Multi-parameter Eigenvalue Problems in Systems and Control

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June 1, 2022

Introduction



Figure 1: System identification and model reduction in practice.

Outline

1 | Multi-parameter eigenvalue problem

2 | Block Macaulay matrix approach

3 | Applications in systems and control

4 | Conclusions and future work

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Multi-parameter eigenvalue problem (MEVP)

Multi-parameter eigenvalue problem

$$\sum_{i=0}^{s} A_i m_i\left(\lambda_1, \ldots, \lambda_n\right) z = 0,$$

with monomials $m_i\left(\lambda_1, \ldots, \lambda_n\right) = \lambda_1^{k_1} \cdots \lambda_n^{k_n}$

k_j : degree

λ_j : eigenvalue

z : eigenvector

Multi-parameter eigenvalue problem (MEVP)

Standard eigenvalue problem	Multi-parameter eigenvalue problem
$Az = \lambda z$	$A_0 z = \sum_{i=1}^{s} A_i m_i \left(\lambda_1, \ldots, \lambda_n\right) z$
$\left(A - I\lambda\right) z = 0$	$\sum_{i=0}^{s} A_i m_i \left(\lambda_1, \ldots, \lambda_n\right) z = 0$
one eigenvalue	multiple eigenvalues

Multi-parameter eigenvalue problem (MEVP)

Example: quadratic multi-parameter eigenvalue problem

$$\left(A_0 + A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_1^2 + A_4\lambda_1\lambda_2 + A_5\lambda_2^2\right) z = 0$$

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Block Macaulay matrix

Quadratic MEVP

$$\left(A_0 + A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_1^2 + A_4\lambda_1\lambda_2 + A_5\lambda_2^2\right) z = 0$$

Block Macaulay matrix

Quadratic MEVP

$$\left(A_0 + A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_1^2 + A_4\lambda_1\lambda_2 + A_5\lambda_2^2\right) z = 0$$

$\hookrightarrow$ Multiply with the eigenvalues λ_1 and λ_2

$$\left(A_0 + A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_1^2 + A_4\lambda_1\lambda_2 + A_5\lambda_2^2\right) z\lambda_1 = 0$$
$$\left(A_0 + A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_1^2 + A_4\lambda_1\lambda_2 + A_5\lambda_2^2\right) z\lambda_2 = 0$$
$$\left(A_0 + A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_1^2 + A_4\lambda_1\lambda_2 + A_5\lambda_2^2\right) z\lambda_1^2 = 0$$
$$\vdots$$

Block Macaulay matrix

Quadratic MEVP

Block Macaulay matrix $M(d)$

$$M(d) = \begin{bmatrix} z & z\lambda_1 & z\lambda_2 & z\lambda_1^2 & z\lambda_1\lambda_2 & z\lambda_2^2 & z\lambda_1^3 & \\ A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & 0 & \cdots \\ 0 & A_0 & 0 & A_1 & A_2 & 0 & A_3 & \cdots \\ 0 & 0 & A_0 & 0 & A_1 & A_2 & 0 & \cdots \\ 0 & 0 & 0 & A_0 & 0 & 0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Block Macaulay matrix

Quadratic MEVP

Rewrite the MEVP using the block Macaulay matrix

$$\begin{bmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & 0 & \cdots \\ 0 & A_0 & 0 & A_1 & A_2 & 0 & A_3 & \cdots \\ 0 & 0 & A_0 & 0 & A_1 & A_2 & 0 & \cdots \\ 0 & 0 & 0 & A_0 & 0 & 0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} z \\ \hline z\lambda_1 \\ z\lambda_2 \\ \hline z\lambda_1^2 \\ z\lambda_1\lambda_2 \\ z\lambda_2^2 \\ \hline z\lambda_1^3 \\ \vdots \end{bmatrix} = 0$$

Block Macaulay matrix

Quadratic MEVP

Rewrite the MEVP using the block Macaulay matrix

$$\begin{bmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & 0 & \cdots \\ 0 & A_0 & 0 & A_1 & A_2 & 0 & A_3 & \cdots \\ 0 & 0 & A_0 & 0 & A_1 & A_2 & 0 & \cdots \\ 0 & 0 & 0 & A_0 & 0 & 0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} z \\ \hline z\lambda_1 \\ z\lambda_2 \\ \hline z\lambda_1^2 \\ z\lambda_1\lambda_2 \\ z\lambda_2^2 \\ \hline z\lambda_1^3 \\ \vdots \end{bmatrix} = 0$$

The null space appears to have a special structure

Block Macaulay matrix

- Intimate link with the classical Macaulay matrix
- Build recursively from the MEVP
- Its null space appears to have a special structure

Block multi-shift-invariant null space

Quadratic MEVP

- Solutions generate vectors in the null space of M

$$MK = 0$$

- Corank corresponds to the number of solutions m_b

$$m_b = \prod_{i=1}^{n} d_i$$

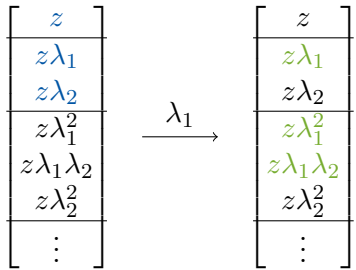
- Null space has a **block multi-shift-invariant** structure

Block multivariate Vandermonde basis K

$$\left[\begin{array}{ccc} z & \cdots & z \\ \hline z\lambda_1 & \cdots & z\lambda_1 \\ z\lambda_2 & \cdots & z\lambda_2 \\ \hline z\lambda_1^2 & \cdots & z\lambda_1^2 \\ z\lambda_1\lambda_2 & \cdots & z\lambda_1\lambda_2 \\ z\lambda_2^2 & \cdots & z\lambda_2^2 \\ \hline z\lambda_1^3 & \cdots & z\lambda_1^3 \\ \vdots & \vdots & \vdots \end{array}\right]$$

Multidimensional realization theory

Quadratic MEVP



$$S_1 k \lambda_1 = S_2 k$$

Multidimensional realization theory

Quadratic MEVP

$$\begin{bmatrix} z & \cdots & z \\ \hline z\lambda_1 & \cdots & z\lambda_1 \\ z\lambda_2 & \cdots & z\lambda_2 \\ \hline z\lambda_1^2 & \cdots & z\lambda_1^2 \\ z\lambda_1\lambda_2 & \cdots & z\lambda_1\lambda_2 \\ z\lambda_2^2 & \cdots & z\lambda_2^2 \\ \hline z\lambda_1^3 & \cdots & z\lambda_1^3 \\ \vdots & \vdots & \vdots \end{bmatrix} \xrightarrow{\lambda_1} \begin{bmatrix} z & \cdots & z \\ \hline z\lambda_1 & \cdots & z\lambda_1 \\ z\lambda_2 & \cdots & z\lambda_2 \\ \hline z\lambda_1^2 & \cdots & z\lambda_1^2 \\ z\lambda_1\lambda_2 & \cdots & z\lambda_1\lambda_2 \\ z\lambda_2^2 & \cdots & z\lambda_2^2 \\ \hline z\lambda_1^3 & \cdots & z\lambda_1^3 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$S_1 K D_{\lambda_1} = S_2 K$$

Multidimensional realization theory

Quadratic MEVP

Consider m_a affine and simple solutions

Realization theory for the unknown λ_1:

$$S_1 K D_{\lambda_1} = S_2 K,$$

where S_1 and S_2 select blocks from K

- Generalized eigenvalue problem, with I the matrix of eigenvectors
- $S_1 K$ has to be of full rank for the eigenvalue problem to be not degenerate:

Select m_a linear independent rows from K

Multidimensional realization theory

Quadratic MEVP

$$S_1 K D_{\lambda_1} = S_2 K$$

- In practice, the solutions are not known in advance
- Consider a numerical null space Z (T is a non-singular matrix)

$$K = ZT$$

- This results in

$$(S_1 Z)\, T D_{\lambda_1} = (S_2 Z)\, T$$

Multidimensional realization theory

Quadratic MEVP

Realization theory for the unknown λ_1:

$$(S_1 Z)\, T D_{\lambda_1} = (S_2 Z)\, T,$$

where S_1 and S_2 select blocks from Z

- Generalized eigenvalue problem, with T the matrix of eigenvectors
- We can rewrite this as a standard eigenvalue problem

$$T D_{\lambda_1} T^{-1} = (S_1 Z)^{\dagger} (S_2 Z)$$

Multidimensional realization theory

Quadratic MEVP

- It is possible to shift with λ_2 as well

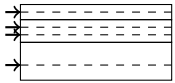
$$(S_1 Z)\, T D_{\lambda_2} = (S_2 Z)\, T,$$

- This leads to the same eigenvectors T

Multidimensional realization theory

1| Solutions at infinity

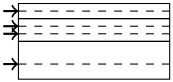
$d = 3$



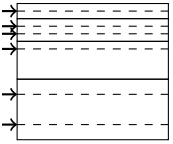
Multidimensional realization theory

1| Solutions at infinity

$d = 3$

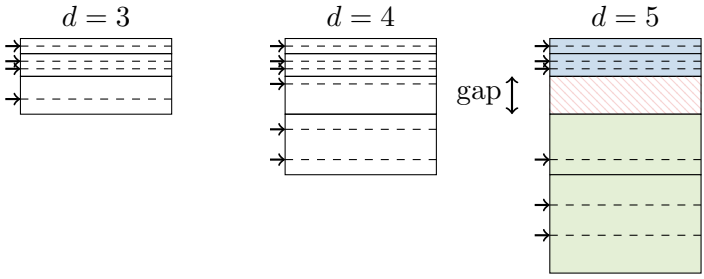


$d = 4$



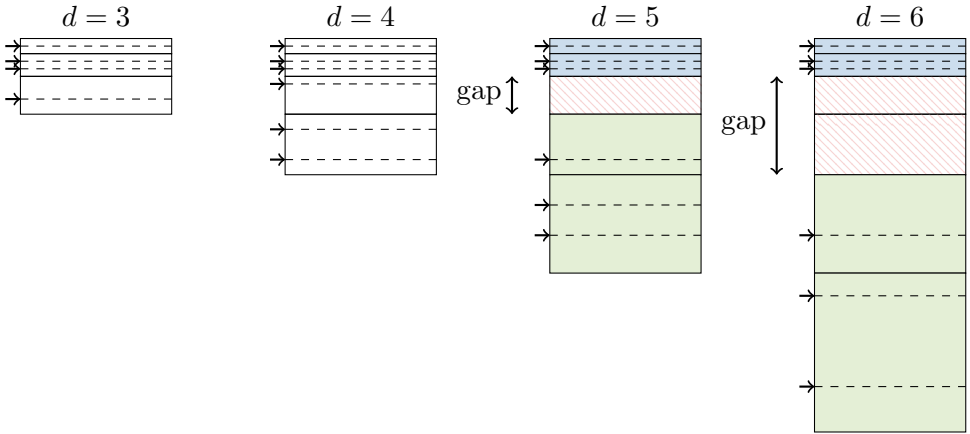
Multidimensional realization theory

1| Solutions at infinity



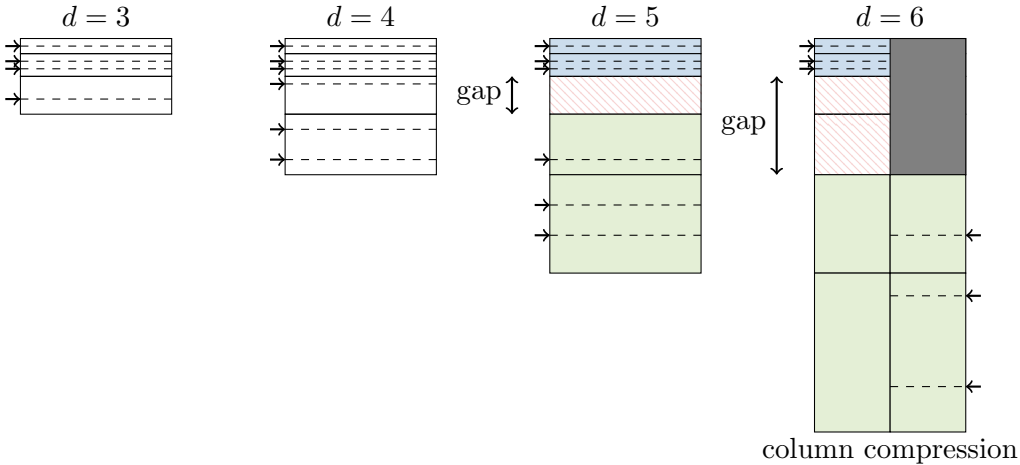
Multidimensional realization theory

1| Solutions at infinity



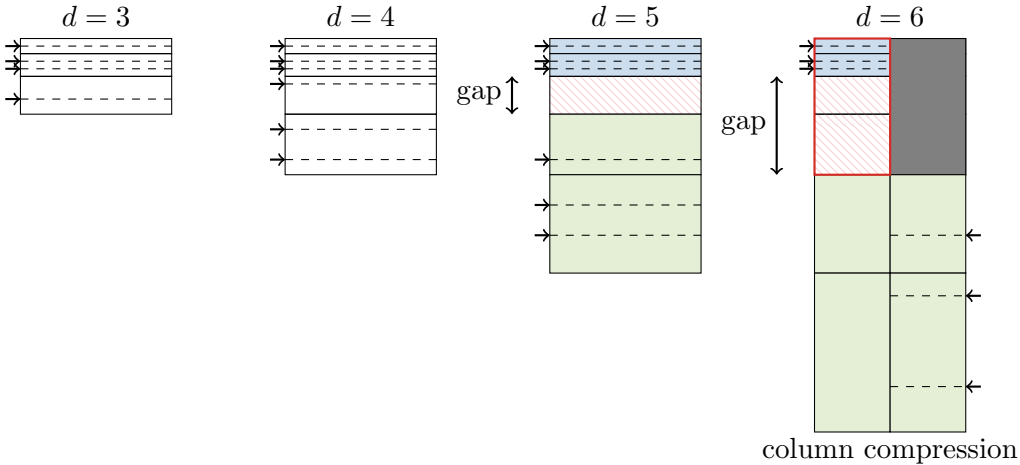
Multidimensional realization theory

1| Solutions at infinity



Multidimensional realization theory

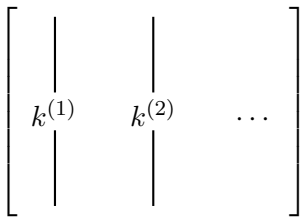
1| Solutions at infinity



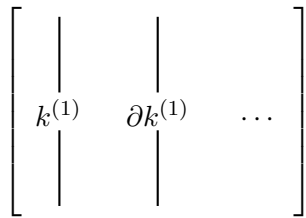
Multidimensional realization theory

2| Multiplicity higher than one

Simple roots



Multiple roots



Outline

1 | Multi-parameter eigenvalue problem

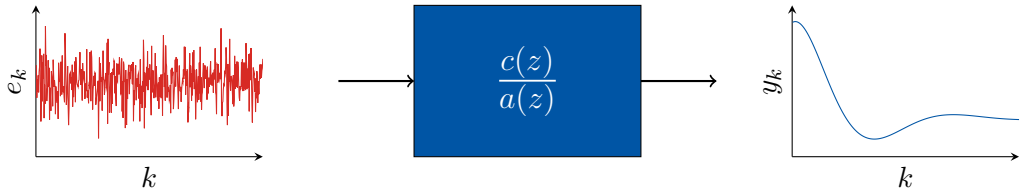
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Identification of linear time-invariant models

Autoregressive moving-average models



$$\sum_{i=0}^{n_a} \alpha_i y_{k-i} = \sum_{j=0}^{n_c} \gamma_j e_{k-j}$$

Identification of linear time-invariant models

Autoregressive moving-average models

$$\min_{a,c,e} \|e\|_2^2$$

$$\text{subject to } T_a y = T_c e$$

Identification of linear time-invariant models

Autoregressive moving-average models

A simple first order ARMA(1, 1) model

$$\underbrace{\begin{bmatrix} y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & y^{\mathrm{T}}T_a^{\alpha\mathrm{T}} & 0 \\ 0 & y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & 0 \\ D_c & 0 & 0 & T_a^{\alpha}y \\ 0 & D_c & D_c^{\gamma} & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix}}_{A(\alpha,\gamma)} \underbrace{\begin{bmatrix} f^{\alpha} \\ f^{\gamma} \\ f \\ -1 \end{bmatrix}}_{z} = 0$$

Identification of linear time-invariant models

Autoregressive moving-average models

A simple first order ARMA(1, 1) model

$$\underbrace{\begin{bmatrix} y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & y^{\mathrm{T}}T_a^{\alpha\mathrm{T}} & 0 \\ 0 & y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & 0 \\ D_c & 0 & 0 & T_a^{\alpha}y \\ 0 & D_c & D_c^{\gamma} & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix}}_{A(\alpha,\gamma)} \underbrace{\begin{bmatrix} f^{\alpha} \\ f^{\gamma} \\ f \\ -1 \end{bmatrix}}_{z} = 0$$

$$\sum_{i=0}^{s} A_i m_i\left(\alpha,\gamma\right) z = 0$$

Identification of linear time-invariant models

Autoregressive moving-average models

A simple first order ARMA(1, 1) model

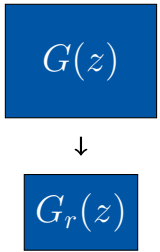
$$\underbrace{\begin{bmatrix} y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & y^{\mathrm{T}}T_a^{\alpha\mathrm{T}} & 0 \\ 0 & y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & 0 \\ D_c & 0 & 0 & T_a^{\alpha}y \\ 0 & D_c & D_c^{\gamma} & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix}}_{A(\alpha,\gamma)} \underbrace{\begin{bmatrix} f^{\alpha} \\ f^{\gamma} \\ f \\ -1 \end{bmatrix}}_{z} = 0$$

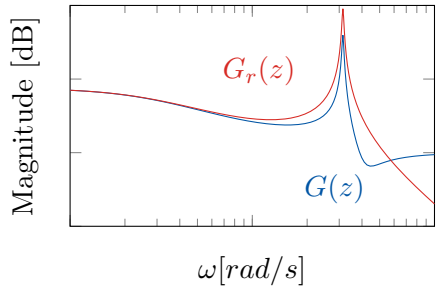
$$\sum_{i=0}^{s} A_i m_i\left(\alpha,\gamma\right) z = 0$$

Eigenvector (z) and 2-tuple of eigenvalues (α and γ)

Model order reduction

Optimal $\mathcal{H}_2$-norm reduction





Model order reduction

Optimal $\mathcal{H}_2$-norm reduction

A simple reduced second order model

$$\begin{bmatrix} CT_1^{-1}H_iT_1^{-1}B & CT_1^{-1}H_iT_2^{-1}B \\ CT_2^{-1}H_iT_1^{-1}B & CT_2^{-1}H_iT_2^{-1}B \end{bmatrix} \begin{bmatrix} q_1^{(i)} \\ q_2^{(i)} \end{bmatrix} = 0$$

From interpolatory model reduction

Model order reduction

Optimal $\mathcal{H}_2$-norm reduction

A simple reduced second order model

$$\underbrace{\begin{bmatrix} -\mathbb{B} & \mathbb{T} & 0 \\ 0 & -H_i & \mathbb{T} \\ 0 & 0 & C \end{bmatrix}}_{A(\lambda_1,\lambda_2)} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}}_{z} = 0$$

Model order reduction

Optimal $\mathcal{H}_2$-norm reduction

A simple reduced second order model

$$\underbrace{\begin{bmatrix} -\mathbb{B} & \mathbb{T} & 0 \\ 0 & -H_i & \mathbb{T} \\ 0 & 0 & C \end{bmatrix}}_{A(\lambda_1,\lambda_2)} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}}_{z} = 0$$

$$\sum_{i=0}^{s} A_i m_i \left(\lambda_1, \lambda_2\right) z = 0$$

Model order reduction

Optimal $\mathcal{H}_2$-norm reduction

A simple reduced second order model

$$\underbrace{\begin{bmatrix} -\mathbb{B} & \mathbb{T} & 0 \\ 0 & -H_i & \mathbb{T} \\ 0 & 0 & C \end{bmatrix}}_{A(\lambda_1,\lambda_2)} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}}_{z} = 0$$

$$\sum_{i=0}^{s} A_i m_i\left(\lambda_1, \lambda_2\right) z = 0$$

Eigenvector (z) and 2-tuple of eigenvalues (λ_1 and λ_2)

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Conclusions and future work

- A new approach to **solve MEVPs**

Conclusions and future work

- A new approach to **solve MEVPs**
- With many applications in systems and control

Conclusions and future work

- A new approach to **solve MEVPs**
- With many applications in systems and control
- However, there's **no** such thing as a **free lunch**