A Near-Optimal Linear Crosstalk Canceler for VDSL

Raphael Cendrillon, George Ginis, Etienne Van den Bogaert, Marc Moonen

***Abstract*— Crosstalk is the major source of performance degradation in VDSL. Several crosstalk cancelers have been proposed to address this. Unfortunately they suffer from error propagation, high complexity and long latency. In this paper we present a simple, linear zero forcing (ZF) crosstalk canceler. This design has a low complexity, no latency and does not suffer from error propagation. Furthermore, due to the well conditioned structure of the VDSL channel matrix, the ZF design causes negligible noise enhancement. A lower bound on the performance of the linear ZF canceler is derived. This allows performance to be predicted without explicit knowledge of the crosstalk channels, which simplifies service provisioning considerably. This bound shows that the linear ZF canceler operates close to the single-user bound. So the linear ZF canceler is a low complexity, low latency design with predictable, near-optimal performance. The combination of spectral optimization and crosstalk cancellation is also considered. Spectra optimization in a multi-access channel generally involves a highly complex optimization problem. Since the linear ZF canceler decouples transmission on each line, the spectrum on each modem can be optimized independently, leading to a significant reduction in complexity.**

***Index Terms*— Crosstalk cancellation, diagonal dominance, digital subscriber lines, dynamic spectrum management, linear, reduced complexity, vectoring**

***EDICS*— 3-TDSL**

I. Introduction

Next generation DSL systems such as VDSL aim at providing extremely high data-rates, up to 52 Mbps in the downstream. Such high data-rates are supported by operating over short loop lengths and transmitting in frequencies up to 12 MHz. Unfortunately, the use of such high frequency ranges causes significant electromagnetic coupling between neighbouring twisted pairs within a binder. This coupling creates interference, referred to as crosstalk, between the DSLs within a network. Over short loop lengths crosstalk is typically 10-15 dB larger than the background noise and is *the* dominant source of performance degradation.

In upstream communications the receiving modems are often co-located at the *central office* (CO) or at an *optical network unit* (ONU) located at the end of the street. This allows joint reception of the signals transmitted on the different lines, thereby enabling crosstalk cancellation.

Several crosstalk canceler designs have been proposed. A decision feedback structure was shown to achieve close to the theoretical channel capacity[1] and is described in more detail in Sec. IV. Unfortunately this structure suffers from error propagation. To minimize the effects of error propagation each user's data-stream must be decoded before decisions are fed back. This leads to a high computational complexity and a latency that grows with the number of users in the binder. Binders can contain hundreds of lines. As a result, it is difficult to apply this design in real-time applications such as voice over IP or video conferencing.

Other cancellation techniques use turbo coding principles to facilitate cancellation[2][3] or exploit the cyclostationarity of crosstalk[4][5]. The advantage of these methods is that they do not require signal coordination, and can instead be applied independently on each modem. Unfortunately these techniques are extremely complex and give poor performance when more than one crosstalker exists. Other techniques use joint linear processing at both the transmit and receive side of the link[6][7]. This requires co-location of both CO and *customer premises* (CP) modems, which is typically not possible since different customers are situated at different locations. Furthermore, it has been shown that the theoretical channel capacity is achievable with receiver-side coordination only, so using coordination on both ends of the link does not improve performance[8].

In this paper we present a simple, linear *zero forcing* (ZF) crosstalk canceler. This design has a low complexity, no latency and does not suffer from error propagation. Furthermore since it is based on a ZF criterion it removes all crosstalk. Despite these advantages it is well known that ZF criteria can lead to severe noise enhancement in ill-conditioned channels.

To address this concern, this paper analyzes the performance of the linear ZF canceler in the VDSL environment. It is shown that due to the well conditioned structure of the VDSL channel matrix ZF designs cause negligible noise enhancement. As a result, this simple linear structure achieves near-optimal performance. We develop bounds to show that the linear ZF canceler operates close to the single-user bound in VDSL channels. These bounds allow the performance of the linear

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R. Cendrillon and M. Moonen are with the Katholieke Universiteit Leuven, Kasteelpark Arenberg 10, Leuven-Heverlee 3001, Belgium. Email: {raphael.cendrillon, marc.moonen}@esat.kuleuven.ac.be. George Ginis is with the Broadband Communications Group, Texas Instruments, 2043 Samaritan Drive, San Jose, CA 95124. Email: gginis@ti.com. Etienne Van den Bogaert is with the DSL Research and Innovation Group, Alcatel Bell, Francis Wellesplein 1, Antwerp 2018, Belgium. Email: etienne.van_den_bogaert@alcatel.be

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ZF canceler to be predicted without explicit knowledge of the crosstalk channels, which simplifies service provisioning significantly.

The rest of this paper is organized as follows. The system model for a network of VDSL modems transmitting to a single CO/ONU is given in Sec. II. A property of the upstream VDSL channel, known as *column-wise diagonal dominance* (CWDD), is explored. As described in Sec. III, from an information theoretical perspective, the upstream VDSL channel is a *multi-access channel* (MAC). This allows the single-user bound to be applied to upper bound the capacity of the channel. Sec. IV describes the current state-of-the-art solution, the multi-user *decision feedback canceler* (DFC), and the problems it has with error propagation, high complexity and latency.

To address the problems of the DFC, Sec. V describes a much simpler linear design that has a low complexity, no latency and is free from error propagation. Sec. V uses the CWDD property to formulate a lower bound on the performance of the linear ZF canceler. This bound shows that the linear canceler operates close to the single-user bound. Sec. VI describes power loading algorithms for use with the linear canceler. Existing power loading algorithms for the MAC are extremely complex, having a polynomial complexity in the number of lines and tones. Application of the linear canceler decouples the power allocation problem between lines. As a result the PSD for each line can be found through a low-complexity waterfilling procedure and this simplifies power allocation significantly. Sec. VII compares the performance of the different cancelers based on simulations.

II. System Model

Assuming that the modems are synchronized and *discrete multi-tone* (DMT) modulation is employed we can model transmission independently on each tone

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{z}_k. \tag{1}$$

Synchronization is straight-forward to implement when the receiving modems are co-located, which is the assumption we make here. The vector $\mathbf{x}_k \triangleq \left[x_k^1, \cdots, x_k^N\right]^T$ contains transmitted signals on tone k, where the tone index k lies in the range $1 \ldots K$. There are N lines in the binder and x_k^n is the signal transmitted onto line n at tone k. The vectors $\mathbf{y}_k$ and $\mathbf{z}_k$ have similar structures. The vector $\mathbf{y}_k$ contains the received signals on tone k. The vector $\mathbf{z}_k$ contains the additive noise on tone k and is comprised of thermal noise, alien crosstalk, RFI etc. The $N \times N$ matrix $\mathbf{H}_k$ is the crosstalk channel matrix on tone k. The element $h_k^{n,m} \triangleq [\mathbf{H}_k]_{n,m}$ is the channel from TX m to RX n on tone k. The diagonal elements of $\mathbf{H}_k$ contain the direct-channels whilst the off-diagonal elements contain the crosstalk channels. We denote the transmit PSD of user n on tone k as $s_k^n \triangleq \mathcal{E}\left\{|x_k^n|^2\right\}$.

Since the receiving modems are co-located, the crosstalk signal transmitted from a disturber into a victim must propagate through the full length of the disturber's line. This is depicted in Fig. 1, where CP 1 is the disturber and CO 2 is the victim. The insulation between twisted pairs increases the attenuation. As a result, the crosstalk channel matrix $\mathbf{H}_k$ is CWDD, since on each column of $\mathbf{H}_k$ the diagonal element has the largest magnitude

$$|h_k^{n,m}| \ll |h_k^{m,m}|, \ \forall m \neq n. \tag{2}$$

CWDD implies that the crosstalk channel $h_k^{n,m}$ from a disturber m into a victim n is always weaker than the direct channel of the disturber $h_k^{m,m}$. The degree of CWDD can be characterized with the parameter α_k

$$|h_k^{n,m}| \leq \alpha_k \, |h_k^{m,m}|, \ \forall m \neq n. \tag{3}$$

Note that crosstalk cancellation is based on joint reception. As such it requires the co-location of receiving modems. So in all channels where crosstalk cancellation can be applied the CWDD property holds. CWDD has been verified through extensive measurement campaigns of real binders. In 99% of lines α_k is bounded

$$\alpha_k \leq K_{\text{xf}} \cdot f_k \cdot \sqrt{d_{\text{coupling}}},$$

where $K_{\text{xf}} = -22.5$ dB and f_k is the frequency on tone k in MHz[9]. Here d_{coupling} is the coupling length between the disturber and the victim in kilometers. The coupling length can be upper bounded by the longest line length in the binder. Hence

$$\alpha_k \leq K_{\text{xf}} \cdot f_k \cdot \sqrt{l_{\max}}, \tag{4}$$

where $l_{\max}$ denotes the length of the longest line in the binder. To find a value for α_k that is independent of the particular binder configuration, $l_{\max}$ can be set to 1.2 km, which is the maximum deployment length for VDSL[9][1]. On typical lines α_k is then less than -11.3 dB. The following sections show that CWDD ensures a well-conditioned crosstalk channel matrix. This results in the near-optimality of the linear ZF canceler.

When VDSL modems are distributed from an ONU the noise on each line is typically spatially white and we make this assumption here

$$\mathcal{E}\left\{\mathbf{z}_k \mathbf{z}_k^H\right\} = \sigma_k \mathbf{I}_N. \tag{5}$$

When VDSL modems are distributed from a CO the noise on each line may be correlated due to the presence of strong alien crosstalk. In this case a noise pre-whitening operation must be applied prior to crosstalk cancellation. This noise pre-whitening may destroy the CWDD property of the channel matrix $\mathbf{H}_k$. In this case the linear ZF canceler is no longer optimal, and more complex decision feedback structures must be employed[8]. Nevertheless, most VDSL deployments will occur from the ONU, where the assumption of spatially white noise is valid. The linear ZF canceler developed in this paper then provides a low complexity, low latency, near-optimal design.

[1]Standardization groups are currently considering the deployment of VDSL2 at lengths greater than 1.2 km[10]. However at such distances far-end crosstalk is no longer the dominant source of noise, and the benefits of far-end crosstalk cancellation are reduced considerably.

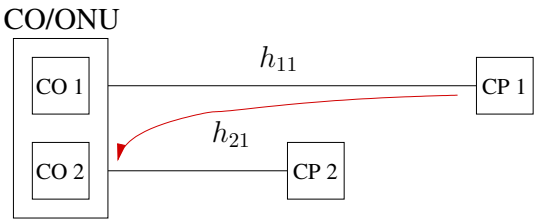

Fig. 1. Column-wise Diagonal Dominance $|h_{11}| \gg |h_{21}|$

III. Theoretical Capacity

We start with a bound on the theoretical capacity of the VDSL channel with coordinated receivers. This will prove useful in evaluating crosstalk canceler performance since it provides an upper bound on the achievable data-rate with any possible crosstalk cancellation scheme.

Theorem 1: The capacity of user n with a fixed transmit spectrum s_k^n is upper bounded by

$$R_n \leq \sum_k b_{k,\text{bnd}}^n(s_k^n) \tag{6}$$

where the data-rate of user n on tone k is upper bounded by

$$b_{k,\text{bnd}}^n(s_k^n) \triangleq \Delta_f \log_2 \left(1 + \Gamma^{-1}\sigma_k^{-1} s_k^n \left|h_k^{n,n}\right|^2 \left[1 + \alpha_k^2 \left(N-1\right)\right]\right) \tag{7}$$

and Δ_f denotes the tone-spacing.

Proof of Theorem 1: CO modems are co-located and do reception in a joint fashion, so from an information theoretical perspective this is a multi-access channel. We start by considering the so-called *single-user bound*, which is the capacity achieved when only one user (CP modem) transmits and all receivers (CO modems) are used to detect that user. Since only one user transmits the received signal at the CO is

$$\mathbf{y}_k = \mathbf{h}_k^n x_k^n + \mathbf{z}_k,$$

where $\mathbf{h}_k^n$ denotes the nth column of $\mathbf{H}_k$. Using the single-user bound the achievable bitloading of user n on tone k is limited to

$$\begin{aligned} b_k^n &\leq \Delta_f I(x_k^n; \mathbf{y}_k), \\ &= \Delta_f \log_2 \left(1 + \sigma_k^{-1} s_k^n \left\|\mathbf{h}_k^n\right\|^2\right), \end{aligned}$$

where $I(a;b)$ denotes the mutual information between a and b and (5) is used in the second line. To account for the sub-optimality of practical coding schemes, we include the SNR-gap to capacity Γ [11]. This results in the following achievable bitloading of user n on tone k.

$$b_k^n = \Delta_f \log_2 \left(1 + \Gamma^{-1}\sigma_k^{-1} s_k^n \left\|\mathbf{h}_k^n\right\|^2\right). \tag{8}$$

In the single-user case with spatially white noise, the single-user bound can be achieved by applying a matched filter to the received vector $\mathbf{y}_k$. The estimate of the transmitted symbol is then

$$\begin{aligned} \hat{x}_k^n &= \left\|\mathbf{h}_k^n\right\|_2^{-2} \left(\mathbf{h}_k^n\right)^H \mathbf{y}_k, \\ &= x_k^n + \left\|\mathbf{h}_k^n\right\|_2^{-2} \left(\mathbf{h}_k^n\right)^H \mathbf{z}_k, \end{aligned}$$

where $(\cdot)^H$ denotes the Hermitian transpose. In the multi-user case, the single-user bound can be achieved by detecting a user last in a *successive interference cancellation* (SIC) structure[12], [8].

The CWDD property (3) leads to the bound

$$\begin{aligned} \left\|\mathbf{h}_k^n\right\|_2^2 &= \left|h_k^{n,n}\right|^2 + \sum_{m \neq n} \left|h_k^{m,n}\right|^2, \\ &\leq \left|h_k^{n,n}\right|^2 \left[1 + \alpha_k^2 \left(N-1\right)\right], \end{aligned} \tag{9}$$

which leads to (6). ∎

Multi-user techniques are often used in wireless systems and lead to large increases in the *signal power* at the receiver. The observation is that if the path from transmit antenna n to receive antenna n is weak, then the path from transmit antenna n to receive antenna m might be strong. The result is a statistical averaging across spatial dimensions, an effect known as spatial diversity, which leads to large improvements in performance[13].

In VDSL channels there is, unfortunately, no equivalent to spatial diversity. This can be seen in equation (9). Here the CWDD of $\mathbf{H}_k$ implies that very little increase can be made in the signal power through the use of multiple VDSL receivers. This is the case since, when receivers are co-located, the crosstalk channel from transmitter n to receiver m is always much weaker than the direct channel from transmitter n to receiver n. Note that the benefit, although small, increases with the crosstalk channel strength α_k and the number of crosstalkers N.

Although spatial diversity is negligible, the use of co-ordinated reception is by no means fruitless. Instead of benefiting through spatial diversity, the primary benefit in VDSL channels is crosstalk cancellation. That is, co-ordinated reception does not increase signal power in VDSL, but instead decreases *interference power*.

IV. Decision Feedback Canceler

Decision feedback equalizers are traditionally used to cancel *inter-symbol interference* (ISI) in frequency selective channels. In a multi-user context the same principle can be applied to remove *inter-user interference*, otherwise known as crosstalk. In this case the decision feedback operates across users rather than time[1].

The structure of the *decision feedback canceler* (DFC) is now described. Consider the QR decomposition of the crosstalk channel matrix

$$\mathbf{H}_k \overset{\text{qr}}{=} \mathbf{Q}_k \mathbf{R}_k, \tag{10}$$

where $\mathbf{Q}_k$ is a unitary matrix and $\mathbf{R}_k$ is upper triangular. The DFC applies the linear feed-forward filter $\mathbf{Q}_k^H$ to the received vector to yield

$$\begin{aligned} \widetilde{\mathbf{y}}_k &= \mathbf{Q}_k^H \mathbf{y}_k, \\ &= \mathbf{R}_k \mathbf{x}_k + \widetilde{\mathbf{z}}_k, \end{aligned} \tag{11}$$

where the filtered noise $\widetilde{\mathbf{z}}_k \triangleq \mathbf{Q}_k^H \mathbf{z}_k$[1]. If the noise is spatially white, as described in (5), then filtering with the unitary matrix

$\mathbf{Q}_k^H$ does not alter the noise statistics

$$\begin{aligned}\mathcal{E}\left\{\widetilde{\mathbf{z}}_k\widetilde{\mathbf{z}}_k^H\right\} &= \mathcal{E}\left\{\mathbf{Q}_k\mathbf{z}_k\mathbf{z}_k^H\mathbf{Q}_k^H\right\},\\ &= \sigma_k\mathbf{I}_N,\\ &= \mathcal{E}\left\{\mathbf{z}_k\mathbf{z}_k^H\right\}.\end{aligned}$$

If the noise is spatially coloured then a noise pre-whitening must be applied prior to the DFC, which leads to a more complex receiver structure[8]. However, in ONU distributed VDSL the assumption of spatially white noise is often a valid one.

From (11) it is clear that the transmission channel has been transformed into an upper triangular channel $\mathbf{R}_k$. This channel is causal in the sense that there is an order in the crosstalk of the users: user N experiences crosstalk from no-one; user N-1 experiences crosstalk only from user N; user N-2 experiences crosstalk only from users N and N-1; and so on.

This causal structure admits the use of decision feedback to remove crosstalk. User N experiences no crosstalk. Hence the signal of user N can be detected directly, and the crosstalk it causes to the other components of $\mathbf{y}_k$ can be removed. At this point user N-1 can be detected free from crosstalk, and the crosstalk it causes to the remaining users can be removed. This procedure iterates until all users have been detected. The estimate for user n is thus formed

$$\hat{x}_k^n = \mathrm{dec}\left[\frac{1}{r_k^{n,n}}\left(y_k^n - \sum_{m=n+1}^{N} r_k^{n,m}\hat{x}_k^m\right)\right],$$

where $r_k^{n,m} \triangleq [\mathbf{R}_k]_{n,m}$ and $\mathrm{dec}[\cdot]$ denotes the decision operation[1]. It is typically assumed that no decisions errors are made

$$\hat{x}_k^m = x_k^m,\ \forall m > n, \tag{12}$$

which leads to the following estimate for the symbol of user n

$$\hat{x}_k^n = \mathrm{dec}\left[x_k^n + \frac{\widetilde{z}_k^n}{r_k^{n,n}}\right].$$

The achievable data-rate of user n on tone k is then

$$b_{k,\mathrm{dfc}}^n = \Delta_f \log_2(1 + \Gamma^{-1}\sigma_k^{-1}s_k^n\,|r_k^{n,n}|^2). \tag{13}$$

The CWDD property can be used to show that $|r_k^{n,n}| \simeq |h_k^{n,n}|$[1]. As a result, for small α_k, the DFC operates very close to the single-user bound

$$b_{k,\mathrm{dfc}}^n \simeq b_{k,\mathrm{bnd}}^n.$$

So the DFC has near-optimal performance. It should be noted, however, that this performance analysis is based on the assumption of error-free decisions (12). For this to be valid a perfect channel code must be used, which has infinite decoding complexity and delay[14].

In practice a sub-optimal code will be used, which can lead to decision errors, error propagation and poor performance. Furthermore, decoding of each user's codeword must be done before decisions are fed back. This increases complexity substantially and leads to a latency that grows with the number of lines in the binder. In VDSL systems the codewords are interleaved across the entire DMT block to add robustness against deep frequency nulls, which result from line properties such as bridged taps. Furthermore, the codeword may be interleaved across several DMT blocks to add robustness against impulse noise[15]. This means that the codewords are already quite long, and the latency is typically at the limit required for most applications. Typical binders contain hundreds of lines, where the increase in latency due to the DFC can be substantial. As a result with the DFC it is difficult to support real-time applications such as voice over IP and video conferencing.

V. Near-optimal Linear Canceler

This section describes a simple linear crosstalk canceler. Unlike the DFC, this structure has a low complexity, no latency and hence supports real-time applications. The structure is based on the *zero-forcing* (ZF) criterion, which leads to the following estimate of the transmitted vector

$$\begin{aligned}\widehat{\mathbf{x}}_k &= \mathbf{H}_k^{-1}\mathbf{y}_k, \\ &= \mathbf{x}_k + \mathbf{H}_k^{-1}\mathbf{z}_k.\end{aligned} \tag{14}$$

Each user then experiences a crosstalk free channel, affected only by the filtered background noise.

It is well known that ZF designs lead to severe noise-enhancement when the channel matrix $\mathbf{H}_k$ is ill-conditioned. Fortunately CWDD ensures that the channel matrix is well-conditioned; so the linear ZF canceler leads to negligible noise enhancement and each user achieves a data-rate close to the single-user bound (8). To see this first consider the *singular value decomposition* (SVD) of $\mathbf{H}_k$

$$\mathbf{H}_k \overset{\mathrm{svd}}{=} \mathbf{U}_k\Lambda_k\mathbf{V}_k^H. \tag{15}$$

Here we assume that the elements of Λ_k are non-zero. This is straight-forward to achieve in practice by simply switching off crosstalk cancellation on lines with a null transfer function. The CWDD of $\mathbf{H}_k$ ensures that its columns are approximately orthogonal. That is (2) implies that

$$\mathbf{h}_k^{m\,H}\,\mathbf{h}_k^n \simeq \begin{cases}|h_k^{n,n}|^2, & n = m;\\ 0, & n \neq m.\end{cases}$$

Hence

$$\mathbf{H}_k^H\mathbf{H}_k \simeq \mathrm{diag}\left\{\left|h_k^{1,1}\right|^2, \ldots, \left|h_k^{N,N}\right|^2\right\}.$$

Combining this with (15) leads to the following approximation

$$\mathbf{V}_k\Lambda_k^H\Lambda_k\mathbf{V}_k^H \simeq \mathrm{diag}\left\{\left|h_k^{1,1}\right|^2, \ldots, \left|h_k^{N,N}\right|^2\right\}.$$

This implies that the right singular vectors can be closely approximated as $\mathbf{V}_k \simeq \mathbf{I}_N$. The linear ZF filter can then be approximated as

$$\begin{aligned}\mathbf{H}_k^{-1} &= \mathbf{V}_k\Lambda_k^{-1}\mathbf{U}_k^H, \\ &\simeq \Lambda_k^{-1}\mathbf{U}_k^H.\end{aligned}$$

Since $\mathbf{U}_k$ is unitary it causes no noise enhancement. Furthermore, Λ_k^{-1} is diagonal and thus it scales the noise and signal powers equally. So thanks to the CWDD of $\mathbf{H}_k$, filtering the received signal with the matrix $\mathbf{H}_k^{-1}$ causes negligible noise enhancement. This allows the linear ZF canceler to achieve

near-optimal performance, operating close to the single-user bound in VDSL channels. This observation is made rigorous through the following theorem.

Theorem 2: If $A_{\min}^{(m)} \geq \alpha_k m B_{\max}^{(m)},\ m = 1 \ldots N-1$; then the data-rate achieved by the linear ZF canceler can be lower bounded

$$R_n \geq \sum_k b_{k,\mathrm{zf-bnd}}^n,$$

where

$$b_{k,\mathrm{zf-bnd}}^n \triangleq \Delta_f \log_2 \left(1 + \Gamma^{-1} \sigma_k^{-1} s_k^n \left|h_k^{n,n}\right|^2 f^{-1}\left(N, \alpha_k\right)\right), \tag{16}$$

$$f(N, \alpha_k) \triangleq \left(\frac{A_{\max}^{(N-1)}}{A_{\min}^{(N)}}\right)^2 + (N-1)\left(\frac{B_{\max}^{(N-1)}}{A_{\min}^{(N)}}\right)^2, \tag{17}$$

$$\begin{bmatrix} A_{\max}^{(m)} \\ B_{\min}^{(m)} \end{bmatrix} \triangleq \left(\prod_{i=1}^{m} \begin{bmatrix} 1 & (i-1)\,\alpha_k \\ \alpha_k & (i-1)\,\alpha_k \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{18}$$

and

$$A_{\min}^{(m)} \triangleq 1 - \sum_{i=1}^{m} \alpha_k\,(i-1)\,B_{\max}^{(i-1)}. \tag{19}$$

Proof of Theorem 2: Eq. (14) implies that, after application of the linear ZF canceler, the soft estimate of the transmitted symbol is

$$\widehat{x}_k^n = x_k^n + \left[\mathbf{H}_k^{-1}\right]_{\mathrm{row}\ n} \mathbf{z}_k.$$

Hence the post-cancellation signal power is s_k^n, the post-cancellation interference power is zero and the post-cancellation noise power is

$$\begin{aligned} \widetilde{\sigma}_{k,n} &\triangleq \mathcal{E}\left\{\left|\left[\mathbf{H}_k^{-1}\right]_{\mathrm{row}\ n} \mathbf{z}_k\right|^2\right\}, \\ &= \left\|\left[\mathbf{H}_k^{-1}\right]_{\mathrm{row}\ n}\right\|^2 \sigma_k, \end{aligned} \tag{20}$$

where (5) is applied in the second line. Hence the data-rate achieved by the linear ZF canceler is

$$b_{k,\mathrm{zf}}^n(s_k^n) = \Delta_f \log_2(1 + \Gamma^{-1}\widetilde{\sigma}_{k,n}^{-1} s_k^n). \tag{21}$$

Define the matrix $\mathbf{G}_k \triangleq [g_k^{n,m}]$, where $g_k^{n,m} \triangleq h_k^{n,m}/h_k^{m,m}$. Now

$$\mathbf{H}_k = \mathbf{G}_k \mathrm{diag}\{h_k^{1,1}, \ldots, h_k^{N,N}\},$$

hence

$$\mathbf{H}_k^{-1} = \mathrm{diag}\{h_k^{1,1}, \ldots, h_k^{N,N}\}^{-1} \mathbf{G}_k^{-1}, \tag{22}$$

and

$$\left[\mathbf{H}_k^{-1}\right]_{n,m} = \frac{1}{h_k^{n,n}} \left[\mathbf{G}_k^{-1}\right]_{n,m}. \tag{23}$$

Since the receivers are co-located at the CO, the US channel matrix is CWDD (3). This implies that $\mathbf{G}_k \in \mathbb{A}^{(N)}$, where $\mathbb{A}^{(N)}$ denotes the set of $N \times N$ diagonally dominant matrices, as defined in the appendix. So Theorem 5 from the Appendix can be applied to bound the elements of $\mathbf{G}_k^{-1}$. This implies

$$\left|\left[\mathbf{H}_k^{-1}\right]_{n,m}\right| \leq \begin{cases} \left|h_k^{n,n}\right|^{-1} A_{\max}^{(N-1)}/A_{\min}^{(N)}, & n = m; \\ \left|h_k^{n,n}\right|^{-1} B_{\max}^{(N-1)}/A_{\min}^{(N)}, & n \neq m; \end{cases}$$

where $A_{\max}^{(N)}$ and $B_{\max}^{(N)}$ are defined in (18) and $A_{\min}^{(N)}$ is defined in (19). Hence

$$\left\|\left[\mathbf{H}_k^{-1}\right]_{n,m}\right\|^2 \leq \left|h_k^{n,n}\right|^{-2} f(N, \alpha_k),$$

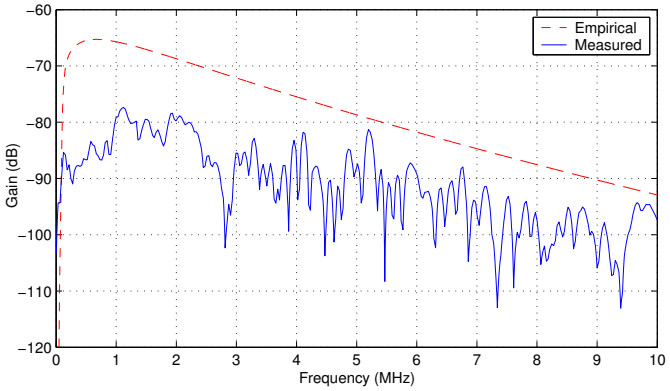

Fig. 2. Crosstalk Channel Transfer Functions (1 km cable, 0.5 mm pairs)

where $f(N, \alpha_k)$ is defined as in (17). Together with (20) this yields

$$\widetilde{\sigma}_{k,n} \leq \sigma_k \left|h_k^{n,n}\right|^{-2} f(N, \alpha_k).$$

Combining this with (21) leads to (16), which concludes the proof. ∎

In practice we have found that $A_{\min}^{(m)} \geq \alpha_k m B_{\max}^{(m)},\ m = 1 \ldots N-1$; holds for N up to 25 and for frequencies up to 12 MHz, so the bound applies in most practical VDSL scenarios.

The function $f(N, \alpha_k)$ can be interpreted as an upper bound on the noise enhancement caused by the linear ZF canceler. In CWDD channels $f(N, \alpha_k) \simeq 1$. As a result each modem operates at a rate

$$b_{k,\mathrm{zf}}^n \simeq \Delta_f \log_2 \left(1 + \Gamma^{-1} \sigma_k^{-1} s_k^n \left|h_k^{n,n}\right|^2\right).$$

So the linear ZF canceler completely removes crosstalk with negligible noise enhancement.

Note that the bound (16) can be used to predict and guarantee a data-rate without explicit knowledge of the crosstalk channels. This is the case because the bound depends only on the binder size, direct channel gain, and background noise power. Good models for these characteristics exist based on extensive measurement campaigns. Crosstalk channels on the other hand are poorly understood and actual channels can deviate significantly from the few empirical models that exist. See for example Fig. 2 which shows a measured crosstalk channel and the predicted crosstalk channel according to empirical models from standardization[15]. This can make provisioning of services difficult. Using the bound (16) allows us to overcome this problem. The bound tells us that the crosstalk channel gain is not important as long as CWDD is observed. CWDD is a well understood and modeled phenomenon. As a result (16) allows provisioning to be done in a reliable and accurate fashion.

A note of explanation may be necessary at this point. It may seem that CWDD allows us to easily predict, or at least bound, the crosstalk power that a receiver experiences. This is *not* true. The crosstalk power that a receiver experiences depends on the magnitude of elements along a *row*, not *column*, of $\mathbf{H}_k$. This in turn depends on the configuration of the other lines within the binder, which varies dramatically from one scenario to another. For example, in the scenario in Fig. 3,

the crosstalk from the 150 m line into the 1200 m line is stronger than the direct signal on the 1200 m line itself. So the crosstalk from the other lines into the 1200 m line cannot be bounded without knowledge of the entire binder configuration. This makes provisioning of services extremely difficult. CWDD, on the other hand, applies to all lines when receivers are co-located. No knowledge of the actual binder configuration is necessary. Using (16) the performance of a line can be estimated using only locally available information about the line itself, such as its direct channel attenuation and background noise.

The value for α_k from (4) is based on worst 1% case models. Hence for 99% of lines α_k will be smaller. So in 99% of lines a data-rate above the bound (16) is achieved. The bound is thus a useful tool not just for theoretical analysis, but for provisioning of services as well.

Simulations in Sec. VII will use this bound to show that the linear ZF canceler operates close to the single-user bound, and hence is a near-optimal design.

VI. Spectra Optimization

Whilst current VDSL standards require the use of spectral masks, there is growing interest in the use of adaptive transmit spectra, a technique known as dynamic spectrum management[16]. This section investigates the optimization of transmit spectra for use with the linear ZF canceler. Each transmitter is subject to a total power constraint

$$\Delta_f \sum_k s_k^n \leq P_n, \ \forall n. \tag{24}$$

The goal is to maximize a weighted sum of the data-rates of the modems within the network

$$\max_{\mathbf{s}_1,\ldots,\mathbf{s}_N} \sum_n w_n R_n \quad \text{s.t. } \Delta_f \sum_k s_k^n \leq P_n, \tag{25}$$

where the vector $\mathbf{s}_n \triangleq [s_1^n, \ldots, s_K^n]$ contains the PSD of user n on all tones. The weights $w_1, \ldots, w_N$ are used to ensure that each modem achieves its target data-rate. The data-rate R_n is a function of the transmit PSDs $\mathbf{s}_1, \ldots, \mathbf{s}_N$, and also depends on the type of crosstalk canceler used. If an optimal, decision-feedback based canceler is used, the objective function becomes convex[17]. Solving (25) then requires the solution of a KN-dimensional convex optimization. Although the cost function is convex, no closed form solution is known[17]. Conventional convex optimization techniques, such as interior point methods, have a polynomial complexity in the dimensionality of the search space. In ADSL $K = 256$, whilst in VDSL $K = 4096$. The resulting search has an extremely high dimensionality, for which conventional optimization techniques are prohibitively complex. A low complexity, iterative algorithm has been proposed for the special case where an unweighted rate-sum is maximized, that is $w_n = 1$ for all n[18]. Unfortunately, since this algorithm cannot optimize a weighted rate-sum, it cannot ensure that the target rates are achieved on each line. These target rates are essential to ensure that each customer obtains their desired service.

In this section a spectra optimization algorithm is developed for use with the linear ZF canceler. Since the linear ZF canceler removes all crosstalk, the spectra optimization decouples into an independent power loading for each user. This reduces complexity considerably. Furthermore, Theorem 2 ensures that this approach leads to a power allocation that operates close to the single-user bound.

A. Theoretical Capacity

We start by extending the single-user bound from Sec. III to VDSL modems that may vary their transmit spectra under a total power constraint. The resulting upper bound is useful for evaluating crosstalk canceler performance with optimized spectra.

Theorem 3: When the transmit PSD s_k^n is allowed to vary under a total power constraint (24), the capacity for user n can be upper bounded

$$R_n \leq \sum_k b_{k,\text{bnd}}^n(s_{k,\text{bnd}}^n), \tag{26}$$

where the single-user waterfilling PSD is defined

$$s_{k,\text{bnd}}^n \triangleq \left[\frac{1}{\lambda_n} - \frac{\Gamma\sigma_k}{|h_k^{n,n}|^2\left[1+\alpha_k^2\,(N-1)\right]}\right]^+, \tag{27}$$

the function $[x]^+ \triangleq \max(0, x)$, and λ_n is chosen such that power constraint on line n is tight, that is

$$\Delta_f \sum_k s_k^n = P_n. \tag{28}$$

This theorem is an intuitively simple extension of the single-user bound from Theorem 1. The proof, however, is not so straightforward and is now detailed. The following Lemma will prove useful in the proof.

Lemma 1: If $g(\mathbf{x}) \geq f(\mathbf{x}), \ \forall \mathbf{x}$, then

$$\max_{\mathbf{vx}\leq p} g(\mathbf{x}) \geq \max_{\mathbf{vx}\leq p} f(\mathbf{x}), \tag{29}$$

where $\mathbf{x}$ is the vector over which the optimization takes place, and the vector $\mathbf{v}$ and scalar p impose a linear constraint on $\mathbf{x}$.

Proof of Lemma 1: Define

$$\mathbf{x}_f \triangleq \arg\max_{\mathbf{vx}\leq p} f(\mathbf{x}).$$

Since $g(\mathbf{x}) \geq f(\mathbf{x}), \ \forall \mathbf{x}$,

$$g(\mathbf{x}_f) \geq f(\mathbf{x}_f). \tag{30}$$

Now define

$$\mathbf{x}_g \triangleq \arg\max_{\mathbf{vx}\leq p} g(\mathbf{x}).$$

The optimality of $\mathbf{x}_g$ in $g(\mathbf{x})$, over the subspace defined by the constraints $\mathbf{vx} \leq p$, implies that

$$\begin{aligned} g(\mathbf{x}_g) &\geq g(\mathbf{x}_f), \\ &\geq f(\mathbf{x}_f), \end{aligned}$$

where (30) is applied in the second line. This implies (29). ■

Corollary 1: Limit the total power of the transmit PSD such that $\Delta_f \sum_k s_k^n \leq P_n$. Under this constraint

$$\max_{\mathbf{s}_n} \sum_k \Delta_f \log_2 \left(1 + \Gamma^{-1} \sigma_k^{-1} s_k^n \left\|\mathbf{h}_k^n\right\|^2\right) \leq \max_{\mathbf{s}_n} \sum_k b_{k,\mathrm{bnd}}^n(s_k^n). \tag{31}$$

Proof of Corollary 1: Let $\mathbf{x} = \mathbf{s}_n$, $p = P_n$, $\mathbf{v} = 1_{1\times K}$,

$$f(\mathbf{s}_n) = \sum_k \Delta_f \log_2 \left(1 + \Gamma^{-1} \sigma_k^{-1} s_k^n \left\|\mathbf{h}_k^n\right\|^2\right),$$

and

$$g(\mathbf{s}_n) = \sum_k b_{k,\mathrm{bnd}}^n(s_k^n).$$

CWDD (2) implies that

$$\left|h_k^{n,n}\right|^2 \left[1 + \alpha_k^2 (N-1)\right] \geq \left\|\mathbf{h}_k^n\right\|^2,$$

hence $g(\mathbf{s}_n) \geq f(\mathbf{s}_n)$, $\forall \mathbf{s}_n$. Lemma 1 now implies (31), which completes the proof. ∎

Proof of Theorem 3: The single-user bound (8) applies. Combining this with the power constraint (24) yields

$$R_n \leq \max_{\sum_k s_k^n \leq P_n} \sum_k \Delta_f \log_2 \left(1 + \Gamma^{-1} \sigma_k^{-1} s_k^n \left\|\mathbf{h}_k^n\right\|_2^2\right).$$

Corollary 1 now implies

$$R_n \leq \max_{\sum_k s_k^n \leq P_n} \sum_k b_{k,\mathrm{bnd}}^n(s_k^n).$$

In this optimization the objective function is concave, and the total power constraint forms a convex set. Hence the *Karush-Kuhn-Tucker* (KKT) conditions are sufficient for optimality. Examining the KKT conditions leads to (26), (27) and (28), which completes the proof. ∎

B. Near-Optimal Linear Canceler

Transmit spectra optimization with the linear ZF canceler is now considered. Equation (21) implies that (25) is equivalent to

$$\max_{\mathbf{s}_1,\ldots,\mathbf{s}_N} \sum_n \sum_k w_n b_{k,\mathrm{zf}}^n(s_k^n) \quad \text{s.t. } \Delta_f \sum_k s_k^n \leq P_n. \tag{32}$$

Observe that, when using the linear ZF canceler, the data-rate of each user depends only on its own transmit PSD. It is independent of the PSDs of the other users since all crosstalk will be removed. The optimization problem is now decoupled between users, allowing the optimal power allocation to be found independently for each user. This also implies that these PSDs are optimal regardless of the choice of weights w_n.

Since the objective function is concave and the constraints form a convex set, the KKT conditions are sufficient for optimality. Examining these leads to the classic waterfilling equation

$$s_{k,\mathrm{zf}}^n = \left[\frac{1}{\lambda_n} - \Gamma \widetilde{\sigma}_{k,n}\right]^+. \tag{33}$$

The waterfilling level λ_n must be chosen such that the total power constraint for user n is tight, that is $\Delta_f \sum_k s_{k,\mathrm{zf}}^n = P_n$.

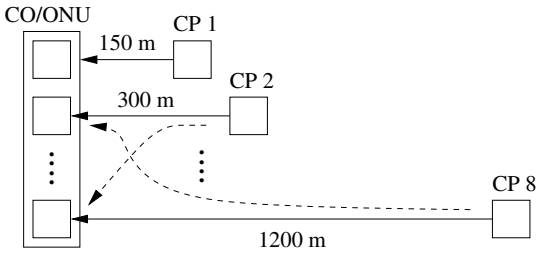

Fig. 3. Upstream VDSL scenario

Conventional waterfilling algorithms can be applied to find the correct waterfilling level with $\mathcal{O}(K \log K)$ complexity. So the overall complexity of power allocation with the linear ZF canceler is $\mathcal{O}(NK \log K)$. This is a significant reduction when compared to existing power allocation algorithms for the multi-access channel, which have $\mathcal{O}(N^4 K \log K)$ complexity in the unweighted rate-sum case, and polynomial complexity in KN in the weighted rate-sum case[18].

Theorem 2 shows that, as a result of CWDD, the linear ZF canceler operates close to the single-user bound. So using the linear ZF canceler in combination with the power allocation (33) gives near-optimal performance. This is confirmed through simulation in the following section.

VII. Performance

This section evaluates the performance of the linear ZF canceler in a binder of 8 VDSL lines. The line lengths range from 150 m to 1200 m in 150 m increments, as shown in Fig. 3. For all simulations the line diameter is 0.5 mm (24-AWG). Direct and crosstalk channels are generated using semi-empirical models[9]. The target symbol error probability is 10^{-7} or less. The coding gain is set to 3 dB and the noise margin is set to 6 dB. As per the VDSL standards the tone-spacing Δ_f is set to 4.3125 kHz[15][9]. The modems use 4096 tones, and the 998 FDD bandplan. Background noise is generated using ETSI noise model A[9]. Performance is compared with the DFC and the single-user bound.

A. Fixed Transmit Spectra

Current VDSL standards require that modems transmit under a spectral mask of -60 dBm/Hz[15][9]. This section evaluates the performance of the linear ZF canceler when all modems are operating at this mask.

Fig. 4 shows the data-rate achieved by each of the lines with the different crosstalk cancelers. The linear ZF canceler achieves substantial gains, typically 30 Mbps or more, over conventional systems with no cancellation. As can be seen the linear ZF canceler achieves near-optimal performance, operating close to the single-user bound. This is a direct result of the CWDD of $\mathbf{H}_k$, which ensures that the linear ZF canceler causes negligible noise enhancement. The noise enhancement caused by the linear ZF canceler on the 600 m line is plotted for each tone in Fig. 5. As can be seen the noise enhancement is less than 0.16 dB, which has negligible impact on performance.

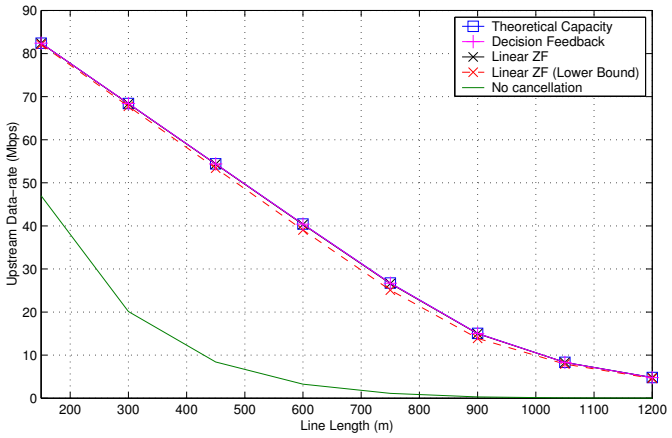

Fig. 4. Data-rate with Different Cancelers

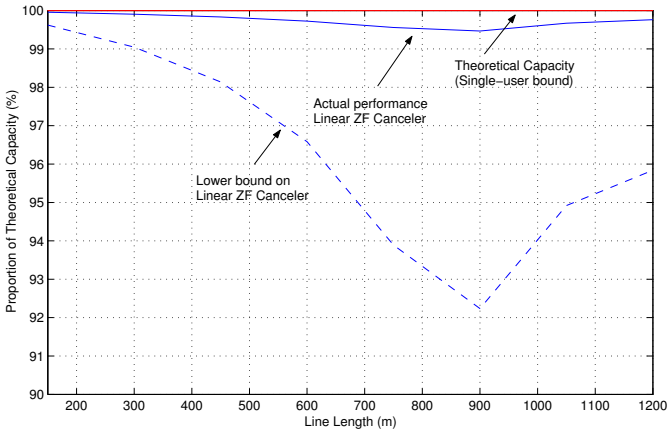

Fig. 6. Proportion of Single-user Bound Achieved by ZF Canceler

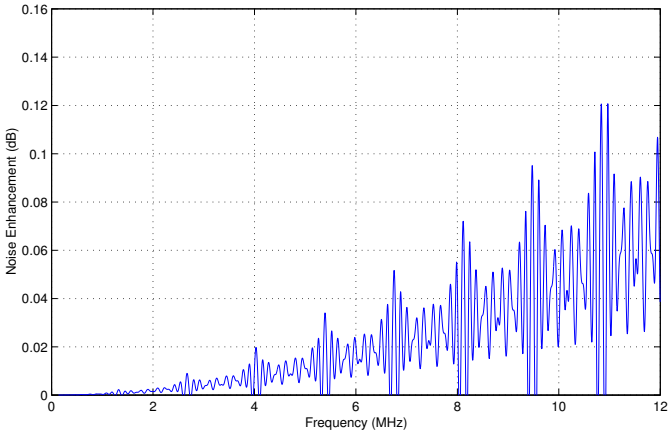

Fig. 5. Noise enhancement of ZF Canceler on 600 m. line

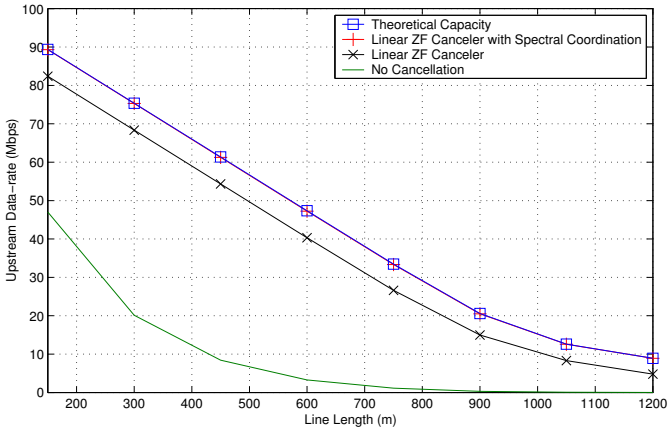

Fig. 7. Data-rate with Optimized Spectra

Fig. 6 shows the data-rate achieved by the linear ZF canceler as a percentage of the single-user bound. Performance does not drop below 99% of the single-user bound. The lower bound on the performance of the linear ZF canceler (16) is also included for comparison. As can be seen the bound is quite tight and guarantees that the linear ZF canceler will achieve at least 92% of the single-user bound.

B. Optimized Transmit Spectra

This section investigates the performance of the linear ZF canceler with optimized spectra (33). A total power constraint of 11.5 dBm/Hz is applied to each modem as per the VDSL standards[15][9]. Spectral mask constraints are not applied. Fig. 7 shows the data-rates achieved on each line. The use of optimized spectra yields a gain of 5-8 Mbps. The benefit is more substantial on the longer lines, where a 5 Mbps gain can double the data-rate.

Fig. 7 shows that spectra optimization gives maximum benefit on long lines. This is to be expected since on long lines the direct channel gain decreases more rapidly with frequency. Note that the benefit of adaptive spectra, when crosstalk has already been cancelled, comes primarily from the modem loading power in the best parts of the channel, which are typically in the lower frequencies.

VIII. Conclusions

This paper investigated the design of crosstalk cancelers for upstream VDSL. Existing designs based on decision feedback suffer from error propagation, high complexity and long latency. A linear ZF canceler is proposed, which has a low complexity and no latency.

An oft-cited problem with the ZF design is that it leads to severe noise enhancement in ill-conditioned channels. Fortunately VDSL channels with co-located receivers are column-wise diagonal dominant. This ensures that the VDSL channel is well conditioned, and noise enhancement caused by the ZF design is negligible.

An upper bound on the capacity of the multi-user VDSL channel was derived. This single-user bound shows that spatial diversity in the VDSL environment is negligible. Therefore the best outcome that a crosstalk canceler can achieve is the complete suppression of crosstalk without noise enhancement.

A lower bound on the performance of the linear ZF canceler was derived. This bound depends only on the binder size, direct channel gain and background noise for which reliable models and statistical data exist. As a result the performance

of the linear ZF canceler can be accurately predicted, which simplifies service provisioning considerably. This bound shows that the linear ZF canceler operates close to the single-user bound. So the linear ZF canceler is a low complexity, low latency design with predictable, near-optimal performance.

The combination of spectral optimization and crosstalk cancellation was considered. The bounds were extended to VDSL systems with optimized spectra. Spectra optimization in a multi-access channel generally involves a highly complex optimization problem. Since the linear ZF canceler decouples transmission on each line, the spectrum on each modem can be optimized independently, leading to a significant reduction in complexity.

APPENDIX

Define the set $\mathbb{A}^{(N)}$ of $N \times N$ matrices, such that for any $\mathbf{A}^{(N)} \in \mathbb{A}^{(N)}$, it holds that

$$\begin{aligned} |a_{n,n}| &= 1; \\ |a_{n,m}| &\leq \alpha_k, \ \forall n \neq m; \end{aligned}$$

where $a_{n,m} \triangleq \left[\mathbf{A}^{(N)}\right]_{n,m}$. Define the set $\mathbb{B}^{(N)}$ of $N \times N$ matrices, such that for any $\mathbf{B}^{(N)} \in \mathbb{B}^{(N)}$, it holds that

$$\begin{aligned} |b_{n,n}| &= 1, \ \forall n < N; \\ |b_{N,N}| &\leq \alpha_k; \\ |b_{n,m}| &\leq \alpha_k, \ \forall n \neq m; \end{aligned}$$

where $b_{n,m} \triangleq \left[\mathbf{B}^{(N)}\right]_{n,m}$.

Theorem 4: Consider any $\mathbf{A}^{(N)} \in \mathbb{A}^{(N)}$ and $\mathbf{B}^{(N)} \in \mathbb{B}^{(N)}$. The magnitude of the determinants of $\mathbf{A}^{(N)}$ and $\mathbf{B}^{(N)}$ can be bounded as follows

$$\left|\det\left(\mathbf{A}^{(N)}\right)\right| \leq A_{\max}^{(N)}, \tag{34}$$

$$\left|\det\left(\mathbf{B}^{(N)}\right)\right| \leq B_{\max}^{(N)}, \tag{35}$$

where $A_{\max}^{(N)}$, $B_{\max}^{(N)}$ and $A_{\min}^{(N)}$ are defined in (18) and (19). Furthermore, if

$$A_{\min}^{(m)} \geq \alpha_k m B_{\max}^{(m)}, \ m = 1 \ldots N-1; \tag{36}$$

then the following bound also holds

$$\left|\det\left(\mathbf{A}^{(N)}\right)\right| \geq A_{\min}^{(N)}. \tag{37}$$

Note that $|\cdot|$ denotes the absolute value operator, whilst $\det(\cdot)$ denotes the determinant operator.

Proof of Theorem 4: The proof is based on induction. Begin by assuming that the bounds (34), (35) and (37) hold for any $N \times N$ matrices of the form $\mathbf{A}^{(N)}$ and $\mathbf{B}^{(N)}$ for some specific value of N. Now consider any matrix $\mathbf{A}^{(N+1)} \in \mathbb{A}^{(N+1)}$. Decompose $\mathbf{A}^{(N+1)}$ as

$$\mathbf{A}^{(N+1)} = \begin{bmatrix} & & & a_{1,N+1} \\ & \mathbf{A}^{(N)} & & \vdots \\ & & & a_{N,N+1} \\ a_{N+1,1} & \cdots & a_{N+1,N} & 1 \end{bmatrix},$$

where $a_{n,m} \triangleq \left[\mathbf{A}^{(N+1)}\right]_{n,m}$ and $\mathbf{A}^{(N)}$ is the submatrix containing the first N rows and columns of $\mathbf{A}^{(N+1)}$. By expanding the determinant along the last row of $\mathbf{A}^{(N+1)}$ it can be seen that

$$\begin{aligned} &\left|\det\left(\mathbf{A}^{(N+1)}\right)\right| \\ &= \left|\det\left(\mathbf{A}^{(N)}\right)\right. \\ &\qquad \left. + \sum_{m=1}^{N} (-1)^{N+1-m} a_{N+1,m} \det\left(\begin{bmatrix} \overline{\mathbf{A}}_m^{(N)} & \mathbf{a}_{N+1} \end{bmatrix}\right)\right|, \end{aligned} \tag{38}$$

$$\leq \left|\det\left(\mathbf{A}^{(N)}\right)\right| + \sum_{m=1}^{N} \alpha_k \left|\det\left(\begin{bmatrix} \overline{\mathbf{A}}_m^{(N)} & \mathbf{a}_{N+1} \end{bmatrix}\right)\right|, \tag{39}$$

where $\overline{\mathbf{A}}_m^{(N)}$ is the sub-matrix formed by removing column m from $\mathbf{A}^{(N)}$ and $\mathbf{a}_{N+1} \triangleq [a_{1,N+1} \ldots a_{N,N+1}]^T$. The second line exploits the fact that row permutation does not affect the magnitude of a determinant. Define the permutation matrix

$$\Pi_m \triangleq [\mathbf{e}_1 \cdots \mathbf{e}_{m-1} \ \mathbf{e}_{m+1} \ldots \mathbf{e}_N \ \mathbf{e}_m],$$

where $\mathbf{e}_m$ is defined as the mth column of the $N \times N$ identity matrix. Note that $\Pi_m^T[\overline{\mathbf{A}}_m^{(N)} \ \mathbf{a}_{N+1}] \in \mathbb{B}^{(N)}$. Using the fact that row permutations have no effect on the magnitude of a determinant, together with (35) and (39) now implies

$$\sum_{m=1}^{N} \alpha_k \left|\det\left(\begin{bmatrix} \overline{\mathbf{A}}_m^{(N)} & \mathbf{a}_{N+1} \end{bmatrix}\right)\right| \leq \alpha_k N B_{\max}^{(N)}. \tag{40}$$

Combining this with (39) and (34) yields

$$\left|\det\left(\mathbf{A}^{(N+1)}\right)\right| \leq A_{\max}^{(N)} + \alpha_k N B_{\max}^{(N)}.$$

Note that by definition

$$A_{\max}^{(N+1)} = A_{\max}^{(N)} + \alpha_k N B_{\max}^{(N)}, \tag{41}$$

hence

$$\left|\det\left(\mathbf{A}^{(N+1)}\right)\right| \leq A_{\max}^{(N+1)}. \tag{42}$$

Now consider any matrix $\mathbf{B}^{(N+1)} \in \mathbb{B}^{(N+1)}$. Decompose $\mathbf{B}^{(N+1)}$ as

$$\mathbf{B}^{(N+1)} = \begin{bmatrix} & & & b_{1,N+1} \\ & \mathbf{C}^{(N)} & & \vdots \\ & & & b_{N,N+1} \\ b_{N+1,1} & \cdots & b_{N+1,N} & b_{N+1,N+1} \end{bmatrix},$$

where $b_{n,m} \triangleq \left[\mathbf{B}^{(N+1)}\right]_{n,m}$ and $\mathbf{C}^{(N)}$ is the submatrix containing the first N rows and columns of $\mathbf{B}^{(N+1)}$. By expanding the determinant along the last row of $\mathbf{B}^{(N+1)}$ it can be seen that

$$\begin{aligned} &\left|\det\left(\mathbf{B}^{(N+1)}\right)\right| = \left|b_{N+1,N+1} \det\left(\mathbf{C}^{(N)}\right)\right. \\ &\qquad \left. + \sum_{m=1}^{N} (-1)^{N+1-m} b_{N+1,m} \det\left(\begin{bmatrix} \overline{\mathbf{C}}_m^{(N)} & \mathbf{b}_{N+1} \end{bmatrix}\right)\right| \end{aligned} \tag{43}$$

where $\overline{\mathbf{C}}_m^{(N)}$ is the sub-matrix formed by removing column m from $\mathbf{C}^{(N)}$ and $\mathbf{b}_{N+1} \triangleq [b_{1,N+1} \ldots b_{N,N+1}]^T$. Note that $\mathbf{C}^{(N)} \in \mathbb{A}^{(N)}$ and

$$\Pi_m^T \begin{bmatrix} \overline{\mathbf{C}}_m^{(N)} & \mathbf{b}_{N+1} \end{bmatrix} \in \mathbb{B}^{(N)}.$$

Using the fact that row permutations have no effect on the magnitude of a determinant, together with (34), (35), and (43) now yields

$$\left|\det(\mathbf{B}^{(N+1)})\right| \leq \alpha_k A_{\max}^{(N)} + \alpha_k N B_{\max}^{(N)}.$$

Note that by definition

$$B_{\max}^{(N+1)} = \alpha_k A_{\max}^{(N)} + \alpha_k N B_{\max}^{(N)}, \tag{44}$$

hence

$$\left|\det(\mathbf{B}^{(N+1)})\right| \leq B_{\max}^{(N+1)}. \tag{45}$$

Combining (41) and (44) in matrix form yields

$$\begin{bmatrix} A_{\max}^{(N+1)} \\ B_{\max}^{(N+1)} \end{bmatrix} = \begin{bmatrix} 1 & \alpha_k N \\ \alpha_k & \alpha_k N \end{bmatrix} \begin{bmatrix} A_{\max}^{(N)} \\ B_{\max}^{(N)} \end{bmatrix}. \tag{46}$$

We now proceed with the inductive proof. First note that $\left|\mathbf{A}^{(1)}\right| = 1$ and $\left|\mathbf{B}^{(1)}\right| \leq \alpha_k$, so (34) and (35) hold for $N = 1$. Hence through induction, (42) and (45) imply that (34) and (35) must hold for all N. This concludes the proof for the upper bounds (34) and (35). We now turn our attention to the lower bound (37). First note that from (38)

$$\begin{aligned} \left|\det\left(\mathbf{A}^{(N+1)}\right)\right| \quad &\geq \quad \left|\left|\det\left(\mathbf{A}^{(N)}\right)\right| \right. \\ &\left. - \sum_{m=1}^{N} \alpha_k \left|\det\left(\begin{bmatrix} \overline{\mathbf{A}}_m^{(N)} & \mathbf{a}_{N+1} \end{bmatrix}\right)\right|\right|. \end{aligned} \tag{47}$$

We assume that (37) holds for some specific value of N. Hence

$$\left|\det(\mathbf{A}^{(N)})\right| \geq A_{\min}^{(N)}. \tag{48}$$

Combining this with (36) and (40) implies

$$\left|\det\left(\mathbf{A}^{(N)}\right)\right| \geq \sum_{m=1}^{N} \alpha_k \left|\det\left(\begin{bmatrix} \overline{\mathbf{A}}_m^{(N)} & \mathbf{a}_{N+1} \end{bmatrix}\right)\right|.$$

So from (47)

$$\begin{aligned} \left|\det\left(\mathbf{A}^{(N+1)}\right)\right| \quad &\geq \quad \left|\det\left(\mathbf{A}^{(N)}\right)\right| \\ &- \sum_{m=1}^{N} \alpha_k \left|\det\left(\begin{bmatrix} \overline{\mathbf{A}}_m^{(N)} & \mathbf{a}_{N+1} \end{bmatrix}\right)\right|. \end{aligned}$$

Combining this with (48) and (40) leads to the bound

$$\left|\det(\mathbf{A}^{(N+1)})\right| \geq A_{\min}^{(N)} - \alpha_k N B_{\max}^{(N)}.$$

Note that by definition

$$A_{\min}^{(N+1)} \triangleq A_{\min}^{(N)} - \alpha_k N B_{\max}^{(N)},$$

hence

$$\left|\det(\mathbf{A}^{(N+1)})\right| \geq A_{\min}^{(N+1)} \tag{49}$$

Now note that $\left|\mathbf{A}^{(1)}\right| = 1$ and $A_{\min}^{(1)} = 1$, so (37) holds for $N = 1$. Hence through induction, (49) implies that (37) holds for all N. This concludes the proof for the lower bound (37). ■

Theorem 5: If $\mathbf{G} \in \mathbb{A}^{(N)}$ and

$$A_{\min}^{(m)} \geq \alpha_k m B_{\max}^{(m)},\ m = 1 \ldots N-1;$$

then the magnitude of the elements of $\mathbf{G}^{-1}$ can be bounded

$$\left|\left[\mathbf{G}^{-1}\right]_{n,m}\right| \leq \begin{cases} A_{\max}^{(N-1)}/A_{\min}^{(N)}, & n = m; \\ B_{\max}^{(N-1)}/A_{\min}^{(N)}, & n \neq m. \end{cases} \tag{50}$$

Proof of Theorem 5: By definition of the matrix inverse

$$\left|\left[\mathbf{G}^{-1}\right]_{n,m}\right| = \left|\det\left(\overline{\mathbf{G}}^{m,n}\right)\right| / \left|\det\left(\mathbf{G}\right)\right|, \tag{51}$$

where $\overline{\mathbf{G}}^{m,n}$ is the sub-matrix formed by removing row m and column n from $\mathbf{G}$. Now $\mathbf{G} \in \mathbb{A}^{(N)}$ so from theorem 4

$$|\det(\mathbf{G})| \geq A_{\min}^{(N)}. \tag{52}$$

If $m = n$ then $\overline{\mathbf{G}}^{m,n} \in \mathbb{A}^{(N-1)}$ and from Theorem 4

$$\left|\det(\overline{\mathbf{G}}^{m,m})\right| \leq A_{\max}^{(N-1)},\ \forall m. \tag{53}$$

If $m \neq n$ then $\Pi_n^T \overline{\mathbf{G}}^{m,n} \Pi_m \in \mathbb{B}^{(N-1)}$ and from Theorem 4

$$\left|\det\left(\overline{\mathbf{G}}^{m,n}\right)\right| = \left|\det\left(\Pi_n^T \overline{\mathbf{G}}^{m,n} \Pi_m\right)\right| \leq B_{\max}^{(N-1)},\ \forall m \neq n. \tag{54}$$

Combining (51), (52), (53) and (54) yields (50), which concludes the proof. ■

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