Simplified Waterfilling for Power Allocation in MIMO-DSL

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Abstract

In the newest generation of DSL systems crosstalk is *the* dominant source of performance degradation. Many crosstalk cancellation schemes have been proposed. These schemes typically employ some form of co-ordination between modems and lead to large performance gains. The use of crosstalk cancellation means that power allocation should be viewed as a multi-user problem. In this contribution we investigate optimal (ie. capacity maximizing) power allocation in DSL systems which employ co-ordination to facilitate crosstalk cancellation. We also describe the optimal transmitter and receiver structures which allow capacity to be achieved with low complexity.

By exploiting certain properties of the DSL channel it is shown that power allocation can be simplified considerably. The result has each user waterfilling against the background noise only, explicitly ignoring the interference from other users. We show this to be near-optimal for upstream DSL when Central Office (CO) modems are co-ordinated and for bonded-DSL systems where co-ordination is used on both ends of the link. Compared with conventional waterfilling which is done against the background noise *and* interference, the performance gains are significant.

Index Terms

MIMO systems, DSL, Multi-user channels, Bonded DSL, Crosstalk Cancellation

I. Introduction

xDSL systems such as ADSL and VDSL offer the potential to bring truly broadband access to the mass-consumer market. The newer generations of xDSL such as VDSL aim at providing data rates up to 52 Mbps in the downstream, enabling a broad range of applications such as video-on-demand, video-conferencing and online education. In VDSL such high data-rates are supported by operating over short loop lengths and transmitting in frequencies up to 12 MHz.

Unfortunately, the use of such high frequency ranges can cause significant electromagnetic coupling between neighbouring twisted-pairs within a binder group. This coupling creates interference, referred to as crosstalk, between the systems operating within a binder. Over short loop lengths crosstalk is typically 10-15 dB larger than the background noise and is *the* dominant source of performance degradation.

Many techniques have been proposed for crosstalk cancellation in DSL [1], [2], [3], [4]. In particular, if Discrete Multi-Tone (DMT) modulation is used, then synchronized transmission allows crosstalk to be canceled on a per-tone basis[1]. This leads to significant performance gains with a realisable complexity.

Another benefit of DMT is that it allows shaping of the transmit spectra, also known as waterfilling to be implemented in a straightforward manner. In highly non-flat channels, like those seen on the twisted-pair medium, waterfilling leads to significant data-rate gains. Waterfilling is traditionally viewed as a single user problem with each user allocating power according to the Channel-SINR (C-SINR). That is, each user's transmit Power Spectral Density (PSD) is found by a waterfilling against the background noise and interference of other systems[5]. When crosstalk cancellation is employed however optimal power allocation requires us to examine the multi-user aspect of the DSL channel. This is the focus of this paper.

In this paper we describe optimal (ie. capacity maximizing) power allocations for the DSL channel. We also examine optimal transmitter and receiver structures which then allow capacity to be achieved with a reasonable complexity.

In particular we investigate power allocation when co-ordination is possible between:

- Receivers Only: This corresponds to the upstream channel where Central Office (CO) modems can use co-ordination to filter out crosstalk.
- Transmitters and Receivers: This corresponds to a bonded-DSL system. In bonded-DSL several lines are multiplexed to form a single high-speed connection. Bonded-DSL is targeted at CO to Remote Terminal (RT) connections and as a competitor to Fiber-To-The-Home (FTTH) services.

As we will show, exploiting certain properties of the DSL channel allows us to simplify the power allocation problem significantly. The result is that each user waterfills against the background noise alone, explicitly ignoring crosstalk from other users.

This property has been noted previously[1] where it was shown that waterfilling against the background noise alone is optimal for a particular receiver structure, namely the Zero Forcing-Decision Feedback Equalizer (ZF-DFE) in upstream communication. Here we show that such a waterfilling scheme is optimal (to within a reasonable approximation for DSL channels) in an information theoretic sense. That is, it maximizes the capacity of the DSL Multi-Access Channel (MAC) when an optimal receiver structure is used. We also extend this result to the bonded-DSL channel where co-ordination is available between receivers *and* transmitters.

Notation

$[\mathbf{a}]_i$: the i'th element of the vector $\mathbf{a}$

$[\mathbf{A}]_{i,j}$: the element on the i'th row and j'th column of $\mathbf{A}$

$[\mathbf{A}]_{\mathrm{row}\,i}$: the i'th row of $\mathbf{A}$

$[\mathbf{A}]_{\mathrm{col}\,i}$: the i'th column of $\mathbf{A}$

$\mathrm{diag}\,\{\mathbf{a}\}$: the diagonal matrix with the vector $\mathbf{a}$ on its diagonal.

$\mathrm{diag}\,\{\mathbf{A}\}$: the matrix $\mathbf{A}$ with its off-diagonal elements set to zero.

$\mathrm{offdiag}\,\{\mathbf{A}\}$: the matrix $\mathbf{A}$ with its diagonal elements set to zero.

$\overline{\mathbf{A}}^{i,j} \triangleq [\mathbf{A}]_{\mathrm{row}\,1:i-1\,i+1:N,\,\mathrm{col}\,1:j-1\,j+1:N}$: the sub-matrix formed by removing row i and column j from $\mathbf{A}$.

$\mathbf{I}_N$: the $N \times N$ identity matrix

$\mathbf{e}_i \triangleq [\mathbf{I}_N]_{\mathrm{col}\,i}$

II. The DSL Channel

A. DMT modulation

In this work we restrict our attention to DSL systems which employ Discrete Multi-Tone (DMT) modulation. This modulation scheme is currently adopted in the ADSL standards[6] as well as the draft VDSL standards[7],

[8]. DMT is effectively a low-complexity implementation of frequency domain transmission. The main benefits of frequency domain transmission come from bitloading and powerloading:

Bitloading allows a DSL modem to dynamically vary the constellation used on a per-tone basis. The constellation employed depends on the SNR at the receiver. Through rate-adaption, the modem can keep the probability of error at a constant value. Furthermore, it can allocate large constellations to tones with high SNRs, ensuring efficient use of the channel.

Powerloading allows the modem to vary the power transmitted at each tone. Through this the modem can strike the optimal balance between transmitting on the tones with the highest SNR and maximizing the transmission bandwidth. Due to the highly non-flat nature of the DSL channel, powerloading yields significant benefit.

B. Crosstalk

To support such high data rates DSL systems use a broad range of frequencies. This is particularly true in the newer DSL generations such as VDSL where the transmission spectrum ranges up to 12 MHz. The use of such high frequencies over a medium originally designed for voice-band (< 4 kHz) communication leads to its own problems. At these high frequencies the signals of neighbouring wires within a binder group can leak into each other as a result of electro-magnetic coupling. This effect, known as crosstalk, is *the* dominant source of performance degradation in VDSL.

Many techniques have been proposed to deal with crosstalk [1], [2], [3], [4]. In particular, so-called Far-End Crosstalk (FEXT) (ie. crosstalk from modems transmitting in the same direction) may be cancelled on a per-tone basis if the modems within a binder are synchronized[1]. This leads to dramatic improvements in performance with reasonable complexity. We thus adopt a channel model which describes crosstalk on a per-tone basis. Transmission of one DMT-block on tone k is modeled as

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{z}_k \tag{1}$$

1) Co-ordinated Receivers: In upstream communications the CO receivers are often co-located which facilitates co-ordinated (ie. joint) reception. In the upstream direction $\mathbf{x}_k$ is the set of QAM-symbols transmitted by each of the Customer Premises (CP) modems on tone k where $x_k^n \triangleq [\mathbf{x}_k]_n$ is the symbol transmitted by modem n. $\mathbf{y}_k$ is the set of received signals on each of the CO modems where $y_k^n \triangleq [\mathbf{y}_k]_n$ is the signal received on modem n. $\mathbf{H}_k$ is the channel matrix where $h_k^{n,m} \triangleq [\mathbf{H}_k]_{n,m}$ is the channel from CP transmitter m into CO receiver n. Note that $h_k^{n,n}$ is the direct channel of user n. The transmit auto-correlation on tone k is $\mathbf{S}_k \triangleq \mathcal{E}\left\{\mathbf{x}_k \mathbf{x}_k^H\right\}$ whose elements are defined $s_k^{n,m} \triangleq [\mathbf{S}_k]_{n,m}$. For convenience we also define $s_k^n \triangleq [\mathbf{S}_k]_{n,n}$

The receivers suffer from additive noise $\mathbf{z}_k$ from sources such as alien crosstalk, RFI and thermal noise. $z_k^n \triangleq [\mathbf{z}_k]_n$ is the noise seen at receiver n which we assume to be Gaussian. There are N users in the binder so $\mathbf{x}_k$, $\mathbf{y}_k$ and $\mathbf{z}_k$ are all vectors of length N, whilst $\mathbf{H}_k$ is a matrix of dimension $N \times N$.

2) Co-ordinated Transmitters and Receivers: We also consider bonded-DSL systems in which several DSL lines are multiplexed to form a single high-speed connection. In these systems downstream (upstream) transmitters are co-located at the CO (CP) *and* several downstream (upstream) receivers are co-located at the CP (CO). This facilitates

co-ordinated transmission *and* reception. In bonded-DSL we use the same channel model (1) however this model can be applied to both upstream and downstream communication.

In this paper we restrict our attention to the AWGN channel where $\mathcal{E}\left\{\mathbf{z}_k\mathbf{z}_k^H\right\} = \sigma^2_{\text{noise}}(k)\mathbf{I}_N$. Note that this is without loss of generality since in the scenarios we consider co-ordination is always possible between receivers. As such, any channel with a noise covariance matrix $\sigma^2_{\text{noise}}(k)\mathbf{F}_k$ can be turned into an equivalent AWGN channel by application of a noise-whitening filter at the receiver $\mathbf{G}_k^{-H}$. $\mathbf{G}_k$ is related to $\mathbf{F}_k$ through the Cholesky decomposition, ie. $\mathbf{G}_k^H\mathbf{G}_k \overset{\text{chol}}{=} \mathbf{F}_k$.

One peculiar property of the DSL channel is that the channel from transmitter n to receiver n will always have a much larger magnitude than the channel from transmitter n to any other receiver. The difference is typically on the order of 15 dB. We refer to this property as *column-wise diagonal dominance* as in [1]. It ensures that a diagonal element of the channel matrix $\mathbf{H}_k$ will always be the largest element of it's column.

$$|h_k^{n,n}| \gg |h_k^{m,n}|, \ \forall m \neq n \tag{2}$$

This property will allow us to simplify power allocation considerably.

C. Power Constraints

The power constraint for DSL systems is on each transmitter (modem) rather than on the total power of all transmitters. Thus the constraints in power allocation are

$$\sum_{k=1}^{K} s_k^n \leq P, \ \forall n \tag{3}$$

where P is typically determined by the analog front end of a modem or by standardization/regulatory bodies. We also have the natural constraint

$$s_k^n \geq 0, \ \forall n, k \tag{4}$$

III. Power Allocation with No Co-ordination

In conventional DSL systems co-ordination is not possible between transmitters or receivers. The lack of co-ordination, and thus crosstalk cancellation is reflected in the power allocation strategies which are traditionally adopted. In the absence of crosstalk cancellation the DSL channel is a so-called 'Interference Channel' from the Information theory perspective. The achievable rate of each user is

$$\begin{aligned} C_n &= \sum_{k=1}^{K} I(x_k^n; y_k^n) \\ &= \sum_{k=1}^{K} \log_2\left(1 + \frac{|h_k^{n,n}|^2 s_k^n}{\sum_{m\neq n} s_k^m |h_k^{n,m}|^2 + \sigma^2_{\text{noise}}(k)\left[\mathbf{F}_k\right]_{n,n}}\right) \end{aligned}$$

where $I(a;b)$ is the mutual information between a and b. Each user is detected in the presence of background noise $\sigma^2_{\text{noise}}(k)\left[\mathbf{F}_k\right]_{n,n}$ and interference from other users $\sum_{m\neq n} s_k^m |h_k^{n,m}|^2$. The term $\left[\mathbf{F}_k\right]_{n,n}$ is present since the lack of receiver co-ordination prevents noise-whitening. Operating at the capacity of an interference channel corresponds to

maximizing a weighted sum of the different users' rates. The weights used reflect the desired trade-off between the data-rates of the different users within the system. The optimal power allocation can found through an optimisation

$$\max_{\{\mathbf{S}_k\}_{k=1,\ldots,K}} \sum_{n=1}^{N} w_n C_n \tag{5}$$

Unfortunately this optimization is non-convex. Due to the high dimensionality of the solution space (e.g. in VDSL $K = 4096$) this problem is computationally intractable.

For this reason power allocation in conventional DSL systems has typically been based upon heuristic approaches. The most common approach is for each user to allocate power independently, waterfilling against the background noise and the interference of the other users within the system[5]. In such systems the power allocation for user n is defined as

$$s_k^n = \left[\frac{1}{\lambda_n} - \frac{\sigma^2_{\text{noise}}(k) + \sum_{m \neq n} s_k^m \left| h_k^{n,m} \right|^2}{\left| h_k^{n,n} \right|^2} \right]^+ \tag{6}$$

where the function $[x]^+ \triangleq \max(0, x)$. Here λ_n is chosen such that (3) is met with equality. Here each user waterfills against the ratio of the noise plus interference term $\sigma^2_{\text{noise}}(k) + \sum_{m \neq n} s_k^m \left| h_k^{n,m} \right|^2$ to the channel gain $\left| h_k^{n,n} \right|^2$. Put another way, each user waterfills against the inverse channel-SINR.

A modified version of this approach was proposed in [9] where the total power constraint (eq. (3)) of each user is varied based on their target data-rate. Waterfilling is done for each user in turn, and iterated across all users until convergence. The algorithm, referred to as 'iterative waterfilling' is based on the proposition that with each user acting in a selfish way; attempting to maximize their own data-rate, the algorithm will converge to a point which is near-optimal from a global perspective, that is in terms of maximizing (5).

Note that Eq. (6), which from now on will be referred to as *conventional waterfilling*, is based on the intrinsic assumption that crosstalk cancellation will *not* be used. Each user is encouraged to allocate power in the regions of the channel where interference is low. When crosstalk cancellation is used a different approach will be necessary.

We now investigate optimal power allocation in two scenarios. In the first scenario co-ordination is available between receivers only. This corresponds to upstream communication where CO modems can use co-ordination to filter out crosstalk. In the second scenario co-ordination is available between both transmitters and receivers. This corresponds to a bonded-DSL system which we describe more in Section V.

IV. Power Allocation with Co-ordinated Receivers

In this section we examine the case where co-ordination is possible between receivers only. This corresponds to the upstream channel where receivers at the CO are co-ordinated and exploit this co-ordination to remove crosstalk from one another's signals.

A. Optimal Power Allocation

In information theory when co-ordination is available between receivers the channel is known as a Multi-Access Channel (MAC). We concern ourselves with maximizing the unweighted rate-sum of the system. In general finding

all optimal operating points requires us to optimize a weighted rate-sum and this is the subject of ongoing research. We have however observed that in DSL channels with crosstalk cancellation varying the weights typically has little effect on the resultant data rates.

Provided an optimal receiver structure is used the achievable rate sum can be shown to be[10]

$$\begin{aligned} C &= \sum_{n=1}^{N}\sum_{k=1}^{K} I\left(x_k^n; \mathbf{y}_k \,|\, x_k^1, \ldots, x_k^{n-1}\right) \\ &= \sum_{k=1}^{K} \log_2 \left|\mathbf{I}_N + \sigma_{\text{noise}}^{-2}(k)\, \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H\right| \end{aligned} \tag{7}$$

where $I(a; b \,|\, c)$ is the mutual information between a and b conditioned on c. The goal is to maximize C as a function of $\{\mathbf{S}_k\}_{k=1\ldots K}$. This optimisation must be done under a total power constraint on each modem (3), plus the non-negativity constraint (4). Since co-ordination is not possible between transmitters we have an additional constraint

$$s_k^{n,m} = 0, \ \forall m \neq n \tag{8}$$

This problem was addressed in [10] where the optimal power allocation was shown to be a vector form of waterfilling which must occur simultaneously for all users within the system.

Theorem 1: Optimal Power Allocation with Co-ordinated Receivers

The optimal power allocation is

$$s_k^n = \left[\frac{1}{\lambda_n} - \frac{1}{\mathbf{h}_k^{n\,H}\left(\sigma_{\text{noise}}^2(k)\mathbf{I}_N + \sum_{m\neq n} s_k^m \mathbf{h}_k^m\, \mathbf{h}_k^{m\,H}\right)^{-1}\mathbf{h}_k^n}\right]^+ \tag{9}$$

where $\mathbf{h}_k^n \triangleq [\mathbf{H}_k]_{\text{col}\, n}$ and $\{\lambda_1 \ldots \lambda_N\}$ are chosen such that the power constraints in (3) are met with equality.

Proof: See [10]. ■

No closed form solution is known for (9) although a cheap iterative algorithm has been proposed which has guaranteed convergence[11]. Whilst this algorithm allows us to find the optimum power allocation in an efficient way, we can exploit the properties of the DSL channel, specifically column-wise diagonal dominance (2) to simplify power allocation even further.

Simplification 1: Optimal Power Allocation with Co-ordinated Receivers in DSL

Under the condition of column-wise diagonal dominance (2) and high SNR, the optimal power allocation is closely approximated by

$$s_k^n = \left[\frac{1}{\lambda_n} - \frac{\sigma_{\text{noise}}^2(k)}{|h_k^{n,n}|^2}\right]^+ \tag{10}$$

where $\{\lambda_1 \ldots \lambda_N\}$ are chosen such that the power constraints in (3) are met with equality.

Proof: See Appendix I. ■

Using the power allocation strategy in (10) each user's PSD can be determined independently, considerably reducing complexity. In contrast to the conventional waterfilling of (6) each user waterfills against their own direct channel and the background noise as if interference is not present, ie. they waterfill against the inverse channel-SNR not

the channel-SINR. This is intuitively satisfying since the high SNR and column-wise diagonal dominance of the DSL channel facilitate near-perfect crosstalk cancellation.

In contrast to (6), (10) allows power allocation to be done with much lower complexity since power allocation of the different users is de-coupled through the receiver co-ordination (ie. crosstalk cancellation).

B. Optimal Receiver Structure

With this power allocation, a low complexity DFE based receiver structure can be used to achieve the full capacity of the channel. Note that the conditioning of the mutual information in (7) on the previous user's symbols $x_k^1, \ldots, x_k^{n-1}$ reflects the successive interference cancellation nature of the optimal receiver structure. See [12] for more details.

V. Power Allocation with Co-ordinated Transmitters and Receivers

We now turn our attention to scenarios where co-ordination is available between transmitters and receivers. This scenario is encountered in bonded-DSL systems which consist of several DSL lines which are multiplexed to form a single high-speed connection. Bonded-DSL is currently targeted at CO to RT connections and as a competitor to FTTH.

In bonded systems we use a combination of pre and post-filtering to remove crosstalk[2], the benefit being that crosstalk is removed without noise-enhancement.

A. Optimal Power Allocation

As in the previous scenario we desire to maximize the total capacity of the bonded system.

$$\begin{aligned} C &= \sum_{k=1}^{K} I(\mathbf{x}_k; \mathbf{y}_k) \\ &= \sum_{k=1}^{K} \log_2 \left| \mathbf{I}_N + \sigma_{\text{noise}}^{-2}(k) \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H \right| \end{aligned} \tag{11}$$

The maximization should be made as a function of $\{\mathbf{S}_k\}_{k=1\ldots K}$ under the power (3) and non-negativity (4) constraints. In contrast to the previous scenario, co-ordination between transmitters allows $s_k^{n,m}$ to be non-zero for any (n, m).

Theorem 2: Optimal Power Allocation with Co-ordinated Transmitters and Receivers

The optimal power allocation is

$$s_k^{n,m} = \begin{cases} \left[\frac{1}{\lambda_n} - t_k^{n,n}\right]^+ & n = m \\ -t_k^{n,m} & n \neq m \end{cases} \tag{12}$$

where $t_k^{n,m} \triangleq \sigma_{\text{noise}}^2(k) \left[\left(\mathbf{H}_k^H \mathbf{H}_k\right)^{-1}\right]_{n,m}$, a form of inverse channel-SINR (C-SINR) for the MIMO channel. $\{\lambda_1 \ldots \lambda_N\}$ are chosen such that the power constraints in (3) are met with equality.

Proof: See Appendix II. ■

Using (12) the off-diagonal elements of the covariance matrix $\mathbf{S}_k$ are known in closed-form. The diagonal elements are found using a conventional waterfilling algorithm with the channel $1/t_k^{n,n}$.

The power allocation in (12) is general and applies to any MIMO channel with a power constraint on each transmitter. Applying the column-wise diagonal dominance property in (2) allows us to simplify power allocation significantly.

Simplification 2: Optimal Power Allocation with Co-ordinated Transmitters and Receivers in DSL

Under the condition of column-wise diagonal dominance (2) the optimal power allocation is closely approximated by

$$s_k^n = \left[\frac{1}{\lambda_n} - \frac{\sigma^2_{\text{noise}}(k)}{|h_k^{n,n}|^2}\right]^+ \tag{13}$$

and $s_k^{n,m} = 0,\ \forall n \neq m$. Here $\{\lambda_1 \dots \lambda_N\}$ are chosen such that the power constraints in (3) are met with equality.

Proof: See Appendix III. ■

Note that (13) is identical to (10) but the high SNR assumption we needed in Simplification 1 is no longer necessary. This occurs because, in contrast to receiver-only co-ordination, bonded transmission allows crosstalk to be removed *without* noise-enhancement. This is the principal advantage of having co-ordination available between transmitters in addition to receivers.

Using (13) we can avoid the matrix inversions necessary in the calculation of $t_k^{n,m}$. This simplifies power allocation considerably. One power allocation is now (near) optimal regardless of the form of co-ordination that is used. So the power allocation algorithm in modems can be designed independently of the implementation of crosstalk cancellation. This is based on the assumption that an optimal crosstalk canceller is employed.

B. Optimal Transmitter and Receiver Structure

We now describe the optimal transmitter and receiver structure which, in combination with the optimal power allocation is used to achieve the capacity of the bonded-channel.

Define $\mathbf{B}_k^H \mathbf{B}_k \overset{\text{chol}}{=} \mathbf{S}_k$ through the Cholesky decomposition. Note that if the simplified waterfilling of (13) is used then $\mathbf{B}_k$ is diagonal with $[\mathbf{B}_k]_{n,n} = \sqrt{s_k^n}$. Define the equivalent channel $\widetilde{\mathbf{H}}_k \triangleq \mathbf{H}_k \mathbf{B}_k^H$ and it's singular value decomposition $\widetilde{\mathbf{H}}_k \triangleq \mathbf{U}_k \Lambda_k \mathbf{V}_k^H$.

Let us begin with a set of normalized, QAM symbols $\widetilde{\mathbf{x}}_k$ which are generated by the encoder at tone k. These are normalized such that $\mathcal{E}\left\{\widetilde{\mathbf{x}}_k \widetilde{\mathbf{x}}_k^H\right\} = \mathbf{I}_N$. Before transmission we apply the pre-coder $\mathbf{P}_k \triangleq \mathbf{B}_k^H \mathbf{V}_k$ to the normalized symbols

$$\mathbf{x}_k = \mathbf{P}_k \widetilde{\mathbf{x}}_k$$

such that the transmitted signal $\mathbf{x}_k$ has the optimal PSD, ie. $\mathcal{E}\left\{\mathbf{x}_k \mathbf{x}_k^H\right\} = \mathbf{S}_k$.

At the receiver we apply the linear crosstalk canceller/equalizer $\mathbf{W}_k \triangleq \Lambda_k^{-1} \mathbf{U}_k^H$. Our estimate of the transmitted symbols is formed

$$\begin{aligned} \widehat{\mathbf{x}}_k &= \mathbf{W}_k \mathbf{y}_k \\ &= \mathbf{W}_k \left(\mathbf{H}_k \mathbf{P}_k \widetilde{\mathbf{x}}_k + \mathbf{z}_k\right) \end{aligned}$$

Proposition 1: Pre-coding with $\mathbf{P}_k$ at the transmitter and filtering with $\mathbf{W}_k$ at the receiver results in the following estimate of $\widetilde{\mathbf{x}}_k$

$$\widehat{\mathbf{x}}_k = \widetilde{\mathbf{x}}_k + \Lambda_k^{-1}\widetilde{\mathbf{z}}_k$$

where $\mathcal{E}\left\{\widetilde{\mathbf{z}}_k\widetilde{\mathbf{z}}_k^H\right\} = \sigma^2_{\text{noise}}(k)\mathbf{I}_N$. When passed through a conventional slicer this transmission/reception structure achieves the theoretical capacity of the channel in (11).

Proof: Straightforward. ■

It is interesting to note here that through the use of simple linear operations on a per-tone basis, and a conventional slicer for each user we can achieve the same capacity as with the optimal, and highly complex Maximum Likelihood (ML) detector.

VI. Performance

A. Co-ordinated Receivers

We now compare the performance of conventional waterfilling (6) and simplified waterfilling (10) against the truly optimal power allocation scheme (9) for the upstream channel with co-ordinated reception.

Our simulation scenario uses the 998 FDD bandplan, ETSI alien noise model A, a coding gain of 3 dB, a noise margin of 6 dB and a total power constraint of 11.5 dBmW on each modem. The target error probability is $< 10^{-7}$ and all lines are 0.5 mm (24-Gauge). Empirical transfer functions are used, details can be found in [7]. Our scenario consists of 4 near-end and 4 far-end users located 300m. and 1200m. from the CO respectively.

Finding the power allocation for conventional waterfilling (6) was done using 'iterative waterfilling' as described in [9] with all users set to full power. Each user waterfills against the interference of the other users in the system. The process is repeated iteratively until convergence. This reflects what would actually occur in a real scenario as the users adapt their power allocations over time. Finding the power allocation using our simplified waterfilling scheme is done using a standard waterfilling algorithm which is done independently for each user as described by (10). The optimal power allocation (9) was found efficiently using an iterative scheme[11].

The PSDs resulting from the different algorithms are shown in Fig. 1. Note that the PSDs of the *near-end* users are identical for all of the schemes. This occurs because the near-end users have high-SINR channels. The result is a flat transmit PSD since for any of the definitions of s_k^n in (6), (9) or (10)

$$\lim_{\text{SINR}\to\infty} s_k^n = \frac{1}{\lambda_n}$$

We now turn our attention to the PSDs of the *far-end* users. First notice that the PSDs found using the optimal and the simplified waterfilling algorithms are virtually identical as predicted (both PSDs overlap in Fig. 1). This was the case for all scenarios we evaluated. Examining the PSD found with conventional waterfilling we see that the introduction of interference into the waterfilling equation in (6) results in a power allocation at lower frequencies. This is logical since crosstalk coupling increases with frequency. As such, the introduction of interference will tend to discourage loading at high frequencies and push the allocated spectra towards DC.

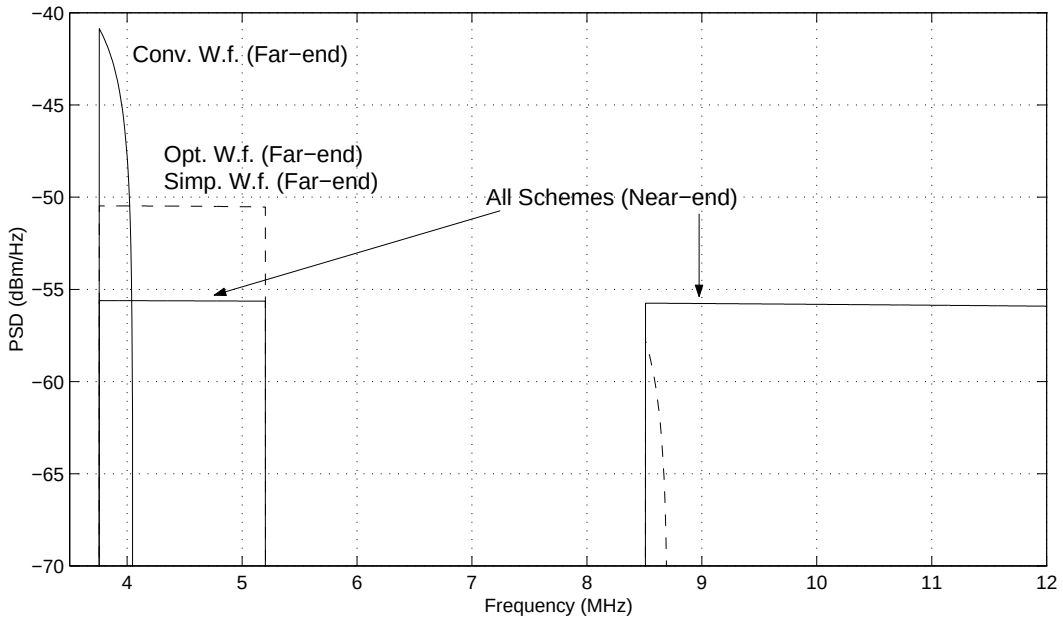


Fig. 1. PSDs of Different Power Allocation Schemes

Scheme	Avg. Far-end Rate	Avg. Near-end Rate
Conv. Waterfilling	2.9 Mbps	59.6 Mbps
Simpl. Waterfilling	10 Mbps	59.6 Mbps
Optimal	10 Mbps	59.6 Mbps

TABLE I

RATES ACHIEVED USING DIFFERENT SCHEMES

To determine the performance of each of the schemes we used these power allocations along with the optimal receiver structure[12] and evaluated the achieved rates. The results are listed in Tab. I. As can be seen, for far-end users conventional waterfilling gives less than 1/3 of the rate achieved using the optimal power allocation. Simplified waterfilling, on the other hand, leads to virtually identical performance to the optimal scheme. Note that crosstalk cancellation was used when evaluating the performance of *all* power allocation schemes including conventional waterfilling.

B. Co-ordinated Transmitters and Receivers

In bonded scenarios all lines have equal length. For this case we found little difference between the various power allocation schemes. Regardless of the scheme used, in the bonded scenario power allocation became primarily dependent on the channel gain with the strength of the interference having little influence. Hence a simple power allocation scheme such as (13) is more attractive than the complex, optimal scheme of (12).

VII. Conclusions

In this paper we investigated optimal power allocation for MIMO channels. We examined 2 cases: when co-ordination is possible between receivers, and when co-ordination is possible between transmitters *and* receivers. We described the optimal transmitter and receiver structures which, in combination with the optimal power allocation allow us to achieve the channel capacity with a reasonable complexity.

We showed that in the DSL environment the property of column-wise diagonal dominance simplifies the problem of power allocation considerably. Whether co-ordination is available between receivers only, or at both ends of the link as in a bonded-DSL system, the optimal power allocation can be well approximated by the same scheme. This scheme consists of a waterfilling against the background noise-only, explicitly ignoring crosstalk. This is intuitively satisfying since the property of column-wise diagonal dominance allows for near-perfect crosstalk cancellation.

With receiver-only co-ordination a high-SNR assumption was required for the near-optimality of the simplified waterfilling scheme. Interestingly, this assumption can be dropped when co-ordination is available between transmitters and receivers. This stems from the fact that, in contrast to receiver-only co-ordination, bonded transmission allows crosstalk to be removed *without* noise-enhancement.

Simulations show minimal performance degradation through the use of the simplified waterfilling scheme. Additionally we noted that power allocation using a conventional waterfilling algorithm (against interference and background noise) leads to poor performance when co-ordination is possible.

In this work we have considered co-ordination between receivers only, and co-ordination between both transmitters and receivers. These correspond to the upstream direction of a DSL system where CO modems are co-ordinated, and a bonded-DSL system respectively. An important extension of this work is to investigate simplified waterfilling schemes when co-ordination is available between transmitters only. This corresponds to the downstream direction of a DSL system where we suspect that the simplified waterfilling algorithm will also be virtually optimal.

Acknowledgments

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Appendix I
Proof of Simplification 1 in Section IV

We begin with the optimal power allocation for the MAC in (9). Define

$$\begin{aligned}\mathbf{Q}_k &\triangleq \sigma^2_{\text{noise}}(k)\mathbf{I}_N + \sum_{m\neq n} s_k^m \mathbf{h}_k^m \, \mathbf{h}_k^{m\,H} \\ &= \sigma^2_{\text{noise}}(k)\mathbf{I}_N + \begin{bmatrix} \overline{\mathbf{h}}_k^1 \\ \vdots \\ \overline{\mathbf{h}}_k^N \end{bmatrix} \begin{bmatrix} \overline{\mathbf{h}}_k^{1\,H} & \cdots & \overline{\mathbf{h}}_k^{N\,H} \end{bmatrix}\end{aligned}$$

where $\overline{\mathbf{h}}_k^i \triangleq \left[\begin{array}{cccccc} h_k^{i,1}\sqrt{s_k^1} & \cdots & h_k^{i,n-1}\sqrt{s_k^{n-1}} & h_k^{i,n+1}\sqrt{s_k^{n+1}} & \cdots & h_k^{i,N}\sqrt{s_k^N} \end{array} \right]$. Using the column-wise diagonal dominance property (2) we can approximate $\mathbf{h}_k^n \simeq \mathbf{e}_n h_k^{n,n}$. Hence

$$s_k^n \simeq \left[\frac{1}{\lambda_n} - \frac{1}{|h_k^{n,n}|^2 \left[\mathbf{Q}_k^{-1}\right]_{n,n}} \right]^+$$

Now $\left[\mathbf{Q}_k^{-1}\right]_{n,n} = \left|\overline{\mathbf{Q}}_k^{n,n}\right| |\mathbf{Q}_k|^{-1}$. Since re-ordering of columns and rows has no effect on the determinant

$$|\mathbf{Q}_k| = \left| \sigma_{\text{noise}}^2(k)\mathbf{I}_N + \left[\begin{array}{c} \overline{\mathbf{h}}_k^n \\ \mathbf{M}^H \end{array} \right] \left[\begin{array}{cc} \overline{\mathbf{h}}_k^{n\,H} & \mathbf{M} \end{array} \right] \right|$$

where $\mathbf{M} \triangleq \left[\begin{array}{cccccc} \overline{\mathbf{h}}_k^{1\,H} & \cdots & \overline{\mathbf{h}}_k^{n-1\,H} & \overline{\mathbf{h}}_k^{n+1\,H} & \cdots & \overline{\mathbf{h}}_k^{N\,H} \end{array} \right]$. Divide $\mathbf{Q}_k$ into sub-matrices

$$|\mathbf{Q}_k| = \left| \begin{array}{cc} a & \mathbf{b}^H \\ \mathbf{c} & \mathbf{D} \end{array} \right|$$

where $a \triangleq \sigma_{\text{noise}}^2(k) + \left\|\overline{\mathbf{h}}_k^n\right\|^2$, $\mathbf{b}^H \triangleq \overline{\mathbf{h}}_k^n \mathbf{M}$, $\mathbf{c} \triangleq \mathbf{M}^H\, \overline{\mathbf{h}}_k^{n\,H}$ and $\mathbf{D} \triangleq \mathbf{M}^H\mathbf{M} + \sigma_{\text{noise}}^2(k)\mathbf{I}_{N-1} = \overline{\mathbf{Q}}_k^{n,n}$. Using the Schur decomposition

$$|\mathbf{Q}_k| = \left|\overline{\mathbf{Q}}_k^{n,n}\right| \left|a - \mathbf{b}^H\mathbf{D}^{-1}\mathbf{c}\right|$$

hence

$$\left[\mathbf{Q}_k^{-1}\right]_{n,n} = \left| \sigma_{\text{noise}}^2(k) + \left\|\overline{\mathbf{h}}_k^n\right\|^2 - \overline{\mathbf{h}}_k^n\mathbf{M}\left(\mathbf{M}^H\mathbf{M} + \sigma_{\text{noise}}^2(k)\mathbf{I}_{N-1}\right)^{-1}\mathbf{M}^H\, \overline{\mathbf{h}}_k^{n\,H} \right|^{-1}$$

Define the singular-value decomposition (SVD) of $\mathbf{M} \stackrel{\text{svd}}{=} \mathbf{U}_M\Lambda_M\mathbf{V}_M^H$. Column-wise diagonal dominance (2) assures us that $\mathbf{M}$ will have full rank hence $\mathbf{U}_M$ and $\mathbf{V}_M$ will be unitary matrices of size $N-1 \times N-1$. Thus

$$\mathbf{M}\left(\mathbf{M}^H\mathbf{M} + \sigma_{\text{noise}}^2(k)\mathbf{I}_{N-1}\right)^{-1}\mathbf{M}^H = \mathbf{U}_M\Lambda_M^2\left(\Lambda_M^2 + \sigma_{\text{noise}}^2(k)\mathbf{I}_{N-1}\right)^{-1}\mathbf{U}_M^H$$

Since the SNR in DSL is high we can approximate $\Lambda_M^2 + \sigma_{\text{noise}}^2(k)\mathbf{I}_{N-1} \simeq \Lambda_M^2$ and

$$\mathbf{M}\left(\mathbf{M}^H\mathbf{M} + \sigma_{\text{noise}}^2(k)\mathbf{I}_{N-1}\right)^{-1}\mathbf{M}^H \simeq \mathbf{I}_{N-1}$$

Hence

$$\left[\mathbf{Q}_k^{-1}\right]_{n,n} \simeq 1/\sigma_{\text{noise}}^2(k)$$

which leads to (10).

APPENDIX II
PROOF OF THEOREM 2 IN SECTION V

We desire to maximize C in (11) as a function of $\{\mathbf{S}_k\}_{k=1,\ldots,K}$ under a total power constraint for each modem (3) and a non-negativity constraint (4). Formulating a Lagrangian

$$J = C + \sum_n \lambda_n \left(P - \sum_k s_k^{n,n} \right) + \sum_n \sum_k \mu_k^n s_k^{n,n}$$

These constraints form a convex set, whilst the capacity function C is concave. As a result, the K.K.T. conditions define an optimal solution:

$$\nabla_{\mathbf{S}_k} J = \mathbf{0},\ \forall k \tag{14}$$

$$\lambda_n \left(P - \sum_k s_k^{n,n} \right) = 0,\ \forall n \tag{15}$$

$$\mu_k^n s_k^{n,n} = 0,\ \forall n,k \tag{16}$$

Using (14)

$$\mathbf{S}_k = \operatorname{diag}\left\{ \left(\lambda_1 - \mu_k^1\right)^{-1}, \ldots, \left(\lambda_N - \mu_k^N\right)^{-1} \right\} - \sigma^2_{\text{noise}}(k) \left(\mathbf{H}_k^H \mathbf{H}_k\right)^{-1} \tag{17}$$

Define $\mathbf{T}_k \triangleq \sigma^2_{\text{noise}}(k) \left(\mathbf{H}_k^H \mathbf{H}_k\right)^{-1}$. First note that $\mathbf{T}_k$ is a positive definite matrix, hence its diagonal elements are positive. Since $\mu_k^n \geq 0$, and $s_k^{n,n} \geq 0$, this implies that $\lambda_n \geq 0$. Now consider two cases

Case 1 $s_k^{n,n} > 0$

Since $s_k^{n,n} > 0$, (16) implies that $\mu_k^n = 0$. Now $\lambda_n \geq 0$, but $\lambda_n = 0$ would result in $s_k^{n,n} \to \infty$ and this clearly violates the power constraint (3). Hence $\lambda_n > 0$. Using (15) this implies that $\sum_k s_k^{n,n} = P$. That is, line n transmits at full power.

Case 2 $s_k^{n,n} = 0$

If $s_k^{n,n} = 0$, (17) implies that $\mu_k^n = \lambda_n - (t_k^{n,n})^{-1}$ where $t_k^{n,m} \triangleq [\mathbf{T}_k]_{n,m}$. Since μ_k^n and $t_k^{n,n}$ are both positive this implies that $\lambda_n > 0$. So again line n transmits at full power. Since $\mu_k^n \geq 0$, this can only occur when $\frac{1}{\lambda_n} - t_k^{n,n} \leq 0$.

Combining both cases we find

$$\mathbf{S}_k = \left[\operatorname{diag}\left\{ (\lambda_1)^{-1}, \ldots, (\lambda_N)^{-1} \right\} - \operatorname{diag}\left\{\mathbf{T}_k\right\} \right]^+ - \operatorname{offdiag}\left\{\mathbf{T}_k\right\}$$

Here $\{\lambda_1 \ldots \lambda_N\}$ are chosen such that all lines transmit at full power. This leads to (12).

APPENDIX III
PROOF OF SIMPLIFICATION 2 IN SECTION V

Define $\mathbf{A}_k \triangleq \mathbf{H}_k^H \mathbf{H}_k$. Hence $t_k^{n,m} = \sigma^2_{\text{noise}}(k) \left[\mathbf{A}_k^{-1}\right]_{n,m} = \sigma^2_{\text{noise}}(k) \left|\overline{\mathbf{A}}_k^{n,m}\right| |\mathbf{A}_k|^{-1}$. Now

$$\mathbf{A}_k = \begin{bmatrix} \mathbf{h}_k^{1\,H} \\ \vdots \\ \mathbf{h}_k^{N\,H} \end{bmatrix} \begin{bmatrix} \mathbf{h}_k^1 & \cdots & \mathbf{h}_k^N \end{bmatrix}$$

Using (2) $\mathbf{h}_k^n \simeq \mathbf{e}_n h_k^{n,n}$ and

$$\mathbf{h}_k^{n\,H} \mathbf{h}_k^m \simeq \begin{cases} |h_k^{n,n}|^2 & m = n \\ 0 & m \neq n \end{cases} \tag{18}$$

Hence $\mathbf{A}_k \simeq \operatorname{diag}\left\{ \left|h_k^{1,1}\right|^2, \ldots, \left|h_k^{N,N}\right|^2 \right\}$ and

$$|\mathbf{A}_k| \simeq \prod_m |h_k^{m,m}|^2$$

Turning our attention to $\overline{\mathbf{A}}_k^{n,m}$

$$\overline{\mathbf{A}}_k^{n,m} = \begin{bmatrix} \mathbf{h}_k^{1\,H} \\ \vdots \\ \mathbf{h}_k^{n-1\,H} \\ \mathbf{h}_k^{n+1\,H} \\ \vdots \\ \mathbf{h}_k^{N\,H} \end{bmatrix} \begin{bmatrix} \mathbf{h}_k^1 & \cdots & \mathbf{h}_k^{m-1} & \mathbf{h}_k^{m+1} & \cdots & \mathbf{h}_k^N \end{bmatrix} \tag{19}$$

Consider 3 cases.

Case 1 $n = m$: Using (18) and (19)

$$\overline{\mathbf{A}}_k^{n,n} \simeq \operatorname{diag}\left\{ \left|h_k^{1,1}\right|^2, \ldots, \left|h_k^{n-1,n-1}\right|^2 \ \left|h_k^{n+1,n+1}\right|^2, \ldots, \left|h_k^{N,N}\right|^2 \right\}$$

Hence

$$\left|\overline{\mathbf{A}}_k^{n,n}\right| \simeq \prod_{m \neq n} |h_k^{m,m}|^2$$

Case 2 $n < m$: In this case it can be shown that

$$\overline{\mathbf{A}}_k^{n,m} \simeq \operatorname{diag}\left\{ \left|h_k^{1,1}\right|^2, \ldots, \left|h_k^{n-1,n-1}\right|^2 \ \left|h_k^{n+1,n+1}\right|^2, \ldots, \left|h_k^{m-1,m-1}\right|^2, 0, \left|h_k^{m+1,m+1}\right|^2, \ldots, \left|h_k^{N,N}\right|^2 \right\}$$

Hence

$$\left|\overline{\mathbf{A}}_k^{n,m}\right| \simeq 0, \ \forall\, n < m$$

Case 3 $n > m$: Similar to Case 2

$$\left|\overline{\mathbf{A}}_k^{n,m}\right| \simeq 0, \ \forall\, n > m$$

Combining the 3 cases

$$\left[\mathbf{A}_k^{-1}\right]_{n,m} \simeq \begin{cases} 1 \,/\, |h_k^{n,n}|^2 & n = n \\ 0 & n \neq m \end{cases}$$

Recalling $t_k^{n,m} = \sigma^2_{\text{noise}}(k) \left[\mathbf{A}_k^{-1}\right]_{n,m}$, (12) simplifies to (13).

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