Solving Multivariate Polynomial Optimization Problems via Numerical Linear Algebra

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1 Introduction

Within system identification and model reduction, multivariate polynomial optimization problems emerge frequently, e.g., when determining the model parameters from input-output data or minimizing the error of a reduced-order model. In this research, we explore a new approach to solve these optimization problems, using numerical linear algebra techniques. Although many heuristic approaches exist, state-of-the-art algorithms, mostly based on nonlinear optimization, do not guarantee to converge to the globally optimal solution. Therefore, the main goal of our research is to solve these optimization problems exactly, i.e., to find the globally optimal solution. Some work has already been done by Dreesen et al. [1] and during this research we continue their efforts.

2 Research methodology

Multivariate polynomial optimization problems, as in Equation (1), are optimization problems for which the objective function $J(\mathbf{x}) \in \mathscr{P}(\mathbb{R})$ and constraints $g_i(\mathbf{x}) \in \mathscr{P}(\mathbb{R})$ are multivariate polynomial equations with real coefficients in the unknown optimization variable $\mathbf{x} \in \mathbb{C}^n$. Although the restrictions, these problems arise in numerous engineering applications, e.g., system identification, control theory, computer vision, robotics, and computational biology [1].

$$\inf_{\mathbf{x}} J(\mathbf{x}) \quad \text{s.t.} \begin{cases} g_1(\mathbf{x}) = 0 \\ \vdots \\ g_s(\mathbf{x}) = 0 \end{cases} \tag{1}$$

We tackle these optimization problems by embedding the objective function and the constraints, using Lagrange multipliers $\mu_i \in \mathbb{R}$, into a Lagrangian, of which the partial derivatives with respect to the components of $\mathbf{x}$ and μ lead to a (potentially large) system of multivariate polynomial equations [1]. This system corresponds to a set of difference equations (see Dreesen et al. [2]). Consequently, the kernel of the Macaulay matrix (constructed from the coefficients of the multivariate polynomials), which is spanned by Vandermonde-like vectors constructed from the roots, is a multidimensional observability matrix. Thus, we find these roots via the construction of an autonomous multidimensional (nD) linear state space model (realization theory), followed by a large eigenvalue calculation.

We do not need to find all the roots (which might be many, including at infinity), but only the optimal one. Therefore, we use a little trick. We define $M = \frac{1}{J(\mathbf{x})} \in \mathbb{R}$ and then add one extra equation to the system, namely $MJ(\mathbf{x}) = 1$. We can reformulate now, after some technical steps, the eigenvalue problem as to find the maximal eigenvalue M. This implies that the power method (or more advanced iterative eigenvalue solvers, e.g., Arnoldi and Lanczos [3]) can be used to find, in an iterative way, the optimizing root of our original multivariate polynomial optimization problem.

3 Presentation outline

In our presentation, we will elaborate on how one can set up, from the Macaulay matrix, an eigenvalue problem, of which one of the eigenvalues is the optimal value of the objective function. Furthermore, we will develop two algorithms to find this optimal solution. The first one starts from the kernel of the Macaulay matrix, the second one (or dual approach) works directly on the columns of the Macaulay matrix.

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References

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