# About (Rectangular) Multiparameter Eigenvalue Problems

42nd Benelux Meeting on Systems and Control

Christof Vermeersch, Sarthak De, and Bart De Moor

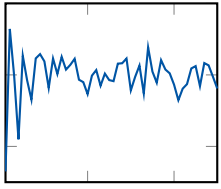
christof.vermeersch@esat.kuleuven.be

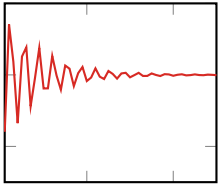



March 22, 2023

# Solving the least-squares realization problem

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}}_{\text{known}} \longrightarrow \underbrace{\begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_N \end{bmatrix}}_{\text{unknown}}$$





such that $\hat{\boldsymbol{y}}_k = \boldsymbol{C}\boldsymbol{A}^k\boldsymbol{x}_0$ is the output of an $n$th-order autonomous system

$$\min_{\hat{\boldsymbol{y}},\boldsymbol{\alpha}} \|\boldsymbol{y} - \hat{\boldsymbol{y}}\|_2^2$$

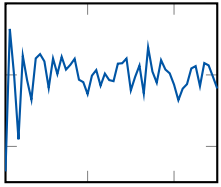
$$\text{subject to } \boldsymbol{T}_\alpha \hat{\boldsymbol{y}} = \boldsymbol{0}$$

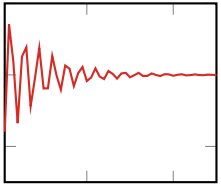
(De Moor, 2019, 2020)

# Solving the least-squares realization problem

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}}_{\text{known}} \longrightarrow \underbrace{\begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_N \end{bmatrix}}_{\text{unknown}}$$

$$\boldsymbol{T}_\alpha = \begin{bmatrix} \alpha_2 & \alpha_1 & 1 & 0 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 & 1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \alpha_1 & 1 & 0 \\ 0 & 0 & 0 & \alpha_2 & \alpha_1 & 1 \end{bmatrix}$$



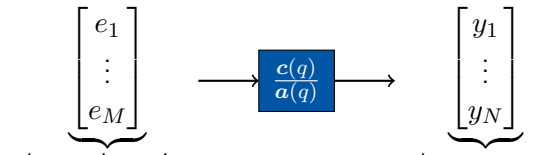


$$\min_{\hat{\boldsymbol{y}},\boldsymbol{\alpha}} \|\boldsymbol{y} - \hat{\boldsymbol{y}}\|_2^2$$

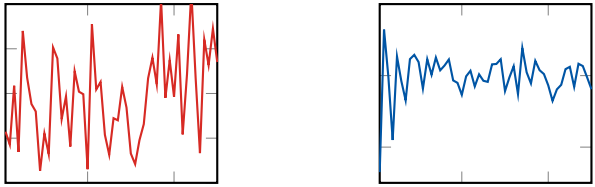
$$\text{subject to } \boldsymbol{T}_\alpha \hat{\boldsymbol{y}} = \boldsymbol{0}$$

(De Moor, 2019, 2020)

# Identifying the parameters of an ARMA model




such that $\sum_{i=0}^{n_a} \alpha_i y_{k-i} = \sum_{i=0}^{n_c} \gamma_i e_{k-i}$

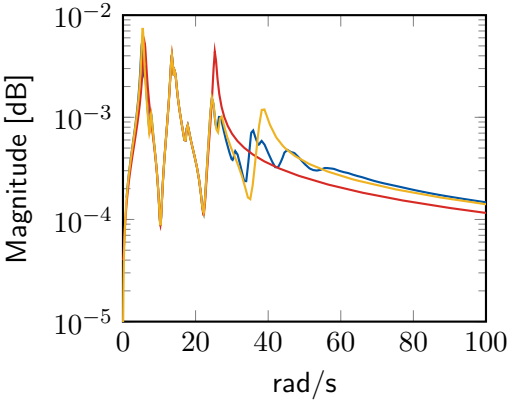


$$\min_{\boldsymbol{e},\boldsymbol{\alpha},\boldsymbol{\gamma}} \|\boldsymbol{e}\|_2^2$$

$$\text{subject to } \boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$$

(Vermeersch and De Moor, 2019)

# Model order reduction




Transfer function of a $45$th-order model (—) and two reduced-order approximations, namely $r = 18$ (—) and $r = 12$ (—)

(Alsubaie, 2019; Agudelo et al., 2021)

$$\mathcal{F}_{\mathsf{H}}(s) = \frac{a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}$$

$$\downarrow \text{with } r << n$$

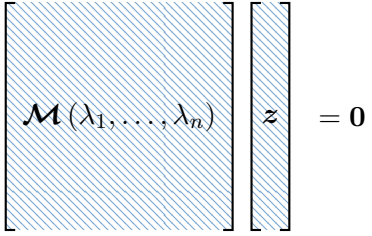
$$\mathcal{F}_{\mathsf{R}}(s) = \frac{\tilde{a}_{r-1}s^{r-1} + \cdots + \tilde{a}_1 s + \tilde{a}_0}{s^r + \tilde{b}_{r-1}s^{r-1} + \cdots + \tilde{b}_1 s + \tilde{b}_0}$$

$$\min_{\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}}} \|\mathcal{E}(s)\|_{\mathcal{H}_2}^2$$

$$\text{subject to } \mathcal{E}(s) = \mathcal{F}_{\mathsf{H}}(s) - \mathcal{F}_{\mathsf{R}}(s)$$

# Common problem!

These three problems have one thing in common: they are **rectangular multiparameter eigenvalue problems***!




But, there are many other problems that also fit into this framework: vibration analysis, prediction error methods, optimization in complex variables, etc.

* . . . and in a sense also one-parameter eigenvalue problems.
(Batselier et al., 2012; Tisseur and Meerbergen, 2001; Vermeersch et al., 2023)

# Outline

1 | Introduction

2 | Multiparameter Eigenvalue Problem

3 | Solution Approaches

4 | Numerical Example

5 | Conclusion and Future Work

# Outline

1 | Introduction

2 | Multiparameter Eigenvalue Problem

3 | Solution Approaches

4 | Numerical Example

5 | Conclusion and Future Work

# Multiparameter eigenvalue problem

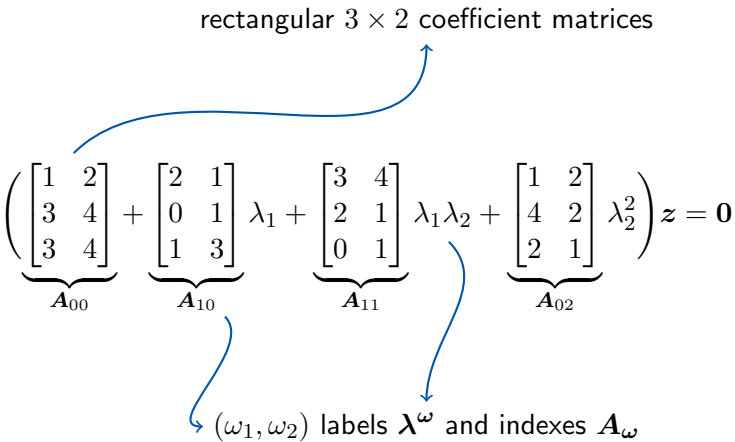
The **rectangular multiparameter eigenvalue problem (rectangular MEP)** consists in finding all $n$-tuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and corresponding vectors $\boldsymbol{z} \in \mathbb{C}^{l \times 1} \setminus \{\boldsymbol{0}\}$, so that

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\, \boldsymbol{z} = \left( \sum_{\{\boldsymbol{\omega}\}} \boldsymbol{A}_{\boldsymbol{\omega}} \boldsymbol{\lambda}^{\boldsymbol{\omega}} \right) \boldsymbol{z} = \boldsymbol{0}.$$

- integer multi-index $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$
- rectangular coefficient matrices $\boldsymbol{A}_{\boldsymbol{\omega}} \in \mathbb{C}^{k \times l}$ (with $k \geq l + n - 1$)
- full normal column rank matrix pencil $\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})$
- eigentuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and eigenvectors $\boldsymbol{z}$ (with $\|\boldsymbol{z}\| = 1$)

(Vermeersch and De Moor, 2019, 2022)

# Example: polynomial two-parameter eigenvalue problem



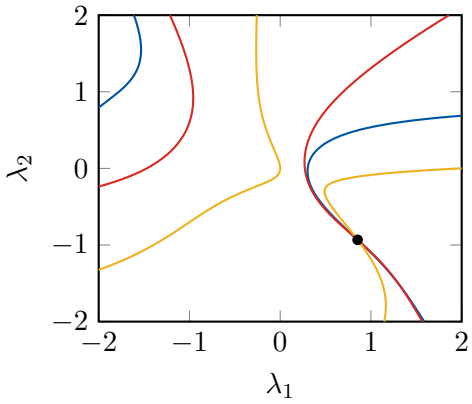
# Alternative formulation

We can also phrase the rectangular MEP by considering the eigentuples for which the rank drops below the normal rank of the rectangular matrix pencil.

Given a rectangular matrix pencil $\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})$, we say that the tuple $\boldsymbol{\lambda} \in \mathbb{C}$ is an eigentuple if

$$\operatorname{rank}(\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})) < \operatorname{nrank}(\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})).$$

# Example: polynomial two-parameter eigenvalue problem



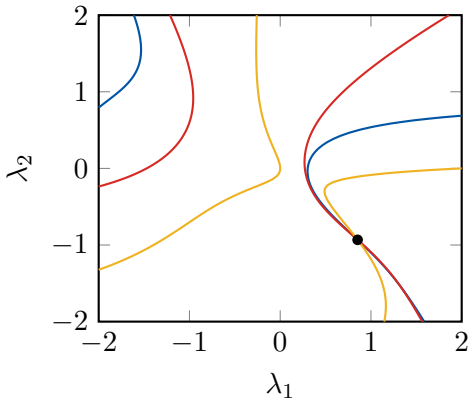

Real picture of the variety of the determinants $p_1(\boldsymbol{\lambda})$ (——), $p_2(\boldsymbol{\lambda})$ (——), and $p_3(\boldsymbol{\lambda})$ (——), which contains $9$ affine points (only one of which is real) and $3$ points at infinity

$$\boldsymbol{D}_1 \uparrow$$

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda}) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix}$$

$$p_i(\boldsymbol{\lambda}) = \det(\boldsymbol{D}_i) = 0$$
$$i = 1, 2, 3$$

# Example: polynomial two-parameter eigenvalue problem

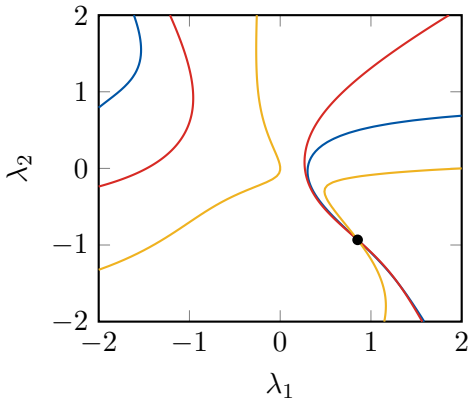



Real picture of the variety of the determinants $p_1(\boldsymbol{\lambda})$ (——), $p_2(\boldsymbol{\lambda})$ (——), and $p_3(\boldsymbol{\lambda})$ (——), which contains $9$ affine points (only one of which is real) and $3$ points at infinity

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda}) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix} \quad \boldsymbol{D}_2$$

$$p_i(\boldsymbol{\lambda}) = \det(\boldsymbol{D}_i) = 0$$
$$i = 1, 2, 3$$

# Example: polynomial two-parameter eigenvalue problem




Real picture of the variety of the determinants $p_1(\boldsymbol{\lambda})$ (——), $p_2(\boldsymbol{\lambda})$ (——), and $p_3(\boldsymbol{\lambda})$ (——), which contains $9$ affine points (only one of which is real) and $3$ points at infinity

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda}) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix} \rightarrow \boldsymbol{D}_3$$

$$p_i(\boldsymbol{\lambda}) = \det(\boldsymbol{D}_i) = 0$$
$$i = 1, 2, 3$$

# Unifying framework for (multiparameter) eigenvalue problems

Different types of (multiparameter) eigenvalue problems*

| Spectral parameter(s) | Linear | Polynomial |
|---|---|---|
| Eigenvalues ($n = 1$) | Type I | Type II |
| | $\{1, \lambda\}$ | $\lambda^{\omega}$ |
| | $\boldsymbol{A} - \boldsymbol{B}\lambda$ | $\boldsymbol{A}_0 + \boldsymbol{A}_1\lambda + \cdots + \boldsymbol{A}_d\lambda^d$ |
| | SEP/GEP | PEP |
| Eigentuples ($n > 1$) ($i = 1, \ldots, n$) | Type III | Type IV |
| | $\lambda_i$ | $\boldsymbol{\lambda}^{\boldsymbol{\omega}} = \prod_{i=1}^{n} \lambda_i^{\omega_i}$ |
| | $\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\lambda_1 + \boldsymbol{A}_{01}\lambda_2$ | $\boldsymbol{A}_{00} + \boldsymbol{A}_{11}\lambda_1\lambda_2 + \boldsymbol{A}_{03}\lambda_2^3$ |
| | linear MEP | polynomial MEP |

* SEP = standard eigenvalue problem – GEP = generalized eigenvalue problem – PEP = polynomial eigenvalue problem

# Square problems?

**Not a presentation about square multiparameter eigenvalue problems!**

Volkmer's square multiparameter eigenvalue problem (square MEP):

$$\begin{cases} \left( \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \lambda_2 \right) \boldsymbol{x}_1 = \mathbf{0} \\ \left( \begin{bmatrix} 20 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix} \lambda_1 + \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \lambda_2 \right) \boldsymbol{x}_2 = \mathbf{0} \end{cases}$$

There exist quite a few relations between both manifestations.

(Volkmer, 1988)

# Relations between square and rectangular problems

$$\begin{cases} \boldsymbol{\mathcal{W}}_1(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \boldsymbol{0} \\ \boldsymbol{\mathcal{W}}_2(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \boldsymbol{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1\boldsymbol{z} = \boldsymbol{\Delta}_0\lambda_1\boldsymbol{z} \\ \boldsymbol{\Delta}_2\boldsymbol{z} = \boldsymbol{\Delta}_0\lambda_2\boldsymbol{z} \end{cases} \Leftrightarrow \left( \begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Delta}_0 \\ \boldsymbol{0} \end{bmatrix} \lambda_1 - \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\Delta}_0 \end{bmatrix} \lambda_2 \right) \boldsymbol{z} = \boldsymbol{0}$$

**square to rectangular**

---

**rectangular to square**

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \boldsymbol{0} \Rightarrow \begin{cases} \boldsymbol{P}_1\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \boldsymbol{0} \\ \boldsymbol{P}_2\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \boldsymbol{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1\boldsymbol{y} = \boldsymbol{\Delta}_0\lambda_1\boldsymbol{y} \\ \boldsymbol{\Delta}_2\boldsymbol{y} = \boldsymbol{\Delta}_0\lambda_2\boldsymbol{y} \end{cases} \Rightarrow \begin{cases} \hat{\boldsymbol{\Delta}}_1\boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0\lambda_1\boldsymbol{z} \\ \hat{\boldsymbol{\Delta}}_2\boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0\lambda_2\boldsymbol{z} \end{cases}$$

(Vermeersch and De Moor, 2022; Hochstenbach et al., 2022)

# Relations between square and rectangular problems

associated system of coupled GEPs

$$\begin{cases} \boldsymbol{\mathcal{W}}_1(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \boldsymbol{0} \\ \boldsymbol{\mathcal{W}}_2(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \boldsymbol{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1\boldsymbol{z} = \boldsymbol{\Delta}_0\lambda_1\boldsymbol{z} \\ \boldsymbol{\Delta}_2\boldsymbol{z} = \boldsymbol{\Delta}_0\lambda_2\boldsymbol{z} \end{cases} \Leftrightarrow \left( \begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Delta}_0 \\ \boldsymbol{0} \end{bmatrix} \lambda_1 - \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\Delta}_0 \end{bmatrix} \lambda_2 \right) \boldsymbol{z} = \boldsymbol{0}$$

**square to rectangular**

---

**rectangular to square**

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \boldsymbol{0} \Rightarrow \begin{cases} \boldsymbol{P}_1\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \boldsymbol{0} \\ \boldsymbol{P}_2\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \boldsymbol{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1\boldsymbol{y} = \boldsymbol{\Delta}_0\lambda_1\boldsymbol{y} \\ \boldsymbol{\Delta}_2\boldsymbol{y} = \boldsymbol{\Delta}_0\lambda_2\boldsymbol{y} \end{cases} \Rightarrow \begin{cases} \hat{\boldsymbol{\Delta}}_1\boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0\lambda_1\boldsymbol{z} \\ \hat{\boldsymbol{\Delta}}_2\boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0\lambda_2\boldsymbol{z} \end{cases}$$

(Vermeersch and De Moor, 2022; Hochstenbach et al., 2022)

# Relations between square and rectangular problems

associated system of coupled GEPs

$$\begin{cases} \boldsymbol{\mathcal{W}}_1(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \boldsymbol{0} \\ \boldsymbol{\mathcal{W}}_2(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \boldsymbol{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1\boldsymbol{z} = \boldsymbol{\Delta}_0\lambda_1\boldsymbol{z} \\ \boldsymbol{\Delta}_2\boldsymbol{z} = \boldsymbol{\Delta}_0\lambda_2\boldsymbol{z} \end{cases} \Leftrightarrow \left( \begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Delta}_0 \\ \boldsymbol{0} \end{bmatrix} \lambda_1 - \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\Delta}_0 \end{bmatrix} \lambda_2 \right) \boldsymbol{z} = \boldsymbol{0}$$

**square to rectangular**

---

**rectangular to square**

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \boldsymbol{0} \Rightarrow \begin{cases} \boldsymbol{P}_1\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \boldsymbol{0} \\ \boldsymbol{P}_2\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \boldsymbol{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1\boldsymbol{y} = \boldsymbol{\Delta}_0\lambda_1\boldsymbol{y} \\ \boldsymbol{\Delta}_2\boldsymbol{y} = \boldsymbol{\Delta}_0\lambda_2\boldsymbol{y} \end{cases} \Rightarrow \begin{cases} \hat{\boldsymbol{\Delta}}_1\boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0\lambda_1\boldsymbol{z} \\ \hat{\boldsymbol{\Delta}}_2\boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0\lambda_2\boldsymbol{z} \end{cases}$$

randomized/structured sketching

(Vermeersch and De Moor, 2022; Hochstenbach et al., 2022)

# Relations between square and rectangular problems

associated system of coupled GEPs

$$\begin{cases} \boldsymbol{\mathcal{W}}_1(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \boldsymbol{0} \\ \boldsymbol{\mathcal{W}}_2(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \boldsymbol{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1\boldsymbol{z} = \boldsymbol{\Delta}_0\lambda_1\boldsymbol{z} \\ \boldsymbol{\Delta}_2\boldsymbol{z} = \boldsymbol{\Delta}_0\lambda_2\boldsymbol{z} \end{cases} \Leftrightarrow \left( \begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Delta}_0 \\ \boldsymbol{0} \end{bmatrix} \lambda_1 - \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\Delta}_0 \end{bmatrix} \lambda_2 \right) \boldsymbol{z} = \boldsymbol{0}$$

**square to rectangular**

---

**rectangular to square**

$$\boldsymbol{y} = \boldsymbol{x}_1 \otimes \boldsymbol{x}_2 \neq \boldsymbol{z}$$

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \boldsymbol{0} \Rightarrow \begin{cases} \boldsymbol{P}_1\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_1 = \boldsymbol{0} \\ \boldsymbol{P}_2\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{x}_2 = \boldsymbol{0} \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{\Delta}_1\boldsymbol{y} = \boldsymbol{\Delta}_0\lambda_1\boldsymbol{y} \\ \boldsymbol{\Delta}_2\boldsymbol{y} = \boldsymbol{\Delta}_0\lambda_2\boldsymbol{y} \end{cases} \Rightarrow \begin{cases} \hat{\boldsymbol{\Delta}}_1\boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0\lambda_1\boldsymbol{z} \\ \hat{\boldsymbol{\Delta}}_2\boldsymbol{z} = \hat{\boldsymbol{\Delta}}_0\lambda_2\boldsymbol{z} \end{cases}$$

randomized/structured sketching

(Vermeersch and De Moor, 2022; Hochstenbach et al., 2022)

# Outline

1 | Introduction

2 | Multiparameter Eigenvalue Problem

3 | Solution Approaches

4 | Numerical Example

5 | Conclusion and Future Work

# Three approaches to compute the eigentuples

Number of eigentuples?

- linear MEP

$$m_b = \binom{l+n-1}{l}$$

- polynomial MEP

$$m_b = d^n \binom{l+n-1}{l}$$

Three solution approaches exist:

- block Macaulay matrix approach – (Vermeersch and De Moor, 2022, 2023)
- reduction to one-parameter problem – (Alsubaie, 2019)
- transformation into square MEP – (Hochstenbach et al., 2022)

(Shapiro and Shapiro, 2009; Hochstenbach et al., 2022)

# Outline

1 | Introduction

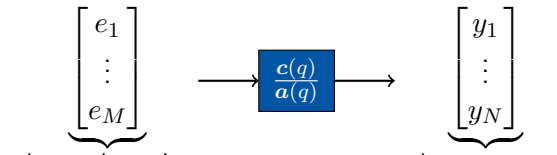
2 | Multiparameter Eigenvalue Problem

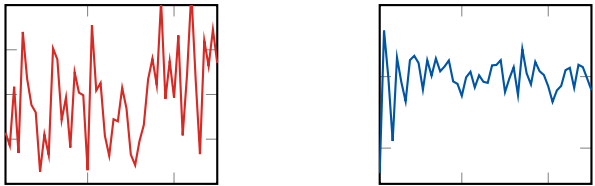
3 | Solution Approaches

4 | Numerical Example

5 | Conclusion and Future Work

# Identifying the parameters of an ARMA(1,1) model



such that

$$y_k + \alpha y_{k-1} = e_k + \gamma e_{k-1}$$

$$\min_{\boldsymbol{e},\alpha,\gamma} \|\boldsymbol{e}\|_2^2$$

$$\text{subject to } \boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$$

(Vermeersch and De Moor, 2019)

# Identifying the parameters of an ARMA(1,1) model

$$\min_{\boldsymbol{e},\alpha,\gamma} \|\boldsymbol{e}\|_2^2$$

$$\text{subject to } \boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$$

$$\Downarrow$$

$$\begin{bmatrix} \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\alpha\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{D}_\gamma & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{T}_\alpha^\alpha \boldsymbol{y} \\ \boldsymbol{0} & \boldsymbol{D}_\gamma & \boldsymbol{D}_\gamma^\gamma & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_\gamma & \boldsymbol{T}_\alpha \boldsymbol{y} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^\alpha \\ \boldsymbol{f}_1^\gamma \\ \boldsymbol{f} \\ -1 \end{bmatrix} = \boldsymbol{0}$$

# Identifying the parameters of an ARMA(1,1) model

$$\min_{\boldsymbol{e},\alpha,\gamma} \|\boldsymbol{e}\|_2^2$$

$$\text{subject to } \boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$$

$$\Downarrow$$

$$\begin{bmatrix} \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\alpha\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{y}^{\mathrm{T}}\boldsymbol{T}_\alpha^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{D}_\gamma & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{T}_\alpha^\alpha \boldsymbol{y} \\ \boldsymbol{0} & \boldsymbol{D}_\gamma & \boldsymbol{D}_\gamma^\gamma & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_\gamma & \boldsymbol{T}_\alpha \boldsymbol{y} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^\alpha \\ \boldsymbol{f}_1^\gamma \\ \boldsymbol{f} \\ -1 \end{bmatrix} = \boldsymbol{0}$$

$$\Downarrow$$

$$\left(\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\alpha + \boldsymbol{A}_{01}\gamma + \boldsymbol{A}_{02}\gamma^2\right)\boldsymbol{z} = \boldsymbol{0}$$

# Identifying the parameters of an ARMA(1,1) model

$$\boldsymbol{y} \in \mathbb{R}^8$$

$$\downarrow$$

$$\min \|\boldsymbol{e}\|_2^2$$

subject to $\boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$

$$\downarrow$$

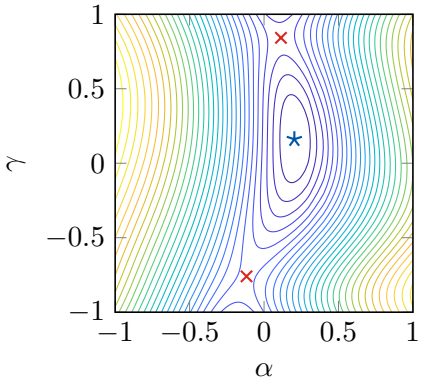
$$\left(\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\alpha + \boldsymbol{A}_{01}\gamma + \boldsymbol{A}_{02}\gamma^2\right)\boldsymbol{z} = \boldsymbol{0}$$

$$\downarrow$$

one of the solution approaches

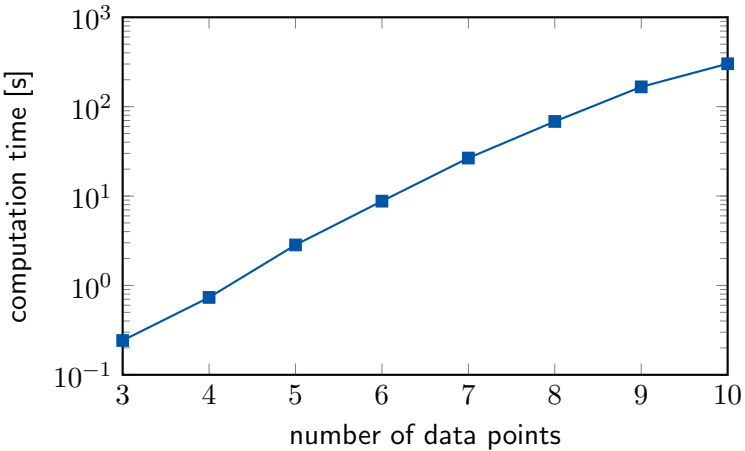
$$\downarrow$$

parameters $\alpha$ and $\gamma$




Contour plot of the cost function with one minimum (★) and two saddle points (×)

# Computational bottleneck




Computation time (—■—) of identifying the parameters of a first-order ARMA model via the MEP approach for a given number of data points

# Outline

1 | Introduction

2 | Multiparameter Eigenvalue Problem

3 | Solution Approaches

4 | Numerical Example

5 | Conclusion and Future Work

# Conclusion and future work

- Multiparameter eigenvalue problems are omnipresent in systems and control:
  - system identification,
  - model order reduction,
  - and partial differential equations.
- There exist relations between square and rectangular manifestations.
- Scalability remains an active research problem*!

* All algorithms are implemented in MATLAB and available at `www.macaulaylab.net`.




Christof Vermeersch[†‡]
Bart De Moor[†]

STADIUS
Center for Dynamical Systems, Signal Processing, and Data Analytics

fwo

[†] KU Leuven, Dept. of Electrical Engineering (ESAT), Center for Dynamical Systems, Signal Processing, and Data Analytics (STADIUS), Kasteelpark Arenberg 10, 3001 Leuven, Belgium (christof.vermeersch@esat.kuleuven.be, bart.demoor@esat.kuleuven.be)

[‡] Supported by FWO Strategic Basic Research fellowship under grant SB/1SA1319N

# References


Mauricio O. Agudelo, Christof Vermeersch, and Bart De Moor. Globally optimal $\mathcal{H}_2$-norm model reduction: A numerical linear algebra approach. **IFAC-PapersOnLine**, 54(9):564–571, 2021. Part of special issue: 24th International Symposium on Mathematical Theory of Networks and Systems (MTNS).

Fawwaz Fayiz F. Alsubaie. $\mathcal{H}_2$ **Optimal Model Reduction for Linear Dynamic Systems and the Solution of Multiparameter Matrix Pencil Problems**. PhD thesis, Imperial College London, London, UK, 2019.

Kim Batselier, Philippe Dreesen, and Bart De Moor. Prediction error method identification is an eigenvalue problem. In **Proc. of the 16th IFAC Symposium on System Identification**, pages 221–226, Brussels, Belgium, 2012.

Bart De Moor. Least squares realization of LTI models is an eigenvalue problem. In **Proc. of the 18th European Control Conference (ECC)**, pages 2270–2275, Naples, Italy, 2019.

# References


Bart De Moor. Least squares optimal realisation of autonomous LTI systems is an eigenvalue problem. **Communications in Information and Systems**, 20(2): 163–207, 2020.

Michiel E. Hochstenbach, Tomaž Košir, and Bor Plestenjak. On the solution of rectangular multiparameter eigenvalue problems. Technical report, TU Eindhoven, Eindhoven, The Netherlands, 2022.

Boris Shapiro and Michael Shapiro. On eigenvalues of rectangular matrices. **Proc. of the Steklov Institute of Mathematics**, 267(1):248–255, 2009.

Françoise Tisseur and Karl Meerbergen. The quadratic eigenvalue problem. **SIAM Review**, 43(2):235–286, 2001.

Christof Vermeersch and Bart De Moor. Globally optimal least-squares ARMA model identification is an eigenvalue problem. **IEEE Control Systems Letters**, 3(4): 1062–1067, 2019.

# References


Christof Vermeersch and Bart De Moor. Two complementary block Macaulay matrix algorithms to solve multiparameter eigenvalue problems. **Linear Algebra and its Applications**, 654:177–209, 2022.

Christof Vermeersch and Bart De Moor. Two double recursive block Macaulay matrix algorithms to solve multiparameter eigenvalue problems. **IEEE Control Systems Letters**, 7:319–324, 2023.

Christof Vermeersch, Sibren Lagauw, and Bart De Moor. Multivariate polynomial optimization in complex variables is a (rectangular) multiparameter eigenvalue problem. Technical report, KU Leuven, Leuven, Belgium, 2023. Submitted for publication.

Hans Volkmer. **Multiparameter Eigenvalue Problems and Expansion Theorems**, volume 1356 of **Lecture Notes in Mathematics**. Springer, Berlin, Germany, 1988.