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Storage functions for systems described by PDE's

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1. Introduction

The notion of a dissipative system is one of the useful concepts in systems theory. Many results involving stability of systems and design of robust controllers make use of this notion. Until now, the theory of dissipative systems has been developed for systems that have time as its only independent variable (1-D systems). However, models of physical systems often have several independent variables (i.e., they are n-D systems), for example, time and space variables. In this chapter we develop the theory of dissipative systems for n-D systems.

The central problem in the theory of dissipative systems is the construction of a storage function. Examples of storage functions include Lyapunov functions in stability analysis and internal energy, entropy in thermodynamics, etc. The construction of storage functions for 1-D systems is well understood [4, Part 1] for general nonlinear systems and for linear systems with quadratic supply rates [4, Part 2] and [7]. In this chapter, we obtain analogous results for n-D systems described by linear constant coefficient partial differential equations with quadratic differential forms as supply rates. However, there are some important differences between the 1-D and the n-D case. The most important one being the dependence of storage functions on the unobservable (or hidden) latent variables.

A few words about notation. We use the standard notation $\mathbb{R}^n$, $\mathbb{R}^{n_1 \times n_2}$, etc., for finite-dimensional vectors and matrices. When the dimension is not specified (but, of course, finite), we write $\mathbb{R}^\bullet$, $\mathbb{R}^{n \times \bullet}$, $\mathbb{R}^{\bullet \times \bullet}$, etc. In order to enhance readability, we typically use the notation $\mathbb{R}^{\mathrm{w}}$ when functions taking their values in that vector space are denoted by w. Real polynomials in the indeterminates $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ are denoted by $\mathbb{R}[\xi]$ and real rational functions by $\mathbb{R}(\xi)$, with obvious modifications for the matrix case. The space of infinitely differentiable functions with domain $\mathbb{R}^n$ and co-domain $\mathbb{R}^{\mathrm{w}}$ is denoted by $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$, and its subspace consisting of elements with compact support by $\mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$.

2. n-D SYSTEMS

We view a system as a triplet $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$. Here $\mathbb{T}$ is the indexing set and stands for the set of "independent" variables (for example, time, space, time and space). $\mathbb{W}$ stands for the set of "dependent" variables, i.e., where the variables take on their values – these are often called the signal space or the space of field variables. Finally the "behavior" $\mathfrak{B}$ is viewed as a subset of the family of all trajectories that map the set of independent variables into the set of dependent variables. In fact, the behavior $\mathfrak{B}$ consists of the set of admissible trajectories that satisfy the system laws (for example, the set of partial differential equations that constitute the system laws). In this chapter, we consider systems with $\mathbb{T} = \mathbb{R}^n$ (n-D systems). We assume throughout that $\mathbb{W}$ is a finite-dimensional real vector space, $\mathbb{W} = \mathbb{R}^{\mathrm{w}}$.

In this chapter, we look at behaviors that arise from a system of partial differential equations. More precisely, if there exists a real polynomial matrix $R \in \mathbb{R}^{\bullet\times \mathrm{w}}[\xi]$ in n indeterminates, $\xi = (\xi_1, \ldots, \xi_n)$, then we consider $\mathfrak{B}$ to be the $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$-solutions of

$$R\left(\frac{d}{d\mathbf{x}}\right) w = 0, \tag{1}$$

where $\frac{d}{d\mathbf{x}} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. The assumption about $\mathfrak{C}^\infty$ solutions is made for ease of exposition. The results of this chapter also hold for other solution concepts like distributions, though the mathematics needed is more involved. Systems $\Sigma = (\mathbb{R}^n, \mathbb{R}^{\mathrm{w}}, \mathfrak{B})$ that are defined by a set of constant coefficient partial differential equations (equivalently, behaviors that arise as a consequence of a set of constant coefficient partial differential equations) will be called differential systems and denoted as $\mathfrak{L}_n^{\mathrm{w}}$. We often abuse the notation by stating $\mathfrak{B} \in \mathfrak{L}_n^{\mathrm{w}}$, as the indexing set and the signal space are then obvious.

Whereas we have *defined* the behavior of a system in $\mathfrak{L}_n^{\mathrm{w}}$ as the set of solutions of a system of PDE's in the system variables, often, in applications, the specification of the behavior involves other, auxiliary variables, which we call *latent variables*. Specifically, consider the system of PDE's

$$R\left(\frac{d}{dx}\right) w = M\left(\frac{d}{dx}\right) \ell \tag{2}$$

with $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$ and $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^\ell)$ and with $R \in \mathbb{R}^{\bullet\times \mathrm{w}}[\xi]$ and $M \in \mathbb{R}^{\bullet\times \ell}[\xi]$ polynomial matrices with the same number of rows. The set

$$\mathfrak{B}_f = \left\{(w, \ell) \in \mathfrak{C}^\infty\left(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}+\ell}\right) \mid (2) \text{ holds}\right\} \tag{3}$$

obviously belongs to $\mathfrak{L}_n^{\mathrm{w}+\ell}$. It follows from a classical result in the theory of PDEs, *the fundamental principle*, that the set

$$\left\{w \in \mathfrak{C}^\infty\left(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}}\right) \mid \exists \ell \in \mathfrak{C}^\infty\left(\mathbb{R}^n, \mathbb{R}^\ell\right) \colon (w, \ell) \in \mathfrak{B}_f\right\} \tag{4}$$

belongs to $\mathfrak{L}_n^{\mathrm{w}}$. We call (2) a latent variable representation with manifest variables w and latent variables ℓ, of the system with full behavior (3) and manifest

behavior (4). Correspondingly, we call (1) a kernel representation of the system with the behavior $\ker(R(\frac{d}{d\mathbf{x}}))$. We shall soon meet another sort of representation, the image representations, in the context of controllability.

3. Controllability and observability

Two very influential classical properties of dynamical systems are those of controllability and observability. These properties were defined for 1-D systems in a behavioral setting in [5] and in [3] generalizations to n-D systems have been introduced. We discuss these concepts here exclusively in the context of systems described by linear constant coefficient PDE's.

Definition 1. A system $\mathfrak{B} \in \mathfrak{L}_n^{\mathrm{w}}$ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ and for all sets $U_1, U_2 \subset \mathbb{R}^n$ with disjoint closure, there exists a $w \in \mathfrak{B}$ such that $w|_{U_1} = w_1|_{U_1}$ and $w|_{U_2} = w_2|_{U_2}$.

Thus controllable PDE's are those in which the solutions can be 'patched up' from solutions on subsets.

Though there are several characterizations of controllability, the characterization that is important for the purposes of this chapter is the equivalence of controllability with the existence of an image representation. Consider the following special latent variable representation

$$w = M\left(\frac{d}{d\mathbf{x}}\right)\ell \tag{5}$$

with $M \in \mathbb{R}^{\mathrm{w}\times\ell}[\xi]$. Obviously, by the elimination theorem, its manifest behavior $\mathfrak{B} \in \mathfrak{L}_n^{\mathrm{w}}$. Such special latent variable representations often appear in physics, where the latent variables involved in such a representation are called *potentials*. Obviously $\mathfrak{B} = \mathrm{im}(M(\frac{d}{d\mathbf{x}}))$ with $M(\frac{d}{d\mathbf{x}})$ viewed as a map from $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^\ell)$ to $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$. For this reason, we call (5) an *image representation* of its manifest behavior. Whereas every $\mathfrak{B} \in \mathfrak{L}_n^{\mathrm{w}}$ allows (by definition) a kernel representation and hence trivially a latent variable representation, not every $\mathfrak{B} \in \mathfrak{L}_n^{\mathrm{w}}$ allows an image representation. In fact:

Theorem 2. *$\mathfrak{B} \in \mathfrak{L}_n^{\mathrm{w}}$ admits an image representation if and only if it is controllable.*

We denote the set of controllable systems in $\mathfrak{L}_n^{\mathrm{w}}$ by $\mathfrak{L}_{n,\mathrm{cont}}^{\mathrm{w}}$.

Observability is the property of systems that have two kinds of variables – the first set of variables are the 'observed' set of variables, and the second set of variables are the ones that are 'to-be-deduced' from the observed variables. Every variable that can be deduced uniquely from the manifest variables of a given behavior will be called an *observable*. So observability is not an intrinsic property of a given behavior. One has to be given a partition of the variables in the behavior into two

classes before one can say whether one class of variables in the behavior can actually be deduced from the other class of variables (which were observed).

Definition 3. Let $w = (w_1, w_2)$ be a partition of the variables in $\Sigma = (\mathbb{R}^n, \mathbb{R}^{w_1+w_2}, \mathfrak{B})$. Then w_2 is said to be *observable* from w_1 in $\mathfrak{B}$ if given any two trajectories $(w_1', w_2'), (w_1'', w_2'') \in \mathfrak{B}$ such that $w_1' = w_1''$, then $w_2' = w_2''$.

A natural situation to use observability is when one looks at the latent variable representation of a behavior. Then one may ask whether the latent variables are observable from the manifest variables. If this is the case, then we call the latent variable representation *observable*.

As we have already mentioned, every controllable behavior has an image representation. Whereas every controllable behavior has an observable image representation in 1-D systems, this is no longer true for n-D systems.

4. Quadratic differential forms

It was shown in [6,7] that for systems described by *one-variable* polynomial matrices, the appropriate tool to express quadratic functionals are *two-variable* polynomial matrices. In this chapter we will use polynomial matrices in $2n$ variables to express quadratic functionals for functions of n variables.

For convenience, let ζ denote $(\zeta_1, \ldots, \zeta_n)$ and η denote $(\eta_1, \ldots, \eta_n)$. Let $\mathbb{R}^{w_1 \times w_2}[\zeta, \eta]$ denote the set of real polynomial matrices in the $2n$ indeterminates ζ and η. We will consider quadratic forms of the type $\Phi \in \mathbb{R}^{w_1 \times w_2}[\zeta, \eta]$. Explicitly,

$$\Phi(\zeta, \eta) = \sum_{\mathbf{k},\mathbf{l}} \Phi_{\mathbf{k},\mathbf{l}} \zeta^{\mathbf{k}} \eta^{\mathbf{l}}.$$

This sum ranges over the non-negative multi-indices $\mathbf{k} = (k_1, k_2, \ldots, k_n), \mathbf{l} = (l_1, l_2, \ldots, l_n) \in \mathbb{N}^n$ and the sum is assumed to be finite. Moreover, $\Phi_{\mathbf{k},\mathbf{l}} \in \mathbb{R}^{w_1 \times w_2}$. The polynomial matrix Φ induces a *bilinear differential form* (BLDF), that is, the map

$$L_\Phi : \mathfrak{C}^\infty\big(\mathbb{R}^n, \mathbb{R}^{w_1}\big) \times \mathfrak{C}^\infty\big(\mathbb{R}^n, \mathbb{R}^{w_2}\big) \to \mathfrak{C}^\infty\big(\mathbb{R}^n, \mathbb{R}\big)$$

defined by

$$L_\Phi(v, w)(\mathbf{x}) := \sum_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{k}} v}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x}) \right)^{\mathrm{T}} \Phi_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{l}} w}{d\mathbf{x}^{\mathbf{l}}}(\mathbf{x}) \right),$$

where $\frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}} = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}$ and analogously for $\frac{d^{\mathbf{l}}}{d\mathbf{x}^{\mathbf{l}}}$. Note that ζ corresponds to differentiation of terms to the left and η refers to differentiation of the terms to the right.

If $w_1 = w_2 = w$, then Φ induces the *quadratic differential form* (QDF)

$$Q_\Phi : \mathfrak{C}^\infty\big(\mathbb{R}^n, \mathbb{R}^{w}\big) \to \mathfrak{C}^\infty\big(\mathbb{R}^n, \mathbb{R}\big)$$

defined by

$$Q_\Phi(w) := L_\Phi(w, w).$$

Define the * operator

$$^* : \mathbb{R}^{w\times w}[\zeta, \eta] \to \mathbb{R}^{w\times w}[\zeta, \eta]$$

by

$$\Phi^*(\zeta, \eta) := \Phi^{\mathrm{T}}(\eta, \zeta).$$

If $\Phi = \Phi^*$, then Φ is called *symmetric*. For the purposes of QDF's induced by polynomial matrices, it suffices to consider the symmetric quadratic differential forms, since $Q_\Phi = Q_{\Phi^*} = Q_{\frac{1}{2}(\Phi+\Phi^*)}$.

We also consider vectors $\Psi \in (\mathbb{R}^{w\times w}[\zeta, \eta])^n$, i.e. $\Psi = (\Psi_1, \ldots, \Psi_n)$. Analogous to the quadratic differential form Φ, Ψ induces a *vector of quadratic differential forms* (VQDF)

$$Q_\Psi(w) : \mathfrak{C}^\infty\big(\mathbb{R}^n, \mathbb{R}^w\big) \to \big(\mathfrak{C}^\infty\big(\mathbb{R}^n, \mathbb{R}\big)\big)^n$$

defined by $Q_\Psi = (Q_{\Psi_1}, \ldots, Q_{\Psi_n})$.

Finally, we define the "div" (divergence) operator that associates with the VQDF induced by Ψ, the scalar QDF

$$(\operatorname{div} Q_\Psi)(w) := \frac{\partial}{\partial x_1} Q_{\Psi_1}(w) + \cdots + \frac{\partial}{\partial x_n} Q_{\Psi_n}(w).$$

The theory of QDF's have been developed in much detail in [6,7] for 1-D systems.

5. Lossless and dissipative systems

Quadratic functionals play an important role in control theory. Quite often, the rate of supply of some physical quantity (for example, the rate of energy, i.e., the power) delivered to a system is given by a quadratic functional. We make use of quadratic differential forms defined earlier to define such supply rates for controllable systems $\mathfrak{B} \in \mathfrak{L}^w_{n,\mathrm{cont}}$.

Let $\Phi = \Phi^* \in \mathbb{R}^{w\times w}[\zeta, \eta]$ and $\mathfrak{B} \in \mathfrak{L}^w_{n,\mathrm{cont}}$. We consider the quadratic differential form $Q_\Phi(w)$ as a supply rate for trajectories $w \in \mathfrak{B}$. More precisely, we consider $Q_\Phi(w)(\mathbf{x})$ (with $\mathbf{x} \in \mathbb{R}^n$) as the rate of supply of some physical quantity delivered to the system at the point $\mathbf{x}$. Thus, $Q_\Phi(w)(\mathbf{x})$ being positive implies that the system absorbs the physical quantity that is being supplied.

Definition 4. The system $\mathfrak{B} \in \mathfrak{L}^w_{n,\mathrm{cont}}$ is said to be *lossless* with respect to the supply rate Q_Φ induced by $\Phi = \Phi^* \in \mathbb{R}^{w\times w}[\zeta, \eta]$ if $\int_{\mathbb{R}^n} Q_\Phi(w)\, d\mathbf{x} = 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^w)$ (i.e., trajectories in the behavior $\mathfrak{B}$ with compact support).

The system $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{n,\mathrm{cont}}$ is said to be *dissipative with respect to* Q_Φ (briefly Φ-dissipative) if

$$\int_{\mathbb{R}^n} Q_\Phi(w)\, d\mathbf{x} \geqslant 0$$

for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$.

We now explain the physical interpretation of the definition above.

$\int_{\mathbb{R}^n} Q_\Phi(w)\, d\mathbf{x}$ denotes the net amount of supply that the system absorbs integrated over "time" and "space". So the system is lossless with respect to the quadratic differential form if this integral is zero, since any supply absorbed at some time or place is temporarily stored but eventually recovered (perhaps at some other time or space). On the other hand, if the integral is non-negative, then the net amount of supply is absorbed (at least for some of the trajectories) by the system. Thus the system is dissipative.

Note that the conditions in the above definitions are defined only for compactly supported trajectories in the behavior $\mathfrak{B}$. The intuitive reason for such a restriction is because for controllable systems, compactly supported trajectories are completely representative of all trajectories. It also gets us through the technical difficulty that might arise if the integral $\int_{\mathbb{R}^n} Q_\Phi(w)\, d\mathbf{x}$ is not well defined. Since we consider only compact trajectories, such a complication does not arise.

We shall first look at lossless systems. The following theorem gives some equivalent conditions for a system to be lossless.

Theorem 5. *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{n,\mathrm{cont}}$. *Let* $R \in \mathbb{R}^{\bullet\times\mathrm{w}}[\xi]$ *and* $M \in \mathbb{R}^{\mathrm{w}\times\bullet}[\xi]$ *induce respectively a kernel and image representation of* $\mathfrak{B}$*; i.e.* $\mathfrak{B} = \ker(R(\frac{d}{d\mathbf{x}})) = \mathrm{im}(M(\frac{d}{d\mathbf{x}}))$. *Let* $\Phi = \Phi^* \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta, \eta]$ *induce a QDF on* $\mathfrak{B}$. *Then the following conditions are equivalent*:

(1) $\mathfrak{B}$ *is lossless with respect to the QDF* Q_Φ;
(2) $\Phi'(-\xi, \xi) = 0$ *where* $\Phi'(\zeta, \eta) := M^{\mathrm{T}}(\zeta)\Phi(\zeta, \eta)M(\eta)$;
(3) *there exists a VQDF* Q_Ψ, *with* $\Psi \in (\mathbb{R}^{m\times m}[\zeta, \eta])^n$, *where* m *is the number of columns of* M, *such that*

$$\text{(6)} \qquad \mathrm{div}\, Q_\Psi(\ell) = Q_{\Phi'}(\ell) = Q_\Phi(w)$$

for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ *and* $w = M(\frac{d}{d\mathbf{x}})\ell$.

Note that the condition (1) in the above theorem states that $\mathfrak{B}$ is lossless with respect to Q_Φ, i.e. that

$$\text{(7)} \qquad \int_{\mathbb{R}^n} Q_\Phi(w)\, d\mathbf{x} = 0$$

for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$. This is a global statement about the concerned trajectory $w \in \mathfrak{B}$. On the other hand, condition (3) of the above theorem states that $\mathfrak{B}$ admits an image representation $w = M(\frac{d}{d\mathbf{x}})\ell$ and there exists some VQDF Ψ such that

$$\text{(8)} \qquad \text{div } Q_\Psi(\ell) = Q_\Phi(w)$$

for all $w \in \mathfrak{B}$ and ℓ such that $w = M(\frac{d}{d\mathbf{x}})\ell$. This statement gives a local characterization of losslessness. This equivalence of the global version of losslessness (7) with the local version (8) is a recurrent theme in the theory of dissipative systems.

The local version states that there is a function, $Q_\Psi(\ell)(\mathbf{x})$ that plays the role of amount of supply stored at $\mathbf{x} \in \mathbb{R}^n$. Thus (8) says that for lossless systems, it is possible to define a *storage function* Q_Ψ such that the *conservation equation*

$$\text{(9)} \qquad \text{div } Q_\Psi(\ell) = Q_\Phi(w)$$

is satisfied for all w, ℓ such that $w = M(\frac{d}{d\mathbf{x}})\ell$.

At this point it is worth emphasizing some basic differences between 1-D and n-D systems. Since every controllable 1-D behavior has an observable image representation, it can be shown that the conservation equation can be rewritten in the form

$$\frac{d}{dt} Q_{\Psi'}(w) = Q_\Phi(w)$$

with some quadratic differential form $Q_{\Psi'}$ that acts on the manifest variables. Here t is assumed to be the independent variable for the concerned 1-D behavior. On the other hand, since every controllable n-D behavior need not necessarily have an observable image representation, there may not exist any storage function of the form $Q_{\Psi'}(w)$, that depend only on the manifest variables. Thus, the storage function in the conservation equation (9) may involve "hidden" (i.e., non-observable) variables. Another important difference between 1-D and n-D behaviors is the non-uniqueness of the vector of quadratic differential forms Q_Ψ involved in the conservation equation (9) for the n-D case. As a result of this non-uniqueness, there will be several possible storage functions in the n-D case that satisfy the conservation equation.

We now formally define the concept of a storage function and the associated notion of a dissipation rate. As we have already seen in the context of lossless systems, the storage function is in general a function of the unobservable latent variables that appear in an image representation of the behavior $\mathfrak{B}$. We now incorporate this in the definition and show later that the function Q_Ψ defined in the conservation equation (9) is indeed a storage function.

Definition 6. Let $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{n,\mathrm{cont}}$, $\Phi = \Phi^* \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta, \eta]$ and $w = M(\frac{d}{d\mathbf{x}})\ell$ be an image representation of $\mathfrak{B}$ with $M \in \mathbb{R}^{\mathrm{w}\times\ell}[\xi]$. Let $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ with $\Psi_k = \Psi_k^* \in \mathbb{R}^{\ell\times\ell}[\zeta, \eta]$ for $k = 1, 2, \ldots, n$. The VQDF Q_Ψ is said to be a *storage function* for $\mathfrak{B}$ with respect to Q_Φ if

$$\text{(10)} \qquad \text{div } Q_\Psi(\ell) \leqslant Q_\Phi(w)$$

for all $\ell \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^\ell)$ and $w = M(\frac{d}{d\mathbf{x}})\ell$.

$\Delta = \Delta^* \in \mathbb{R}^{\ell\times\ell}[\zeta, \eta]$ is said to be a *dissipation rate* for $\mathfrak{B}$ with respect to Q_Φ if

$$Q_\Delta \geqslant 0 \quad \text{and} \quad \int_{\mathbb{R}^n} Q_\Delta(\ell)\, d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(w)\, d\mathbf{x}$$

for all $\ell \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^\ell)$ and $w = M(\frac{d}{d\mathbf{x}})\ell$.

We define $Q_\Delta \geqslant 0$ if $Q_\Delta(w(\mathbf{x})) \geqslant 0$ for all $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$ evaluated at every $\mathbf{x} \in \mathbb{R}^n$. This defines a pointwise positivity condition. Thus $\int_\Omega Q_\Delta(w)\, d\mathbf{x} \geqslant 0$ for every $\Omega \subset \mathbb{R}^n$ if $Q_\Delta \geqslant 0$.

In the case of lossless systems, we had obtained the conservation equation

$$\text{div } Q_\Psi(\ell) = Q_\Phi(w).$$

Clearly, this Q_Ψ qualifies to be a storage function as it satisfies the inequality stated in the definition above.

From the above definitions, it is also easy to see that there is a relation between a storage function for $\mathfrak{B}$ with respect to Q_Φ and a dissipation rate for $\mathfrak{B}$ with respect to Q_Φ, given by

$$\text{div } Q_\Psi(\ell) = Q_\Phi\left(M\left(\frac{d}{d\mathbf{x}}\right)\ell\right) - Q_\Delta(\ell). \tag{11}$$

The above definitions of the storage function and the dissipation rate, combined with (11), yield intuitive interpretations. The dissipation rate can be thought of as the rate of supply that is dissipated in the system and the storage function as the rate of supply stored in the system. Intuitively, we could think of the QDF Q_Φ as measuring the power going into the system. Φ-dissipativity would imply that the net power flowing into a system is non-negative which in turn implies that the system dissipates energy. Of course, locally the flow of energy could be positive or negative, leading to variations in $Q_\Psi(\ell)$ (in many practical situations $Q_\Psi(\ell)$ play the role of energy density and fluxes). If the system is dissipative, then the rate of change of energy density and fluxes cannot exceed the power delivered into the system. This is captured by the inequality (10) in Definition 6. The excess is precisely what is lost (or dissipated). This interaction between supply, storage and dissipation is formalized by Eq. (11).

When the independent variables are time and space, we can rewrite (11) as

$$\frac{\partial \mathbf{U}(\ell)}{\partial t} = Q_\Phi\left(M\left(\frac{d}{d\mathbf{x}}\right)\ell\right) - \nabla \cdot \mathbf{S}(\ell) - Q_\Delta(\ell), \tag{12}$$

where we substitute $Q_\Psi = (\mathbf{U}, \mathbf{S})$, with $\mathbf{U} = \Psi_t$ the stored energy and $\mathbf{S} = (\Psi_x, \Psi_y, \Psi_z)$ the flux. Moreover $w = M(\frac{d}{d\mathbf{x}})\ell$. The above equation is reminiscent of energy balance equations that appear in several fields like fluid mechanics, thermodynamics, etc. Thus (12) states that the change in the stored energy ($\frac{\partial \mathbf{U}(\ell)}{\partial t}$) in an infinitesimal volume is exactly equal to the difference between the energy supplied

$(Q_\Phi(w))$ into the infinitesimal volume and the energy lost by the infinitesimal volume by means of energy flux flowing out of the volume $(\nabla \cdot \mathbf{S}(\ell))$ and the energy dissipated $(Q_\Delta(\ell))$ within the volume.

The problem we now address is the equivalence of (i) dissipativeness of $\mathfrak{B}$ with respect to Q_Φ, (ii) the existence of a storage function and (iii) the existence of a dissipation rate. Note that this problem also involves the construction of an appropriate image representation. We first consider the case where $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$. In this case, the definition of the dissipation rate requires that for all $\ell \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^\ell)$

$$\int_{\mathbb{R}^n} Q_\Phi(w)\, d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Delta(\ell)\, d\mathbf{x} \tag{13}$$

with $w = M(\frac{d}{d\mathbf{x}})\ell$; $M(\frac{d}{d\mathbf{x}})$ a surjective partial differential operator and $Q_\Delta(\ell) \geqslant 0$ for all $\ell \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^\ell)$. By stacking the variables and their various derivatives to form a new vector of variables, this latter condition can be shown to be equivalent to the existence of a polynomial matrix $D \in \mathbb{R}^{\bullet \times \ell}[\xi]$ such that $\Delta(\zeta, \eta) = D^{\mathrm{T}}(\zeta) D(\eta)$. Using Theorem 5, it follows that (13) is equivalent to the factorization equation

$$M^{\mathrm{T}}(-\xi)\Phi(-\xi, \xi)M(\xi) = D^{\mathrm{T}}(-\xi)D(\xi). \tag{14}$$

A very well known problem in 1-D systems is that of *spectral factorization* which involves the factorization of a matrix $\Gamma(\xi) \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\xi]$ into the form

$$\Gamma(\xi) = F^{\mathrm{T}}(-\xi)F(\xi)$$

with $F \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\xi]$ (the matrix F may have to satisfy some additional conditions like being Hurwitz). It is well known that a polynomial matrix $\Gamma(\xi)$ in one variable ξ admits a solution $F \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\xi]$ if and only if $\Gamma^{\mathrm{T}}(-\xi) = \Gamma(\xi)$ and $\Gamma(i\omega) \geqslant 0$ for all $\omega \in \mathbb{R}$. The above factorization problem for n-D systems (14) is very similar in flavor. We can reformulate the problem as follows: given $\Gamma \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\xi]$, a polynomial matrix in n commuting variables $\xi = (\xi_1, \ldots, \xi_n)$, is it possible to factorize it as

$$\Gamma(\xi) = F^{\mathrm{T}}(-\xi)F(\xi) \tag{15}$$

with $F \in \mathbb{R}^{\bullet\times\mathrm{w}}[\xi]$ itself a polynomial matrix. Quite clearly, $\Gamma^{\mathrm{T}}(-\xi) = \Gamma(\xi)$ and $\Gamma(i\omega) \geqslant 0$ for all $\omega \in \mathbb{R}^n$ are necessary conditions for the existence of a factor $F \in \mathbb{R}^{\bullet\times\mathrm{w}}[\xi]$. The important question is whether these conditions are also sufficient (as in the 1-D case).

If we consider the case when $\mathrm{w} = 1$ (the scalar case), substituting $i\omega$ for ξ, (15) reduces to finding F such that

$$\Gamma(i\omega) = F^{\mathrm{T}}(-i\omega)F(i\omega).$$

Separating the real and imaginary parts of the above equation, the problem further reduces to the case of finding a sum of "two" squares which add up to a given positive (or non-negative) polynomial.

This turns out to be a problem with a long history. It is Hilbert's 17th problem, which deals with the representation of positive definite functions as sums of squares [2]. This investigation of positive definite functions began in the year 1888 with the following "negative" result of Hilbert: If $f(\xi) \in \mathbb{R}[\xi]$ is a positive definite polynomial in n variables, then f need not be a sum of squares of polynomials in $\mathbb{R}[\xi]$, except in the case when $n = 1$. Several examples of such positive definite polynomials which cannot be expressed as sum of squares of polynomials are available in the literature, for example, the polynomial

$$\xi_1^2\xi_2^2\big(\xi_1^2 + \xi_2^2 - 1\big) + 1$$

is not factorizable as a sum of squares of polynomials [1].

Thus the two conditions that we mentioned earlier (namely $\Gamma^{\mathrm{T}}(-\xi) = \Gamma(\xi)$ and $\Gamma(i\omega) \geqslant 0$ for all $\omega \in \mathbb{R}^n$) are not sufficient to guarantee a polynomial factor $F \in \mathbb{R}^{\bullet\times \mathrm{w}}$ (even for the scalar case). However, we have the following result.

Theorem 7. *Assume that* $\Gamma \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\xi]$ *satisfies* $\Gamma^{\mathrm{T}}(-\xi) = \Gamma(\xi)$ *and* $\Gamma(i\omega) \geqslant 0$ *for all* $\omega \in \mathbb{R}^n$. *Then there exists an* $F \in \mathbb{R}^{\bullet\times\mathrm{w}}(\xi)$ *such that* $\Gamma(\xi) = F^{\mathrm{T}}(-\xi)F(\xi)$.

Note that even when Γ is a polynomial matrix, the entries of the matrix F are rational functions in n-indeterminates with real coefficients, whereas for the 1-D case one can obtain an F with polynomial entries. Combining the result of Theorem 7 along with the factorization problem (14), we obtain the following theorem.

Theorem 8. *Let* $\Phi = \Phi^* \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta, \eta]$. *Then the following conditions are equivalent*:

(1) $\int_{\mathbb{R}^n} Q_\Phi(w)\, d\mathbf{x} \geqslant 0$ *for all* $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$.
(2) *there exists a polynomial matrix* $M \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\xi]$ *such that* $M(\frac{d}{d\mathbf{x}})$ *is surjective and* $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ *with* $\Psi_k = \Psi_k^* \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta, \eta]$ *for* $k = 1, 2, \ldots, n$ *such that the VQDF* Q_Ψ *is a storage function, i.e.,*

$$\operatorname{div} Q_\Psi(\ell) \leqslant Q_\Phi(w)$$

for all $\ell \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$ *and* $w = M(\frac{d}{d\mathbf{x}})\ell$.
(3) *There exists a polynomial matrix* $M \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\xi]$ *such that* $M(\frac{d}{d\mathbf{x}})$ *is surjective and a* $\Delta = \Delta^* \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta, \eta]$ *such that* Q_Δ *is a dissipation rate, i.e.,*

$$Q_\Delta \geqslant 0 \quad \text{and} \quad \int_{\mathbb{R}^n} Q_\Delta(\ell)\, d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(w)\, d\mathbf{x}$$

for all $\ell \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathrm{w}})$ *and* $w = M(\frac{d}{d\mathbf{x}})\ell$.

(4) *There exists a polynomial matrix* $M \in \mathbb{R}^{w\times w}[\xi]$ *such that* $M(\frac{d}{d\mathbf{x}})$ *is surjective, a* $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ *with* $\Psi_k = \Psi_k^* \in \mathbb{R}^{w\times w}[\zeta, \eta]$ *for* $k = 1, 2, \ldots, n$ *and a* $\Delta = \Delta^* \in \mathbb{R}^{w\times w}[\zeta, \eta]$ *such that*

$$Q_\Delta \geqslant 0$$

and

$$\text{div } Q_\Psi(\ell) = Q_\Phi(w) - Q_\Delta(\ell) \tag{16}$$

for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$ *and* $w = M(\frac{d}{d\mathbf{x}})\ell$. *Note that this states that the VQDF* Q_Ψ *is a storage function and that* Q_Δ *is a dissipation rate.*

The above theorem considers the case when $\mathfrak{B}$ is all of $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$ and it shows the equivalence of dissipativeness of $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$ with respect to Q_Φ, the existence of a storage function (Q_Ψ) and the existence of a dissipation rate (Q_Δ). The important message of this theorem is the unavoidable emergence of latent variables in the dissipation equation (16) for n-D systems. Also note that the storage and dissipation functions that one obtains using the above theorem are not unique.

Finally, for an arbitrary controllable n-D behavior $\mathfrak{B} \in \mathfrak{L}^w_{n,\text{cont}}$, the above theorem can be modified to obtain the following.

Theorem 9. *Let* $\mathfrak{B} \in \mathfrak{L}^w_{n,\text{cont}}$ *and* $\Phi = \Phi^* \in \mathbb{R}^{w\times w}[\zeta, \eta]$. *The following conditions are equivalent*:

(1) $\mathfrak{B}$ *is* Φ*-dissipative, i.e.,* $\int_{\mathbb{R}^n} Q_\Phi(w)\, d\mathbf{x} \geqslant 0$ *for all* $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^w)$,
(2) *there exists an integer* $l \in \mathbb{N}$, *a polynomial matrix* $M \in \mathbb{R}^{w\times l}[\xi]$ *such that* $M(\frac{d}{d\mathbf{x}})$ *is an image representation of* $\mathfrak{B}$, *a* $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ *with* $\Psi_k = \Psi_k^* \in \mathbb{R}^{l\times l}[\zeta, \eta]$ *for* $k = 1, 2, \ldots, n$ *and a* $\Delta = \Delta^* \in \mathbb{R}^{l\times l}[\zeta, \eta]$ *such that*

$$Q_\Delta \geqslant 0$$

and

$$\text{div } Q_\Psi(\ell) = Q_\Phi(w) - Q_\Delta(\ell) \tag{17}$$

with $w = M(\frac{d}{d\mathbf{x}})\ell$.

6. Conclusions

In this chapter, we dealt with n-D systems described by constant coefficient linear partial differential equations. We started by defining controllability for such systems, in terms of patching up of feasible trajectories. We then explained that it is exactly the controllable systems which allow an image representation, i.e., a representation in terms of what in physics is called a potential function. Subsequently, we turned to lossless and dissipative systems.

For lossless systems, we proved the equivalence with the existence of a conservation law involving the storage function. Important features of the storage function are (i) the fact that it depends on latent variables that are in general hidden (i.e., non-observable), and (ii) its non-uniqueness.

For dissipative systems, we proved the equivalence with the existence a storage function and a dissipation rate. The problem of constructing a dissipation rate led to the question of factorizability of certain polynomial matrices in n variables. We reduced this problem to Hilbert's 17th problem, the representation of a non-negative rational function in n variables as a sum of squares of rational functions.

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