Improved frequency domain equalizers for zero-padded single-carrier block transmission

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Abstract

This paper focuses on the equalization of single-carrier block transmissions with zero padding or known signal padding. If the channel order does not exceed the padding length, it is possible to construct zero-forcing equalizers (ZFEs). The optimum performance is achieved using a minimum mean squared error (MMSE) equalizer, but both this solution and the ZFEs are computationally intensive. While the frequency domain implementation of a ZFE has a lower complexity, in the presence of spectral zeros it can be very ill conditioned, resulting in noise enhancement. The MMSE equalizer in the frequency domain performs better by limiting the noise but still loses all information stored at the spectral zeros. This paper presents a method to exploit the redundancy of the padding to recover this lost information, leading to two new structures, a ZFE and an MMSE-like equalizer. These have a performance close to the time-domain solutions, while maintaining the low complexity of a frequency domain solution.

Index Terms

equalization, spectral zero, frequency domain, ZFE, MMSE, single-carrier block transmission, zero padding

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I. Introduction

Single carrier (SC) block transmission with frequency domain equalization has been recognized as an alternative to orthogonal frequency division multiplexing (OFDM) [1]. Since one can be converted into the other by an orthogonal precoding, they have many similarities and mathematical techniques for one system can usually be adapted for use with the other. The differences mostly concern practical implementation issues: because SC systems generally have a lower peak to average power ratio (PAPR) and a better robustness against non-linearities or carrier offset ([1], [2], [3]), their demands on the analog hardware are lower than for OFDM. While the focus in this article is on SC block transmission, most of the references are related to OFDM.

In block transmissions, a guard interval (GI) is inserted in between blocks to prevent inter block interference (IBI) and to allow to convert the linear channel convolution into a cyclic one. Because the corresponding circular channel matrix can be diagonalized by a discrete Fourier transform (DFT) and inverse DFT (IDFT) matrix, the frequency selective channel is converted into a collection of flat-fading channels. The equalization can therefore be performed in the frequency domain, resulting in a significant complexity reduction. The GI can either be a cyclic prefix (CP), i.e. a copy of the last P samples of the block, a zero pad (ZP) or a known symbol pad (KSP). Specifically a CP system relies on the overlap-save (OLS) technique, while a ZP system uses overlap-add (OLA) to obtain a circular convolution.

Although CP and ZP systems appear to be interchangeable there are important differences. More specifically in a CP system the channel may exhibit spectral zeros on or near the unit circle such that zero-forcing equalization is not possible or leads to severe noise enhancement ([4], [5]). A CP-OFDM system can take this into account by discarding the affected frequencies if the channel is known at the transmitter, or by using forward error correction (FEC) codes if it is not ([2], [5]). The signal projection scheme from [6] offers a fundamental solution but is prohibitive in complexity, while the precoding technique from [7] requires the channel to remain constant over a period of multiple blocks. For a CP-SC system a ZFE in the presence of spectral zeros is even worse, as the enhanced noise spreads out over the entire time domain block. The classical solution is to limit the noise enhancement by using a minimum mean square error (MMSE) equalizer instead ([8])

For a ZP system on the other hand, zero-forcing equalization is always possible ([5], [9]). Unfortunately, using the OLA technique to calculate the ZFE in the frequency domain introduces the same vulnerability to spectral zeros as in CP systems. This results in an ill-conditioned problem and again the need to revert to an MMSE approach.

This paper presents two new structures for a ZP-SC system which operate in the frequency domain and which can recover the information lost in spectral zeros. With minor modifications they can also be used for KSP-SC transmissions. The organization is as follows: in section II the system model will be explained, followed by an overview of existing equalization techniques. In section III the new equalizer structures are introduced. In section IV, simulations results are presented and finally, in section V conclusions are summarized.

II. System model and equalizer structures

Consider a channel $\mathbf{h} = \begin{bmatrix} h_0 & h_1 & \dots & h_{\mathrm{L}} \end{bmatrix}^T$ of order L, which is known by both transmitter and receiver and which can be considered *quasi static* over one block period[1]. The transmitted block $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_{\mathrm{N}} \end{bmatrix}^T$ is followed by a ZP of length P, with $P \geq L$ to avoid IBI. Without loss of generality it is assumed that the variance of $\mathbf{x}$ equals unity, i.e. $E\{\mathbf{x}\mathbf{x}^{\mathcal{H}}\} = \mathbf{I}_{(\mathrm{N})}$. The received signal $\mathbf{y}$ is given as:

$$\mathbf{y} = \underbrace{\begin{bmatrix} h_0 & & \\ & \ddots & \\ \vdots & & \\ h_{\mathrm{L}} & & \\ & \ddots & h_0 \\ & & \vdots \\ & & h_{\mathrm{L}} \end{bmatrix}}_{H_{(\mathrm{N+L})\times \mathrm{N}}} \mathbf{x} + \mathbf{n}, \tag{1}$$

with $\mathbf{H}$ the linear convolution matrix of $\mathbf{h}$ and $\mathbf{n}$ additive noise. Without loss of generality we assume that the noise is white with variance σ_{n}, i.e. $E\{\mathbf{n}\mathbf{n}^{\mathcal{H}}\} = \sigma_{\mathrm{n}}\mathbf{I}_{(\mathrm{N+L})}$.

[1]Bold lowercase letters denote vectors and bold uppercase letters denote matrices. Normal uppercase letters denote constants. $(.)^T$, $(.)^{\mathcal{H}}$, $(.)^{\dagger}$ and $E\{.\}$ denote transpose, Hermitian, pseudo-inverse and expected value respectively and $(.)(:, [l_1 \dots l_{\mathrm{k}}])$ selects columns $l_1 \dots l_{\mathrm{k}}$. The DFT, IDFT, zero and identity matrix of size K are represented by $\boldsymbol{\mathcal{F}}_{\mathrm{K}}$, $\boldsymbol{\mathcal{I}}_{\mathrm{K}}$, $\mathbf{0}_{\mathrm{K}}$, and $\mathbf{I}_{\mathrm{K}}$ respectively. Occasionally, for clarity the dimensions of a matrix are indicated as $[.]_{\mathrm{K}\times\mathrm{L}}$.

A. Time domain equalization

An equalizer $\mathbf{W}$ is a matrix for which $\hat{\mathbf{x}} = \mathbf{W}\mathbf{y}$ is an estimate of $\mathbf{x}$. Moreover, any equalizer $\mathbf{W}_{\text{ZFE}}$ which satisfies the following property

$$\mathbf{W}_{\text{ZFE}}\mathbf{H}_{(\text{N+L})\times\text{N}} = \mathbf{I}_{\text{N}}, \tag{2}$$

is referred to as a *zero forcing equalizer*. In the absence of noise, such structure obviously leads to a perfect reconstruction of the transmitted block. Because $\mathbf{H}$ is a tall Toeplitz matrix, an infinite number of equalizers exist that satisfy eq.(2). Since $\hat{\mathbf{x}} = \mathbf{x} + \mathbf{W}_{\text{ZFE}}\mathbf{n}$, they only differ in the way they treat the noise.

One particular ZFE is obtained as the pseudo inverse of $\mathbf{H}$, which then provides the *minimum-norm* solution of eq.(2). It is given by[2]:

$$\mathbf{W}_{\text{ZFE-TD}} = \mathbf{H}^{\dagger} = (\mathbf{H}^{\mathcal{H}}\mathbf{H})^{-1}\mathbf{H}^{\mathcal{H}}. \tag{3}$$

Here $\mathbf{H}^{\mathcal{H}}\mathbf{H}$ is a full-rank $N \times N$ matrix and hence always invertible. This equalizer is referred to as the ZFE-TD (time domain ZFE) where ”time domain” refers to the absence of a transformation to the frequency domain (see II-B). It is the optimal ZFE because it has the lowest noise amplification.

By dropping the perfect reconstruction constraint of eq. (2) it is possible to do even better. The MMSE equalizer also takes into account the noise statistics and performs a global optimization by minimizing the expected value of the squared error, $E\{||\mathbf{x} - \mathbf{W}_{\text{MMSE-TD}}\mathbf{y}||^2\}$. It can be expressed as:

$$\mathbf{W}_{\text{MMSE-TD}} = (\mathbf{H}^{\mathcal{H}}\mathbf{H} + \sigma_{\text{n}}^2\mathbf{I}_{\text{N}})^{-1}\mathbf{H}^{\mathcal{H}}. \tag{4}$$

The calculation of these equalizers has a complexity of $\mathcal{O}(N^2)$ flops (using structured matrix inversion formulae) while calculating $\hat{\mathbf{x}}$ requires no more than $\mathcal{O}(N^2)$ flops per block [10].

B. Frequency domain equalization based on matrix folding

To exploit frequency domain techniques the linear convolution needs to be converted into a circular one. Accordingly the matrix $\mathbf{H}$ needs to be transformed into a circular matrix. This can be achieved in two ways: either by folding the last L rows back to the top or by extending it with L more columns.

[2]The minimum norm ZFE is often referred to as MMSE-ZFE. In this paper it is called ZFE-TD to stress its calculation in the time domain and avoid confusion with the MMSE solution of eq.(4)

The first method corresponds to an *overlap-add* operation: the last L samples of $\mathbf{y}$ are added to the first L samples and then only the first N samples of the result are kept, resulting in

$$\mathbf{y}_{\mathrm{f}} = \underbrace{\left[\begin{array}{c|c} \mathbf{I}_{\mathrm{N}} & \begin{matrix}\mathbf{I}_{\mathrm{L}}\\ \mathbf{0}\end{matrix} \end{array}\right]}_{\mathbf{G}_{\mathrm{f}}} \mathbf{y} = \underbrace{\begin{bmatrix} h_0 & & & h_{\mathrm{L}} & \dots & h_1 \\ & \ddots & & & \ddots & \vdots \\ \vdots & & & & & h_{\mathrm{L}} \\ h_0 & & & & & \\ & \ddots & & & & \\ & & h_{\mathrm{L}} & \dots & & h_0 \end{bmatrix}}_{\mathbf{H}_{\mathrm{f}(\mathrm{N}\times\mathrm{N})}} \mathbf{x} + \mathbf{n}_{\mathrm{f}}. \tag{5}$$

Note that multiplication with the $(\mathrm{N}\times(\mathrm{N}+\mathrm{L}))$-matrix $\mathbf{G}_{\mathrm{f}}$ reduces the number of matrix rows and hence the information in the system. The existence of a ZFE solution may therefore no longer be guaranteed. Indeed, $\mathbf{H}_{\mathrm{f}}$ is an $N \times N$ matrix which becomes rank deficient if the channel has spectral zeros. If it is regular, the resulting unique ZFE can be written as:

$$\mathbf{W}_{\text{ZFE-TD-FOLD}} = \mathbf{H}_{\mathrm{f}}^{-1}\mathbf{G}_{\mathrm{f}}. \tag{6}$$

To perform the equalization in the frequency domain, note that $\mathbf{H}_{\mathrm{f}}$ can be diagonalized to $\mathbf{\Lambda}_{\mathrm{f}}$, such that eq.(5) becomes:

$$\mathbf{y}_{\mathrm{f}} = \boldsymbol{\mathcal{I}}_{\mathrm{N}}\mathbf{\Lambda}_{\mathrm{f}}\boldsymbol{\mathcal{F}}_{\mathrm{N}}\mathbf{x} + \mathbf{n}_{\mathrm{f}}. \tag{7}$$

The ZFE in the frequency domain can now be written as:

$$\mathbf{W}_{\text{ZFE-FD-FOLD}} = \boldsymbol{\mathcal{I}}_{\mathrm{N}}\mathbf{\Lambda}_{\mathrm{f}}^{-1}\boldsymbol{\mathcal{F}}_{\mathrm{N}}\mathbf{G}_{\mathrm{f}}. \tag{8}$$

The inversion of the diagonal matrix in this equation requires $\mathcal{O}(N)$ flops, whereas the calculation of the estimate for $\mathbf{x}$ requires only $\mathcal{O}(N\log(N))$ flops per block, if the (I)DFT operation is implemented using the (inverse) fast Fourier transform. Obviously an important saving is to be made compared to the ZFE-TD, especially for large values of N. However, this speedup comes at a cost: if the channel has spectral zeros on or near the unit circle, the inversion of $\mathbf{\Lambda}_{\mathrm{f}}$ may be impossible or lead to very large values and hence a very high noise amplification. This problem is well known, and several authors have proposed the MMSE approach to limit the detrimental influence of the noise ([6], [11], [12] for the OFDM case). The MMSE equalizer in the frequency domain can be expressed as

$$\mathbf{W}_{\text{MMSE-FD-FOLD}} = \boldsymbol{\mathcal{I}}_{\mathrm{N}}\left(\mathbf{\Lambda}_{\mathrm{f}}^{\mathcal{H}}\mathbf{\Lambda}_{\mathrm{f}} + \sigma_{\mathrm{n}}^2\left(\frac{N+L}{N}\right)\mathbf{I}_{\mathrm{N}}\right)^{-1}\mathbf{\Lambda}_{\mathrm{f}}^{\mathcal{H}}\boldsymbol{\mathcal{F}}_{\mathrm{N}}\mathbf{G}_{\mathrm{f}}, \tag{9}$$

where the regularisation term $\left(\frac{N+L}{N}\right)$ reflects the difference in noise shaping due to $\mathbf{G}_{\mathrm{f}}$.

C. Frequency domain equalization based on matrix extension

It is also possible to add appropriate columns to $\mathbf{H}$ to extend it to a circular $(N+L)\times(N+L)$ matrix $\mathbf{H}_{\text{e}}$. Equation (1) is then extended to explicitly involve the ZP in the calculations, i.e.[3]:

$$\mathbf{y} = \underbrace{\left[\begin{array}{ccc|ccc} h_0 & & & h_{\text{L}} & \dots & h_1 \\ & \ddots & & & \ddots & \vdots \\ \vdots & & & & & h_{\text{L}} \\ h_{\text{L}} & & & & & \\ & \ddots & h_0 & & & \\ & & \vdots & h_0 & & \\ & & & \vdots & \ddots & \\ & & h_{\text{L}} & h_{\text{L-1}} & \dots & h_0 \end{array}\right]}_{\mathbf{H}_{\text{e}(\text{N+L})\times(\text{N+L})}} \underbrace{\left[\begin{array}{c} x_1 \\ \vdots \\ x_{\text{N}} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right]}_{\mathbf{x}_{\text{ZP}}} + \mathbf{n} \tag{10}$$

Note that $\mathbf{H}_{\text{e}}$ is an $(N+L)\times(N+L)$ matrix of which the rank is minimally N. Now $\mathbf{H}_{\text{e}}$ can be diagonalized to $\mathbf{\Lambda}_{\text{e}}$, having diagonal elements $\lambda_{\text{e}_1}, \lambda_{\text{e}_2}, \dots\ \lambda_{\text{e}_\text{M}}$, yielding:

$$\mathbf{y} = \boldsymbol{\mathcal{I}}_{\text{M}} \mathbf{\Lambda}_{\text{e}} \boldsymbol{\mathcal{F}}_{\text{M}} \mathbf{x}_{\text{ZP}} + \mathbf{n}, \tag{11}$$

with $M = N + L$. The ZFE in the frequency domain can be written as:

$$\mathbf{W}_{\text{ZFE-FD-EXT}} = \left[\mathbf{I}_{\text{N}} \,\middle|\, \mathbf{O}\right] \boldsymbol{\mathcal{I}}_{\text{M}} \mathbf{\Lambda}_{\text{e}}^{-1} \boldsymbol{\mathcal{F}}_{\text{M}} \tag{12}$$

The inversion of the diagonal matrix in this equation requires $\mathcal{O}(M)$ flops, whereas the calculation of the estimate for $\mathbf{x}$ requires $\mathcal{O}(M\log(M))$ flops per block.

Note that, in case of spectral zeros, $\mathbf{\Lambda}_{\text{e}}$ may be a rank-deficient matrix. Again, this can be dealt with by using the MMSE equalizer,

$$\mathbf{W}_{\text{MMSE-FD-EXT}} = \left[\mathbf{I}_{\text{N}} \,\middle|\, \mathbf{O}\right] \boldsymbol{\mathcal{I}}_{\text{M}} \left(\mathbf{\Lambda}_{\text{e}}^{\mathcal{H}} \mathbf{\Lambda}_{\text{e}} + \sigma_{\text{n}}^2 \mathbf{I}_{\text{M}}\right)^{-1} \mathbf{\Lambda}_{\text{e}}^{\mathcal{H}} \boldsymbol{\mathcal{F}}_{\text{M}} \tag{13}$$

At a theoretical level however, it is strikingly curious that the assured possibility of zero-forcing equalization is lost in going from eq.(3) to eq.(12), because the number of equations is unchanged (contrary to the folding method, where information is discarded). Actually the problem arises in eq.(10), where the matrix $\mathbf{H}_{\text{e}}$ may not be of full rank. In the time domain however $\mathbf{H}_{\text{e}}$ needs not be of full rank because the last L elements of $\mathbf{x}_{\text{ZP}}$ are known to be zero. In the frequency domain this additional information

[3]Note that only the first L zeros of the ZP are taken into account. Without loss of generality it can be assumed that $P = L$.

is unavailable. In the next section this problem is solved by feeding back this known information and fixing the possible gaps in $\mathbf{\Lambda}_{\mathrm{e}}$.

III. Improved frequency domain equalization

A. Frequency domain ZFE with zero restoration

If $\mathbf{\Lambda}_{\mathrm{e}}$ is invertible the solutions from eqs. (12) or (13) can be used. If not, assume V diagonal elements of $\mathbf{\Lambda}_{\mathrm{e}}$ are non-zero while the remaining $K = M - V$ elements are zero. Now rearrange the diagonal elements of $\mathbf{\Lambda}_{\mathrm{e}}$ with the K zero elements at the end. Rearrange the rows of the DFT-matrix $\boldsymbol{\mathcal{F}}_{\mathrm{M}}$ in a likewise manner and call the result $\mathbf{F}$. The matrix $\mathbf{H}_{\mathrm{e}}$ can now be diagonalized as

$$\mathbf{H}_{\mathrm{e}} = \underbrace{\left[\begin{array}{c|c} \mathbf{F}_{\mathrm{A}}^{\mathcal{H}} & \mathbf{F}_{\mathrm{C}}^{\mathcal{H}} \\ \hline \mathbf{F}_{\mathrm{B}}^{\mathcal{H}} & \mathbf{F}_{\mathrm{D}}^{\mathcal{H}} \end{array}\right]}_{\mathbf{F}^{\mathcal{H}}} \left[\begin{array}{c|c} \mathbf{\Lambda}_{\mathrm{nz}} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O}_{\mathrm{K}} \end{array}\right] \underbrace{\left[\begin{array}{c|c} \mathbf{F}_{\mathrm{A}} & \mathbf{F}_{\mathrm{B}} \\ \hline \mathbf{F}_{\mathrm{C}} & \mathbf{F}_{\mathrm{D}} \end{array}\right]}_{\mathbf{F}}, \tag{14}$$

in which $\mathbf{\Lambda}_{\mathrm{nz}}$ holds the non-zero diagonal elements of $\mathbf{\Lambda}_{\mathrm{e}}$ and $\mathbf{F}_{\mathrm{A(V\times N)}}$, $\mathbf{F}_{\mathrm{B(V\times L)}}$, $\mathbf{F}_{\mathrm{C(K\times N)}}$ and $\mathbf{F}_{\mathrm{D(K\times L)}}$ are submatrices of $\mathbf{F}$. Because $\mathbf{F}^{\mathcal{H}}\mathbf{F} = \mathbf{I}$, certain relations hold between these submatrices, e.g. $\mathbf{F}_{\mathrm{B}}^{\mathcal{H}}\mathbf{F}_{\mathrm{A}} = -\mathbf{F}_{\mathrm{D}}^{\mathcal{H}}\mathbf{F}_{\mathrm{C}}$ and $\mathbf{F}_{\mathrm{A}}^{\mathcal{H}}\mathbf{F}_{\mathrm{A}} + \mathbf{F}_{\mathrm{C}}^{\mathcal{H}}\mathbf{F}_{\mathrm{C}} = \mathbf{I}$.

Because the diagonal matrix cannot be inverted, it is impossible to construct a ZFE directly. However, by disregarding the spectral zero frequencies and inverting only $\mathbf{\Lambda}_{\mathrm{nz}}$, it is possible to obtain a first approximation of $\mathbf{x}_{\mathrm{ZP}}$,

$$\hat{\mathbf{x}}_{\mathrm{ZP_{temp}}} = \mathbf{F}^{\mathcal{H}} \left[\begin{array}{c|c} \mathbf{\Lambda}_{\mathrm{nz}}^{-1} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right] \mathbf{F}\mathbf{y} \tag{15}$$

$$= \left[\begin{array}{c} \mathbf{F}_{\mathrm{A}}^{\mathcal{H}} \\ \hline \mathbf{F}_{\mathrm{B}}^{\mathcal{H}} \end{array}\right] \mathbf{\Lambda}_{\mathrm{nz}}^{-1} \left[\begin{array}{c|c} \mathbf{F}_{\mathrm{A}} & \mathbf{F}_{\mathrm{B}} \end{array}\right] \mathbf{y}, \tag{16}$$

Note that for high SNR values, this converges to the MMSE-solution of eq.(13). This approximation of $\mathbf{x}_{\mathrm{ZP}}$ can be improved by adding a correction term which lies in the column space of the last K columns of $\mathbf{F}^{\mathcal{H}}$, and which can be written as:

$$\hat{\mathbf{x}}_{\mathrm{ZP_{corr}}} = \left[\begin{array}{c} \mathbf{F}_{\mathrm{C}}^{\mathcal{H}} \\ \hline \mathbf{F}_{\mathrm{D}}^{\mathcal{H}} \end{array}\right] \mathbf{q} \tag{17}$$

The correction term can be determined by taking into account that the last L elements of $\hat{\mathbf{x}}_{\mathrm{ZP}}$ should be zero. This leads to a system of L equations in K unknowns, which can be solved exactly if no noise is present. In practice the correction term can be calculated from the following least-squares (LS) system:

$$\mathbf{F}_{\mathrm{D}}^{\mathcal{H}}\mathbf{q} + \left[\begin{array}{c|c} \mathbf{O} & \mathbf{I}_{\mathrm{L}} \end{array}\right] \hat{\mathbf{x}}_{\mathrm{ZP_{temp}}} \overset{LS}{=} \mathbf{O}_{\mathrm{L}\times 1}, \tag{18}$$

the solution of which is given by:

$$\mathbf{q} = -(\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{D}}^{\mathcal{H}})^{-1}\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{B}}^{\mathcal{H}}\mathbf{\Lambda}_{\mathrm{nz}}^{-1}\left[\begin{array}{c|c}\mathbf{F}_{\mathrm{A}} & \mathbf{F}_{\mathrm{B}}\end{array}\right]\mathbf{y}. \tag{19}$$

The overall expression for $\hat{\mathbf{x}}$ therefore becomes:

$$\begin{aligned}\hat{\mathbf{x}} &= \left[\begin{array}{c|c}\mathbf{I}_N & \mathbf{O}\end{array}\right]\left(\hat{\mathbf{x}}_{\mathrm{ZP_{temp}}} + \hat{\mathbf{x}}_{\mathrm{ZP_{corr}}}\right) \\ &= \underbrace{(\mathbf{F}_{\mathrm{A}}^{\mathcal{H}} - \mathbf{F}_{\mathrm{C}}^{\mathcal{H}}(\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{D}}^{\mathcal{H}})^{-1}\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{B}}^{\mathcal{H}})\mathbf{\Lambda}_{\mathrm{nz}}^{-1}\left[\begin{array}{c|c}\mathbf{F}_{\mathrm{A}} & \mathbf{F}_{\mathrm{B}}\end{array}\right]}_{\mathbf{W}_{\text{ZFE-ZR}}}\mathbf{y}\end{aligned} \tag{20}$$

This equation defines the ZFE-ZR, where ZR refers to *zero restoration*. It can now be shown that $\mathbf{W}_{\text{ZFE-ZR}}$ is indeed a ZFE,

$$\begin{aligned}&\mathbf{W}_{\text{ZFE-ZR}}\mathbf{H} \\ &\quad= \mathbf{W}_{\text{ZFE-ZR}}\mathbf{H}_{\mathrm{e}}\left[\begin{array}{c}\mathbf{I}\\\hline\mathbf{O}\end{array}\right] \\ &\quad= (\mathbf{F}_{\mathrm{A}}^{\mathcal{H}} - \mathbf{F}_{\mathrm{C}}^{\mathcal{H}}(\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{D}}^{\mathcal{H}})^{-1}\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{B}}^{\mathcal{H}})\mathbf{\Lambda}_{\mathrm{nz}}^{-1}\left[\begin{array}{c|c}\mathbf{F}_{\mathrm{A}} & \mathbf{F}_{\mathrm{B}}\end{array}\right]\cdot \\ &\qquad\left[\begin{array}{c|c}\mathbf{F}_{\mathrm{A}}^{\mathcal{H}} & \mathbf{F}_{\mathrm{C}}^{\mathcal{H}}\\\hline\mathbf{F}_{\mathrm{B}}^{\mathcal{H}} & \mathbf{F}_{\mathrm{D}}^{\mathcal{H}}\end{array}\right]\left[\begin{array}{c|c}\mathbf{\Lambda}_{\mathrm{nz}} & \mathbf{O}\\\hline\mathbf{O} & \mathbf{O}\end{array}\right]\left[\begin{array}{c|c}\mathbf{F}_{\mathrm{A}} & \mathbf{F}_{\mathrm{B}}\\\hline\mathbf{F}_{\mathrm{C}} & \mathbf{F}_{\mathrm{D}}\end{array}\right]\left[\begin{array}{c}\mathbf{I}\\\hline\mathbf{O}\end{array}\right] \\ &\quad= (\mathbf{F}_{\mathrm{A}}^{\mathcal{H}} - \mathbf{F}_{\mathrm{C}}^{\mathcal{H}}(\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{D}}^{\mathcal{H}})^{-1}\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{B}}^{\mathcal{H}})\cdot \\ &\qquad\underbrace{\mathbf{\Lambda}_{\mathrm{nz}}^{-1}\left[\begin{array}{c|c}\mathbf{F}_{\mathrm{A}} & \mathbf{F}_{\mathrm{B}}\end{array}\right]\left[\begin{array}{c}\mathbf{F}_{\mathrm{A}}^{\mathcal{H}}\\\hline\mathbf{F}_{\mathrm{B}}^{\mathcal{H}}\end{array}\right]}_{\mathbf{I}}\mathbf{\Lambda}_{\mathrm{nz}}\,\mathbf{F}_{\mathrm{A}} \\ &\quad= \mathbf{F}_{\mathrm{A}}^{\mathcal{H}}\mathbf{F}_{\mathrm{A}} - \mathbf{F}_{\mathrm{C}}^{\mathcal{H}}(\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{D}}^{\mathcal{H}})^{-1}\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{B}}^{\mathcal{H}}\mathbf{F}_{\mathrm{A}} \\ &\quad= \mathbf{F}_{\mathrm{A}}^{\mathcal{H}}\mathbf{F}_{\mathrm{A}} + \mathbf{F}_{\mathrm{C}}^{\mathcal{H}}\underbrace{(\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{D}}^{\mathcal{H}})^{-1}\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{D}}^{\mathcal{H}}}_{\mathbf{I}}\mathbf{F}_{\mathrm{C}} \\ &\quad= \mathbf{I}\end{aligned}$$

While eq. (20) is interesting from a theoretical point of view, it offers no complexity gain compared to time-domain methods. Therefore the practical implementation of the ZFE-ZR starts with the calculation of $\hat{\mathbf{x}}_{\mathrm{ZP_{temp}}}$, having the same complexity as the evaluation of (12) or (13). Next the LS system from (18) is solved, which is typically very small. The calculation of $\mathbf{q}$ requires $(N + L + \frac{K}{3})K^2$ flops ([13], p 238) and evaluating $\hat{\mathbf{x}}_{\mathrm{ZP_{corr}}}$ uses another $(2K-1)N$ flops to multiply $\mathbf{q}$ and $\mathbf{F}_{\mathrm{C}}^{\mathcal{H}}$. The overall computational complexity is thus equal to the complexity of eq. (12) or (13) plus approximately $\mathcal{O}((K^2+2K)N)$ operations and is generally significantly smaller than the complexity of the time domain solutions of eq. (3) and (4).

B. Frequency domain MMSE-like equalization with zero restoration

All ZFE structures so far presented, with the exception of the ZFE-ZR, also have an MMSE equivalent which is known to perform better. It is therefore tempting to investigate if this also holds for the ZFE-ZR. The derivation of such a MMSE-ZR equalizer starts with an MMSE-equivalent of eq. (15):

$$\hat{\mathbf{x}}_{\mathrm{ZP_{temp,\ mmse}}} = \mathbf{F}^{\mathcal{H}} \left[\begin{array}{c|c} \left(\mathbf{\Lambda}_{\mathrm{nz}}^{\mathcal{H}}\mathbf{\Lambda}_{\mathrm{nz}} + \sigma_{\mathrm{n}}^2\mathbf{I}_{\mathrm{V}}\right)^{-1}\mathbf{\Lambda}_{\mathrm{nz}}^{\mathcal{H}} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right] \mathbf{F}\mathbf{y}. \tag{21}$$

It is easy to verify that this is exactly equal to the result obtained using the MMSE-FD-EXT from eq.(13). Following the same reasoning as for the ZFE-ZR, the 'MMSE-ZR' equalizer structure is obtained as:

$$\begin{aligned} \mathbf{W}_{\text{MMSE-ZR}} \;=\; & (\mathbf{F}_{\mathrm{A}}^{\mathcal{H}} - \mathbf{F}_{\mathrm{C}}^{\mathcal{H}}(\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{D}}^{\mathcal{H}})^{-1}\mathbf{F}_{\mathrm{D}}\mathbf{F}_{\mathrm{B}}^{\mathcal{H}}). \\ & \left(\mathbf{\Lambda}_{\mathrm{nz}}^{\mathcal{H}}\mathbf{\Lambda}_{\mathrm{nz}} + \sigma_{\mathrm{n}}^2\mathbf{I}_{\mathrm{V}}\right)^{-1}\mathbf{\Lambda}_{\mathrm{nz}}^{\mathcal{H}}\Big[\mathbf{F}_{\mathrm{A}}\Big|\,\mathbf{F}_{\mathrm{B}}\Big]. \end{aligned} \tag{22}$$

Intuitively it is reasonable to assume that this structure performs better than the ZFE-ZR, and this is also demonstrated by the simulation results below. The proof that this truly is a (constrained) MMSE structure, however, appears not to be straightforward and has not yet been established. What makes this equalizer particularly interesting is that it can be seen as an ordinary MMSE-FD-EXT complemented with a post-processing, typically requiring only very limited resources. As a practical consequence, existing implementations using the MMSE-FD-EXT can be reused. Based on the measurement of the actual channel, it can be decided to use the post-processing step or not.

C. Remarks

Before proceeding to the simulation results, some additional remarks are in place:

1) Threshold: Practical channel knowledge is imperfect because it is based on estimates and furthermore channel notches will never exactly be equal to zero. Therefore a decision needs to be made as to which elements of $\mathbf{\Lambda}_{\mathrm{e}}$ should be treated as zero, e.g. based on a threshold. This can be set at some percentage of the root mean square value of $\mathbf{h}$ or through some other method. In any case it needs to be such that the number of zeros K is not larger than P. A trade-off needs to be made: setting a low threshold and identifying too few 'zeros' can lead to noise enhancement. Setting a high threshold can lead to too many 'zeros', all of whose received information is discarded. In case of non-white noise this threshold should also take into account the SNR level at each frequency.

2) Condition: Related to the previous point, it should be avoided to select many consecutive elements of $\mathbf{\Lambda}_e$ as zeros because this leads to an LS system having consecutive rows of a DFT submatrix, which is known to be ill conditioned. Both points should lead to some heuristic algorithm to determine the selection of the elements which will be treated as zeros.

3) KSP: The zero-restoration equalizerscan also be extended to KSP-based transmission, either by subtracting the influence of the KSP at the receiver and proceeding as in the ZP case, or by replacing $\hat{\mathbf{x}}_{\text{ZP}}$ by $\hat{\mathbf{x}}_{\text{KSP}}$ and modifying the LS system such that the last P samples of $\hat{\mathbf{x}}_{\text{KSP}}$ equal the KSP.

IV. Simulations and discussion

Simulations have been done based on two channels, $h_1(n)$=[-0.3699-j0.5782 -0.4053-j0.5750 -0.0834-j0.0406 0.1587-j0.0156] and $h_2(n)$=[0.707 0 0 0.707]. These correspond to p_2 from [8] and channel A from [6] respectively. These channels are shown in **Fig. 1**. Note that $h_1(n)$ has a zero near the 29th subcarrier and $h_2(n)$ has a zero on the unit circle as well as two other zeros close to this first zero. The

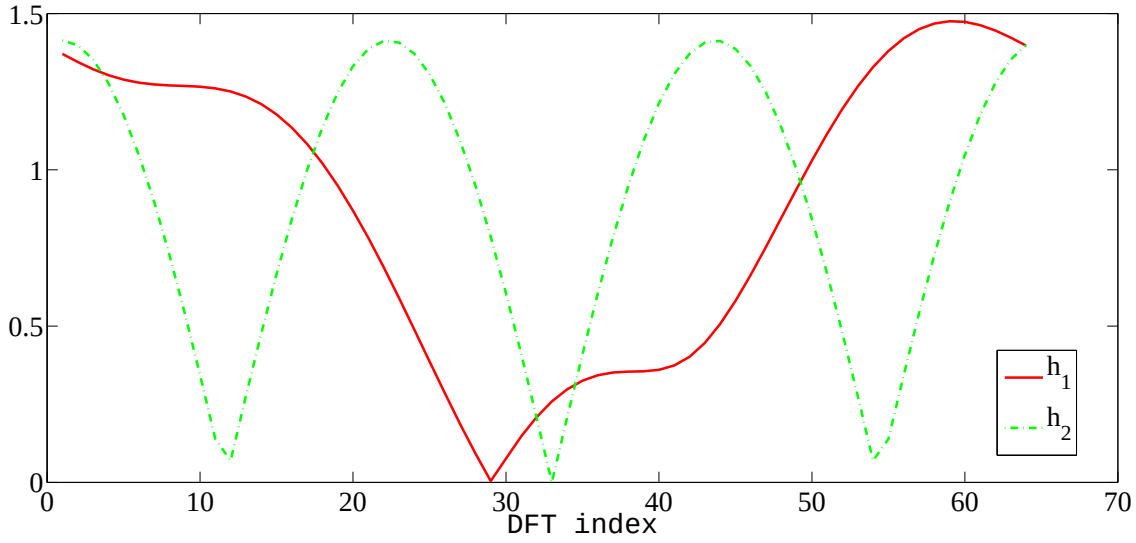

Fig. 1. Magnitude response for $h_1(n)$, $h_2(n)$ (from [8]) and $h_3(n)$ (from [6])

OFDM system in the original papers is replaced by ZP-SC with $M = 64$, $P = L$ and $N = 64 - P$. The noise is white Gaussian with different variances to match the signal-to-noise ratio (SNR). The BER performance is simulated using a Monte-Carlo method with 100000 random blocks, assuming a 16QAM system.

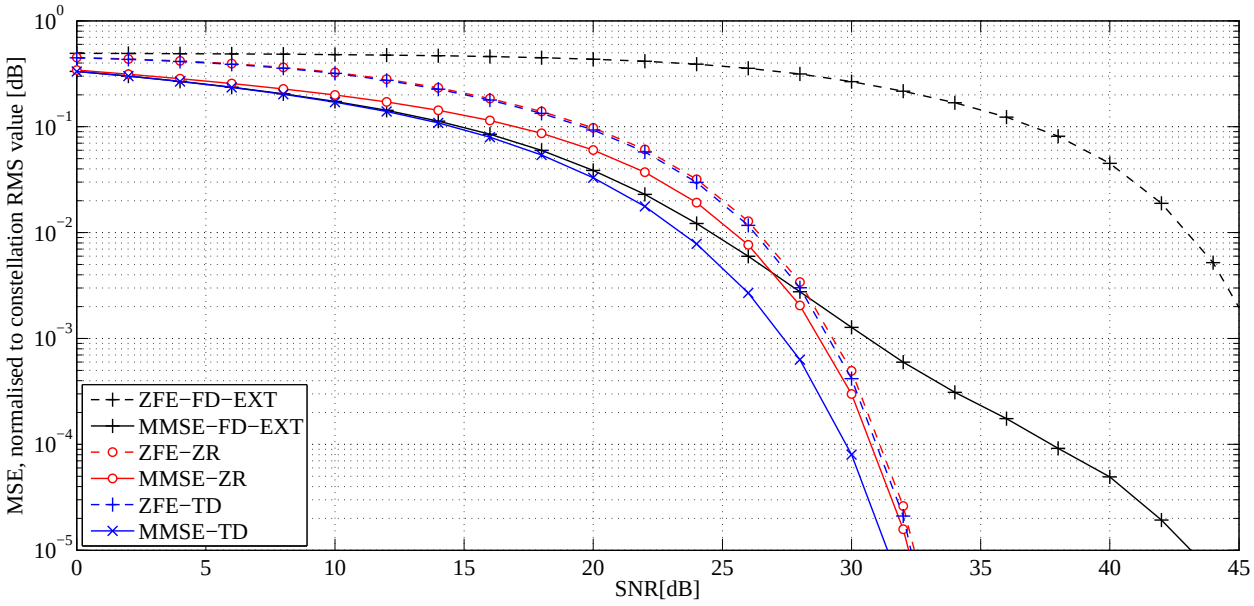

Fig. 2. Performance comparison for channel $h_1(n)$

For the first channel, **Fig. 2** indicates that limiting the noise at the notch frequency, as the MMSE-FD-EXT does, is a good strategy for low SNR. However, because of the noise enhancement at the notch, the MMSE-FD-EXT is unable to benefit fully from increasing SNR levels and at some point the ZFE-TD starts to perform better. Apparently the ZFE-ZR technique manages to stay very close to the ZFE-TD, which obviously forms a lower bound. As expected, the MMSE-ZR performs better than the ZFE-ZR, especially for low SNRs, and it even outperforms the ZFE-TD. Because the zero is not exactly on the unit circle, even the performance of the ZFE-FD-EXT improves as the SNR increases, albeit slower.

The results for channel $h_2(n)$ are shown in **Fig. 3**. Because of the spectral zero, the noise amplification for the ZFE-FD-EXT equals infinity and no information can be transferred, as reflected by a BER of 0.5 for all SNR values. The MMSE-FD-EXT is also severely impacted by the spectral zero. For moderate SNR values its performance is comparable to that of Fig. 2, but in contrast the performance now saturates for high SNR values. Indeed all information is lost at the spectral zero frequency and the best strategy is to discard the noise present at this frequency. Roughly speaking this implies that one M^{th} part of the signal power will be lost, resulting in an irreducible error, no matter how high the SNR. This is also apparent from the mean squared error curves in **Fig.4**, for which this limit would be at 64^{-1} or -18dB. The ZFE-ZR and MMSE-ZR on the other hand assume three zeros and reconstruct the information that

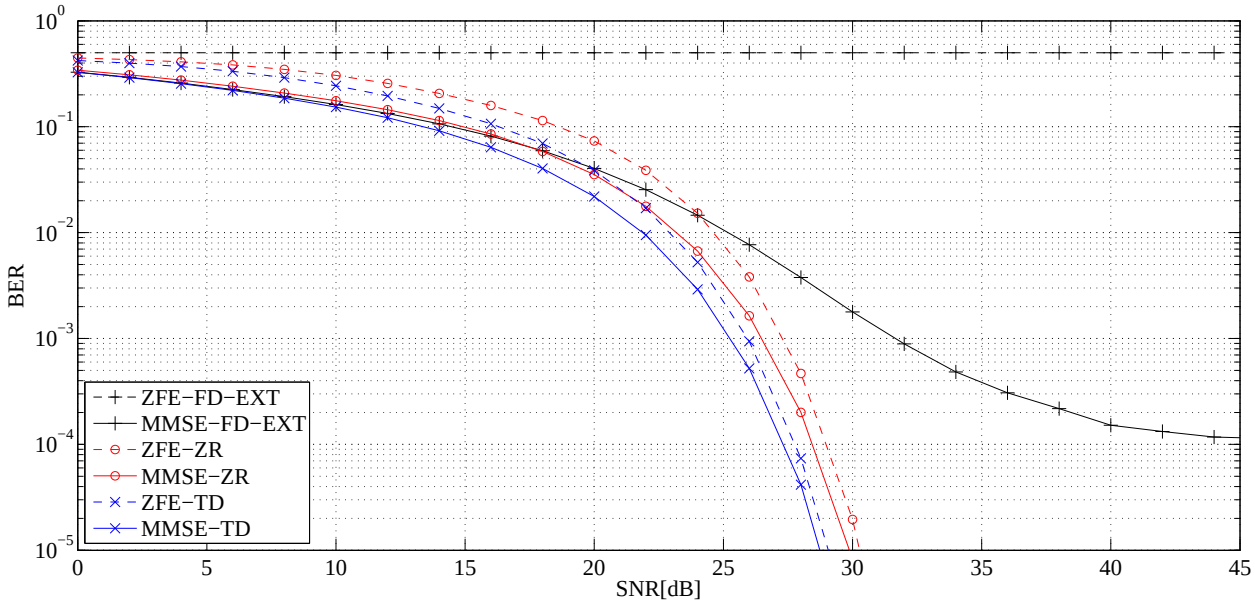

Fig. 3. Performance comparison for channel $h_2(n)$

was stored there. As the SNR levels increase, the estimation of the $\mathbf{q}$ vector in eq.(18) also improves and therefore this system does not show a saturation. These structures achieve a performance close to the ZFE-TD and the MMSE-TD, which are obviously not at all affected by the spectral zero.

V. Conclusion

Two new techniques have been proposed to equalize a ZP-SC system over a channel exhibiting zeros on or near the unit circle, by exploiting the redundancy of the ZP to restore the lost information. The first technique is shown to have the zero forcing property, the second one can be considered as a post processing for MMSE systems. Simulations show a performance comparable to time domain solutions, at a computational complexity comparable to a frequency domain techniques.

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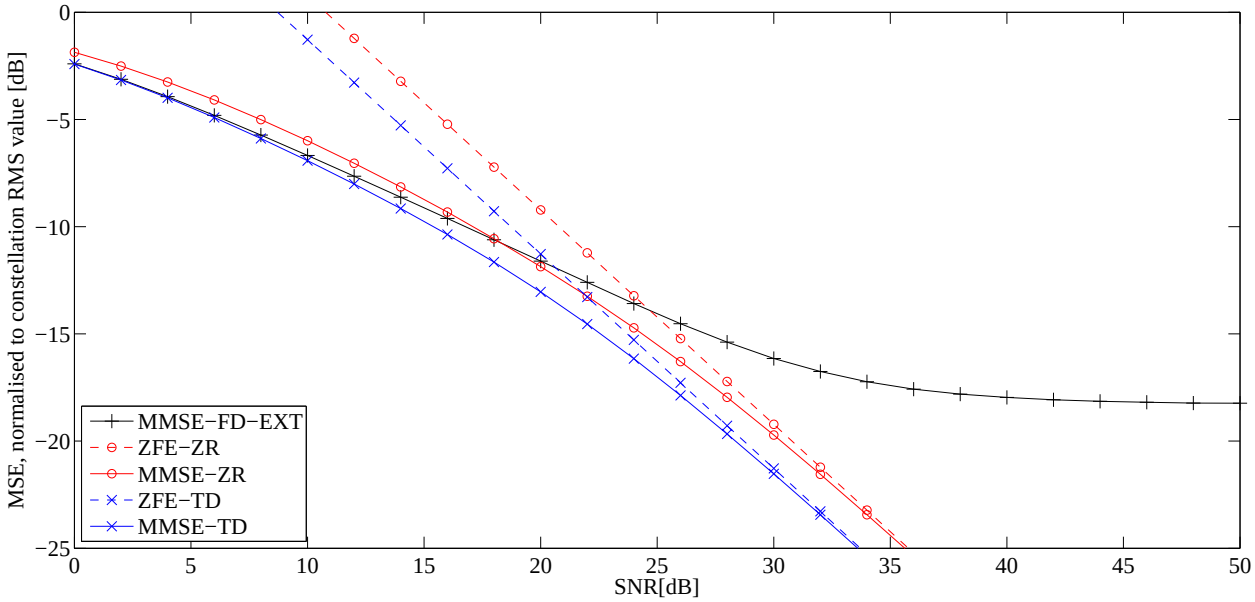

Fig. 4. Mean square error for channel $h_2(n)$

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