unimodular over $\mathbb{R}(\xi)_{\mathscr{P}}$. $M_1, M_2, \ldots, M_{\mathtt{n}} \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}\times\bullet}$ are said to be *left co-prime over* $\mathbb{R}(\xi)_{\mathscr{P}}$ if $M = \operatorname{row}(M_1, M_2, \ldots, M_{\mathtt{n}})$ is left prime over $\mathbb{R}(\xi)_{\mathscr{P}}$. Denote

$$\mathbb{R}(\xi)_{\mathscr{S}} := \left\{ f \in \mathbb{R}(\xi) \,\middle|\, f \text{ proper, no poles in } \overline{\mathbb{C}}_+ \right\}.$$

Other stability domains are of interest, but we stick with the usual 'Hurwitz' domain. It is easy to see that $\mathbb{R}(\xi)_{\mathscr{S}}$ is a ring. $\mathbb{R}(\xi)$ is its field of fractions. An element $U \in$

$\mathbb{R}(\xi)_{\mathscr{J}}^{\mathtt{n}\times\mathtt{n}}$ is said to be *unimodular over* $\mathbb{R}(\xi)_{\mathscr{J}}$ if it has an inverse in $\mathbb{R}(\xi)_{\mathscr{J}}^{\mathtt{n}\times\mathtt{n}}$. This is the case iff $\det(U)$ is bi-proper and miniphase ($:\Leftrightarrow$ it has no poles and no zeros in $\overline{\mathbb{C}}_+$). We denote the unimodular elements of $\mathbb{R}(\xi)_{\mathscr{J}}^{\bullet\times\bullet}$ by $\mathscr{U}_{\mathbb{R}(\xi)_{\mathscr{J}}}$. $M\in\mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_2}$ is said to be *left prime* over $\mathbb{R}(\xi)_{\mathscr{J}}$ if every factorization $M=FM'$ with $F\in\mathbb{R}(\xi)_{\mathscr{J}}^{\mathtt{n}_1\times\mathtt{n}_1}, M'\in\mathbb{R}(\xi)_{\mathscr{J}}^{\mathtt{n}_1\times\mathtt{n}_2}$ is such that $F$ is unimodular over $\mathbb{R}(\xi)_{\mathscr{J}}$. $M_1,M_2,\ldots,M_{\mathtt{n}}\in\mathbb{R}(\xi)_{\mathscr{J}}^{\mathtt{n}\times\bullet}$ are said to be *left co-prime over* $\mathbb{R}(\xi)_{\mathscr{J}}$ if $M=\operatorname{row}(M_1,M_2,\ldots,M_{\mathtt{n}})$ is left prime over $\mathbb{R}(\xi)_{\mathscr{J}}$.

Right (co-)prime, right (co-) prime factorizations, etc., are defined in analogy with their left counterparts.

## II. Polynomial representations

A *dynamical system* is a triple $\Sigma=(\mathbb{T},\mathbb{W},\mathscr{B})$, with $\mathbb{T}\subseteq\mathbb{R}$ the *time-set*, $\mathbb{W}$ the *signal space*, and $\mathscr{B}\subseteq\mathbb{W}^{\mathbb{T}}$ the *behavior*. In this article, we deal with continuous-time systems, $\mathbb{T}=\mathbb{R}$, with a finite dimensional signal space, $\mathbb{W}=\mathbb{R}^{\bullet}$. We assume throughout that our systems are (i) *linear* (meaning that $\mathscr{B}$ is a linear subspace of $(\mathbb{R}^{\bullet})^{\mathbb{R}}$), (ii) *time-invariant* (meaning that $\mathscr{B}=\sigma^t(\mathscr{B})$ for all $t\in\mathbb{R}$, where $\sigma^t$ is defined by $\sigma^t(f)(t'):=f(t'+t)$), and (iii) *differential*. This means that the behavior consists of the set of solutions of a system of differential equations.

Each of the behaviors $\mathscr{B}\subseteq(\mathbb{R}^{\bullet})^{\mathbb{R}}$ which we consider is hence the solution set of a system of linear constant coefficient differential equations. In other words, there exists a polynomial matrix $R\in\mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$ such that $\mathscr{B}$ is the solution set of

$$\boxed{R\left(\tfrac{d}{dt}\right)w=0} \qquad (\mathscr{R})$$

$(\mathscr{R})$ defines the dynamical system $\Sigma=(\mathbb{R},\mathbb{R}^{\bullet},\mathscr{B})$ with

$$\mathscr{B}=\left\{w\in\mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\bullet})\;\middle|\;(\mathscr{R})\text{ is satisfied}\right\}.$$

Note that we may as well denote this as $\mathscr{B}=\ker\left(R\left(\frac{d}{dt}\right)\right)$, since $\mathscr{B}$ is actually the kernel of the differential operator $R\left(\frac{d}{dt}\right):\mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\operatorname{coldim}(R)})\to\mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\operatorname{rowdim}(R)})$. We denote the set of linear time-invariant differential systems or their behaviors by $\mathscr{L}^{\bullet}$, and by $\mathscr{L}^{\mathtt{w}}$ when the number of variables is $\mathtt{w}$.

## III. Rational representations

The aim of this article is to discuss other representations of $\mathscr{L}^{\bullet}$, namely representations by means of matrices of rational functions. Let $G\in\mathbb{R}(\xi)^{\bullet\times\bullet}$, and consider the system of 'differential equations'

$$\boxed{G\left(\tfrac{d}{dt}\right)w=0.} \qquad (\mathscr{G})$$

Since $G$ is a matrix of rational functions, it is not clear when $w:\mathbb{R}\to\mathbb{R}^{\bullet}$ is a solution of $(\mathscr{G})$. This is not a question of smoothness, but a matter of giving a meaning to the equality, since $G(\frac{d}{dt})$ is not a differential operator. We do this as follows.

*Definition 1:* Let $(P,Q)$ be a left co-prime matrix factorization of $G=P^{-1}Q$ over $\mathbb{R}[\xi]$. Define

$$[\![w:\mathbb{R}\to\mathbb{R}^{\bullet}\text{ is a solution of }(\mathscr{G})]\!]:\Leftrightarrow[\![Q(\frac{d}{dt})w=0]\!].$$

■

In analogy of the polynomial case, we denote the set of solutions of $(\mathscr{G})$ by $\ker\left(G(\frac{d}{dt})\right)$. Whence $(\mathscr{G})$ defines the system $\Sigma=\left(\mathbb{R},\mathbb{R}^{\bullet},\ker\left(G(\frac{d}{dt})\right)\right)=\left(\mathbb{R},\mathbb{R}^{\bullet},\ker\left(Q(\frac{d}{dt})\right)\right)\in\mathscr{L}^{\bullet}$. Since the representations $(\mathscr{R})$ are merely a subset of the representations $(\mathscr{G})$, matrices of rational functions form a representation class of $\mathscr{L}^{\bullet}$ that is more redundant than the polynomial matrices. That $\mathscr{B}\in\mathscr{L}^{\bullet}$ admits a representation as the kernel of a polynomial or rational matrix in $\frac{d}{dt}$ is a matter of definition. However, representations using the ring of proper stable rational functions are important in applications. We state this representability in the next proposition.

*Proposition 2:* Let $\mathscr{B}\in\mathscr{L}^{\bullet}$. There exists $G\in\mathbb{R}(\xi)_{\mathscr{J}}^{\bullet\times\bullet}$ such that $\mathscr{B}=\ker\left(G(\frac{d}{dt})\right)$.

## IV. Controllability and stabilizability

In this section, we relate controllability and stabilizability of a system to properties of their rational representations. We first recall the behavioral definitions of these notions.

*Definition 3:* The time-invariant system $\Sigma=(\mathbb{R},\mathbb{R}^{\bullet},\mathscr{B})$ is said to be *controllable* if for all $w_1,w_2\in\mathscr{B}$, there exists $T\geq 0$ and $w\in\mathscr{B}$, such that $w(t)=w_1(t)$ for $t<0$, and $w(t)=w_2(t-T)$ for $t\geq T$.

It is said to be *stabilizable* if for all $w\in\mathscr{B}$, there exists $w'\in\mathscr{B}$, such that $w'(t)=w(t)$ for $t<0$ and $w'(t)\to 0$ for $t\to\infty$.

■

Observe that for $\mathscr{B}\in\mathscr{L}^{\bullet}$, controllability $\Rightarrow$ stabilizability. Denote the controllable elements of $\mathscr{L}^{\bullet}$ by $\mathscr{L}^{\bullet}_{\text{contr}}$, and of $\mathscr{L}^{\mathtt{w}}$ by $\mathscr{L}^{\mathtt{w}}_{\text{contr}}$, and the stabilizable elements of $\mathscr{L}^{\bullet}$ by $\mathscr{L}^{\bullet}_{\text{stab}}$, and of $\mathscr{L}^{\mathtt{w}}$ by $\mathscr{L}^{\mathtt{w}}_{\text{stab}}$. It is easy to derive tests for controllability and stabilizability in terms of kernel representations.

*Proposition 4:* $(\mathscr{G})$ defines a controllable system iff $G$ has no zeros, and a stabilizable one iff $G$ has no zeros in $\overline{\mathbb{C}}_+$.

The following result links controllability and stabilizability of systems in $\mathscr{L}^{\bullet}$ to the existence of left prime representations over the rings $\mathbb{R}[\xi]$ and $\mathbb{R}(\xi)_{\mathscr{J}}$ respectively.

*Theorem 5:*

1) $\mathscr{B}\in\mathscr{L}^{\bullet}$ is controllable iff it admits a representation $(\mathscr{R})$ with $R\in\mathbb{R}[\xi]^{\bullet\times\bullet}$ left prime over $\mathbb{R}[\xi]$.
2) $\mathscr{B}\in\mathscr{L}^{\bullet}$ iff it admits a representation $(\mathscr{G})$ with $G$ left prime over $\mathbb{R}(\xi)_{\mathscr{P}}$.
3) $\mathscr{B}\in\mathscr{L}^{\bullet}$ is stabilizable iff it admits a representation $(\mathscr{G})$ with $G\in\mathbb{R}(\xi)_{\mathscr{J}}^{\bullet\times\bullet}$ left prime over $\mathbb{R}(\xi)_{\mathscr{J}}$.

The above theorem spells out exactly what the condition is for the existence of a kernel representation that is left prime over $\mathbb{R}(\xi)_{\mathscr{J}}$: *stabilizability*. It is of interest to compare this result with the classical results, obtained by Vidyasagar in his book [4] (this builds on a series of earlier results, for example [2], [8], [1]). In these publications, the aim is to obtain a representation of a system that is given as a transfer function to start with,

$$y=F(\frac{d}{dt})u,\quad w=\begin{bmatrix}u\\y\end{bmatrix} \qquad (\mathscr{F})$$

where $F\in\mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}$. This is a special case of $(\mathscr{G})$, and, since $\begin{bmatrix}I_{\mathtt{p}} & -F\end{bmatrix}$ has no zeros, this system is controllable (by

proposition 4), and therefore stabilizable. Thus, by theorem 5 it also admits a representation $G_1(\frac{d}{dt})y = G_2(\frac{d}{dt})u$ with $G_1, G_2 \in \mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{S}}$, and left co-prime over $\mathbb{R}(\xi)_{\mathscr{S}}$. This is an important, classical, result. However, theorem 5 implies that, since we are in the controllable case, there exists a such a representation such that $\begin{bmatrix} G_1 & G_2 \end{bmatrix}$ has no zeros.

The difference of our result from the classical left co-prime factorization results over $\mathbb{R}(\xi)_{\mathscr{S}}$ is that we preserve also the non-controllable part, whereas in the classical approach all stabilizable systems with the same transfer function are identified. By taking a trajectory based definition, the behavioral point of view is able to keep track of all trajectories, also of the non-controllable ones. Loosely speaking, left co-prime factorizations over $\mathbb{R}(\xi)_{\mathscr{S}}$ manage to avoid unstable pole-zero cancellations. Our approach avoids introducing and cancelling common poles and zeros. Since the whole issue of co-prime factorizations started from a need to deal with pole-zero cancellations [8], we feel that our trajectory based mode of thinking offers a useful point of view.

## V. Latent variables

Many models, e.g. first principles models obtained by interconnection and state models, include auxiliary variables in addition to the variables the model aims at. We call the latter *manifest variables*, and the auxiliary variables *latent variables*. In the context of rational models, this leads to the model class

$$\boxed{R\left(\tfrac{d}{dt}\right)w = M\left(\tfrac{d}{dt}\right)\ell} \qquad (\mathscr{RM})$$

with $R, M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$. Since we have reduced the behavior of the system of 'differential equations' $(\mathscr{RM})$ involving rational functions, to one involving only polynomials, the *elimination theorem* [3, theorem 6.2.2] remains valid. Consequently, the *manifest behavior* of $(\mathscr{RM})$, defined as

$$\{w \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet) \;\big|\; \exists \ell \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet) \text{ such that } (\mathscr{RM}) \text{ holds}\},$$

belongs to $\mathscr{L}^\bullet$.

*Definition 6:* The latent variable representation $(\mathscr{RM})$ is said to be *observable* if, whenever $(w,\ell_1)$ and $(w,\ell_2)$ satisfy $(\mathscr{RM})$, then $\ell_1 = \ell_2$. It is said to be *detectable* if, whenever $(w,\ell_1)$ and $(w,\ell_2)$ satisfy $(\mathscr{RM})$, then $\ell_1(t) - \ell_2(t) \to 0$ as $t \to \infty$. ∎

The following proposition is readily obtained.

*Proposition 7:* $(\mathscr{RM})$ is observable iff $M$ has full column rank and has no zeros. It is detectable iff $M$ has full column rank, and has no zeros in $\overline{\mathbb{C}}_+$. ∎

## VI. Image-like representations

Consider now the following special cases of $(\mathscr{R})$ and $(\mathscr{G})$:

$$\boxed{w = M\left(\tfrac{d}{dt}\right)\ell} \qquad (\mathscr{M})$$

with $M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$, and

$$\boxed{w = H(\tfrac{d}{dt})\ell} \qquad (\mathscr{H})$$

with $H \in \mathbb{R}(\xi)^{\bullet\times\bullet}$. Note that the manifest behavior of $(\mathscr{M})$ is the image of the differential operator $M(\frac{d}{dt})$. This representation is hence called an *image representation* of its manifest behavior. It is *not* appropriate, however, to call $(\mathscr{H})$ an image representation of its manifest behavior. Indeed, for a given $\ell \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)$, there are, whenever $H$ has poles, many corresponding solutions $w$: $H(\frac{d}{dt})$ *is not a map*. It is only in the observable case that $(\mathscr{H})$ defines a map from $w$ to $\ell$. Hence if $H$ is of full column rank and has no zeros, $(\mathscr{RM})$ defines a map from $w$ to $\ell$. If it has no poles, it is a map from $\ell$ to $w$. Nevertheless, the well known relation between controllability and these representations remains valid.

*Theorem 8:* The following are equivalent for $\mathscr{B} \in \mathscr{L}^\bullet$.

1) $\mathscr{B}$ is controllable,
2) $\mathscr{B}$ admits an image representation $(\mathscr{M})$,
3) $\mathscr{B}$ admits an observable image representation $(\mathscr{M})$,
4) $\mathscr{B}$ admits an image representation $(\mathscr{M})$ with $M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ right prime over $\mathbb{R}[\xi]$,
5) $\mathscr{B}$ admits a representation $(\mathscr{H})$ with $H \in \mathbb{R}(\xi)^{\bullet\times\bullet}$,
6) $\mathscr{B}$ admits a representation $(\mathscr{H})$ with $H \in \mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{S}}$ right prime over $\mathbb{R}(\xi)_{\mathscr{S}}$,
7) $\mathscr{B}$ admits an observable representation $(\mathscr{H})$ with $H \in \mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{S}}$ right prime over $\mathbb{R}(\xi)_{\mathscr{S}}$.

∎

## VII. Conclusions

The set of solutions of the system of 'differential equations' $G(\frac{d}{dt})w = 0$ with $G$ a matrix of rational functions can be defined very concretely in terms of a left co-prime factorization of $G$. This implies that $G(\frac{d}{dt})w = 0$ defines a linear shift invariant differential behavior. This definition bring the behavioral theory of systems and the theory of representations using proper stable rational functions in line with each other.

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