On the Rectangular Multiparameter Eigenvalue Problem

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Abstract

What are rectangular multiparameter eigenvalue problems? We answer that question by showing how they generalize the well-known one-parameter eigenvalue problem. This paper reviews the current landscape of the rectangular multiparameter eigenvalue problem, focusing on the single polynomial matrix case. We provide an overview of the known theoretical properties, such as its spectral characteristics, homogeneous interpretation, and the number of (isolated) solutions. We incorporate several minor extensions to and reformulations of the literature to bridge existing results and enhance the consistency of the available knowledge. The paper contains a comparison of the currently available numerical algorithms to solve rectangular multiparameter eigenvalue problems and provides a non-exhaustive list of applications that involve such generalizations. It also offers some concrete, interesting research challenges for the (near) future.

Acknowledgments: This work was supported in part by the KU Leuven: Research Fund (projects IBOF/23/064 and IDN/24/013), Industrial Research Fund, and LRD bilateral industrial projects; in part by Flemish Government agencies: FWO (TBM project T001919N), EWI (Flanders AI Research Program), and VLAIO (HBC.2021.0076); and in part by the European Commission (ERC Advanced Grant under grant 885682). *Views and opinions expressed are, however, those of the authors only and do not necessarily reflect those of the European Union or ERC. Neither the European Union nor the granting authority can be held responsible for them.*

Keywords: matrix pencils, multiparameter eigenvalues, eigenvalue-finding algorithms

MSC Classification: 15A22 , 15A18 , 65F15

"For many years I have battled for proper values, and against the one and a half times translated German-English hybrid that is often used to refer to them. I have now become convinced that the war is over, and eigenvalues have won it."

—Paul R. Halmos, *A Hilbert Space Problem Book* (1967)

1 Introduction

Eigenvalues of one-parameter polynomial matrices are indispensable for describing a variety of phenomena, such as capturing the vibration of mechanical structures [1]. Consequently, the past decades have witnessed many advances in numerical methods to solve these one-parameter eigenvalue problems [2, 3]. Regular one-parameter eigenvalue problems form a mature area of numerical linear algebra. For some phenomena, however, the matrix equation

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z} = \boldsymbol{0} \tag{1}$$

depends on a multivariate spectral parameter $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$. Because of this multivariate dependency, we call these generalizations multiparameter eigenvalue problems. One particular generalization is the *rectangular multiparameter eigenvalue problem*. For example,

$$\left(\begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \lambda_1\lambda_2 + \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 2 & 1 \end{bmatrix} \lambda_2^2 + \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 3 & 4 \end{bmatrix} \right) \boldsymbol{z} = \boldsymbol{0} \tag{2}$$

is a rectangular quadratic eigenvalue problem that depends on two spectral parameters. It can be viewed as a matrix-valued mapping,

$$\boldsymbol{\mathcal{A}} : \mathbb{C}^2 \to \mathbb{C}^{3\times 2} : \boldsymbol{\lambda} \mapsto \boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}) = \begin{bmatrix} a_{11}(\boldsymbol{\lambda}) & a_{12}(\boldsymbol{\lambda}) \\ a_{21}(\boldsymbol{\lambda}) & a_{22}(\boldsymbol{\lambda}) \\ a_{31}(\boldsymbol{\lambda}) & a_{32}(\boldsymbol{\lambda}) \end{bmatrix}, \tag{3}$$

that maps a point from the complex space onto a complex matrix value. Every element $a_{ij}(\boldsymbol{\lambda})$ of that matrix is a scalar polynomial in the multiparameter $\boldsymbol{\lambda}$, for example, $a_{11}(\boldsymbol{\lambda}) = 3\lambda_1\lambda_2 + \lambda_2^2 + 2\lambda_1 + 1$. For some points $\boldsymbol{\lambda}^*$, the polynomial matrix drops rank and the matrix equation (2) has a nontrivial solution, for which the vector $\boldsymbol{z}^*$ is not equal to zero. We call these solutions $(\boldsymbol{\lambda}^*, \boldsymbol{z}^*)$, in accordance with the one-parameter case, the eigenvalues and eigenvectors of the polynomial matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$.

Most research on rectangular multiparameter eigenvalue problems has been of theoretical nature. Recently, however, there has been a new focus on developing numerical algorithms to solve these problems, mainly driven by the fact that various applications can be reformulated as a multiparameter eigenvalue problem. For instance, rectangular multiparameter eigenvalue problems have been encountered when solving differential equations [4], analyzing public keys in cryptography [5], identifying least-squares dynamical systems from input-output data [6–9], and reducing the order of a system's transfer function [10–12]. A numerical example is given in Fig. 1.

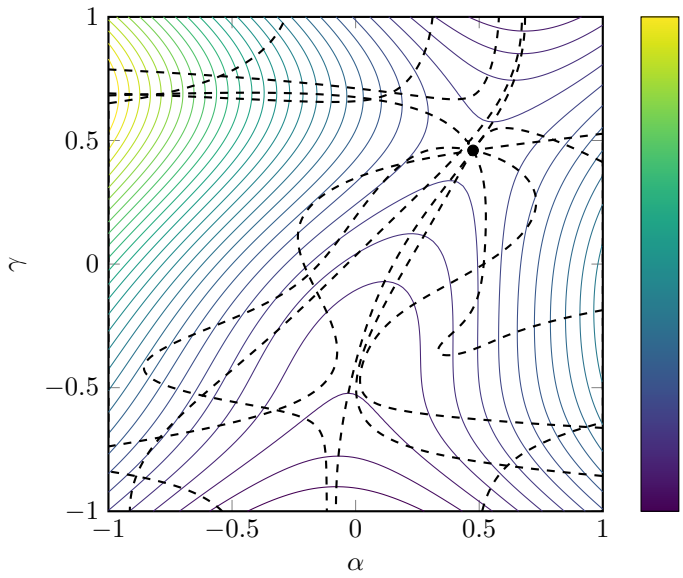

Figure 1 Illustration of a multiparameter eigenvalue application. The contour of the least-square objective function of a first-order autoregressive moving-average model depends on two model parameters, α and γ, which we want to identify from a given data sequence. The minimum of the objective function yields the globally optimal model parameters for that data sequence. The stationary points of the objective function also corresponds to the eigenvalues of a two-parameter eigenvalue problem [6]. The real-valued stationary points (●) correspond to the intersections of the secular equations (- - -) of that two-parameter eigenvalue problem.

Coupled square multiparameter eigenvalue systems

Multiparameter spectral theory is closely linked to solving boundary-value problems for partial differential equations via the separation of variables [13–16]. For example, the vibration of an elliptic membrane in two elliptic coordinates, i.e., the two-dimensional Helmholtz equation, leads to the study of a pair of ordinary differential equations, both of which share two spectral parameters [15]. The multiparameter eigenvalue problems obtained when discretizing the differential equations are systems of coupled, square multiparameter eigenvalue problems. Although these problems share many characteristics with the rectangular multiparameter eigenvalue problem, they have been studied separately in the literature. Boundary-value problems have motivated the long history of theoretical research and resulted in numerical algorithms to solve such problems. Some key references about coupled square multiparameter eigenvalue systems are [17–22].

Contributions and scope

We review the current landscape of the rectangular multiparameter eigenvalue problem. *The main contribution of the paper is that it provides a didactic and comprehensive exposition of the topic.* We provide an overview of the known theoretical properties in the literature, supplemented with several minor extensions and reformulations to bridge existing results and enhance the consistency of the available knowledge. The

theory is far from complete at this point, but we aim to provide a coherent overview of its current status. Furthermore, we compare for the different state-of-the-art numerical algorithms to compute the eigenvalues of the multivariate polynomial matrix and provide a non-exhaustive list of applications that involve such problems. Although the above-mentioned coupled square multiparameter eigenvalue systems are very interesting in their own right, *we restrict ourselves to only the rectangular multiparameter eigenvalue problem*. Therefore, we omit the explicit statement that the problems are rectangular, except in cases where ambiguity may arise.

Outline of this paper

Section 1 continues with a definition of the problem (Section 1.1), an introduction of three didactic examples that are used throughout the text (Section 1.2), and a historical overview (Section 1.3). After the introduction, we discuss the spectrum of the polynomial matrix and its characteristics in Section 2. A homogeneous intermezzo in Section 3 presents the multiparameter eigenvalue problem in the projective space. Afterwards, in Section 4, we focus on the isolated spectrum and a bound the number of zero-dimensional solutions for generic problems. Various numerical algorithms to compute these zero-dimensional solutions are discussed in Section 5, while Section 6 comprises a non-exhaustive list of applications of the multiparameter eigenvalue problem. We conclude this paper with a discussion of some open research challenges in Section 7.

1.1 Multiparameter eigenvalue problem definition

Consider, for given (rectangular[1]) coefficient matrices $\boldsymbol{A}_{\boldsymbol{\kappa}} \in \mathbb{C}^{k\times l}$ with $k \geq l$, the polynomial matrix function that maps an m-tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ from the complex space $\mathbb{C}^m$ onto a complex polynomial matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ parametrized by that m-tuple,

$$\boldsymbol{\mathcal{A}} : \mathbb{C}^m \to \mathbb{C}^{k\times l} : \boldsymbol{\lambda} \mapsto \boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}) = \sum_{\mathcal{K}} \boldsymbol{A}_{\boldsymbol{\kappa}} \boldsymbol{\lambda}^{\boldsymbol{\kappa}} = \begin{bmatrix} a_{11}(\boldsymbol{\lambda}) & \cdots & a_{1l}(\boldsymbol{\lambda}) \\ \vdots & \ddots & \vdots \\ a_{k1}(\boldsymbol{\lambda}) & \cdots & a_{kl}(\boldsymbol{\lambda}) \end{bmatrix}, \tag{4}$$

where the summation runs over all the multi-indices $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_m) \in \mathcal{K}$ that index the coefficient matrices and monomials. Notice that $\mathcal{K} \subset \mathbb{N}^m$ denotes the support of the polynomial matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$, i.e., the set of all integer multi-indices

$$\mathcal{K} = \{\boldsymbol{\kappa} \in \mathbb{N}^m | \boldsymbol{A}_{\boldsymbol{\kappa}} \neq \boldsymbol{0}\}. \tag{5}$$

The degree, σ, of the polynomial matrix is equal to the maximum total degree of all its monomials $\boldsymbol{\lambda}^{\boldsymbol{\kappa}}$, i.e.,

$$\sigma = \max_{\boldsymbol{\kappa}\in\mathcal{K}} |\boldsymbol{\kappa}| = \max_{\boldsymbol{\kappa}\in\mathcal{K}} \kappa_1 + \cdots + \kappa_m. \tag{6}$$

[1] The situation where $k < l$ can be considered by taking the (Hermitian) transpose of the problem.

Every element $a_{ij}(\boldsymbol{\lambda})$ of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ is a multivariate polynomial, with degree less than or equal to σ. The normal rank of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$, denoted by $\operatorname{nrank}(\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}))$, is the smallest integer ρ such that $\operatorname{rank}(\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})) \leq \rho$ for all $\boldsymbol{\lambda} \in \mathbb{C}^m$. We assume that the coefficient matrices of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ are such that the normal rank is full. For some values of $\boldsymbol{\lambda}$, the polynomial matrix becomes rank deficient and the matrix equation

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z} = \mathbf{0} \tag{7}$$

has a nontrivial solution. We formalize this problem in the following definition.

Definition 1. *Given (rectangular) coefficient matrices $\boldsymbol{A}_{\boldsymbol{\kappa}} \in \mathbb{C}^{k\times l}$ (with $k \geq l$) that lead to a polynomial matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ with normal rank $\rho = l$, the **multiparameter eigenvalue problem (MEP)** consists in finding all m-tuples $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_m^*) \in \mathbb{C}^m$ that result in a rank drop, i.e., $\rho^* = \operatorname{rank}(\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}^*)) < \rho$. There exists at least one nonzero vector $\boldsymbol{z}^* \in \mathbb{C}_0^{l\times 1}$ that solves matrix equation (7) for every $\boldsymbol{\lambda}^*$.*

The m-tuples $\boldsymbol{\lambda}^*$ that result in this rank deficiency are the eigenvalues of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$, while the associated (nonzero) vectors $\boldsymbol{z}^* \in \mathbb{C}_0^l$ are called the (right) eigenvectors. Similarly, the left eigenvectors $\boldsymbol{w}^* \in \mathbb{C}_0^k$ can be defined as the nontrivial vectors that satisfy

$$\boldsymbol{w}^{\mathrm{H}}\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}) = \mathbf{0} \tag{8}$$

for an eigenvalue $\boldsymbol{\lambda}^*$ but not for another value of $\boldsymbol{\lambda}$.

Remark 1. *We restrict ourselves to problems for which the normal rank ρ is equal to the number of columns l, i.e., polynomial matrices that have a full column rank. If $\rho < l$, then the polynomial matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ has a null space consisting of right functional solutions $\boldsymbol{z}(\boldsymbol{\lambda})$, which are polynomial vectors that satisfy the equation*

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z}(\boldsymbol{\lambda}) = \mathbf{0} \tag{9}$$

for any value of $\boldsymbol{\lambda}$. It is still possible to define eigenvalues as points where the rank drops below the normal rank, but such problems are more difficult to analyze and solve. A more elaborated discussion of this special case can be found in [23].

Remark 2. *Some references make a difference between regular and singular polynomial matrices. A square ($k = l$) polynomial matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ is called **regular** if $\det(\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})) \not\equiv 0$ is not identically zero. Otherwise, if either $k \neq l$ or $\det(\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})) \equiv 0$, the polynomial matrix is called **singular**. We want to point out that most of the examples and applications in this paper are singular (with $k \geq l + m - 1$).*

1.2 Three didactic examples

Three examples help to illustrate the properties of MEPs throughout the paper.

Example 1. *The following example is a linear two-parameter eigenvalue problem:*

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \lambda_1 - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \lambda_2 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \boldsymbol{z} = \mathbf{0}. \tag{10}$$

Example 2. *The following quadratic two-parameter eigenvalue problem has a full support with all monomials of degree two or lower:*

$$\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \lambda_1^2 + \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda_1\lambda_2 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix} \lambda_2^2 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix} \lambda_2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right) \boldsymbol{z} = \mathbf{0}. \quad (11)$$

Example 3. *The quadratic two-parameter eigenvalue problem in (2) is the third didactic example. Contrary to Example 2, its support misses some of the monomials.*

1.3 Historical overview

To the best of the authors' knowledge, the MEP was discussed for the first time[2] by Khazanov [27, 28] and Kublanovskaya [29]. They examined the spectrum of a multivariate polynomial matrix, along with several other properties, such as matrix factorizations and Jordan vector semilattices. Khazanov [23, 30, 31] also proposed iterative algorithms to determine the eigenvalues of multivariate polynomial matrices. Shapiro and Shapiro [32] explored the multiparameter linear matrix pencil in the context of the higher-order Heine–Stieltjes problem, providing a theoretical analysis of the problem and establishing an expression for the number of affine solutions of generic linear MEPs.

Apart from the iterative optimization-based algorithms in [30, 31], most of the research on MEPs has been theoretical in nature. The iterative optimization-based algorithms, as well as adaptations of earlier approaches described in [24–26, 33], were the first numerical algorithms to solve MEPs. However, there has recently been a growing interest in developing numerical algorithms for solving MEPs. Via operator determinants [12, 34, 35] or randomized sketching [34], these algorithms transform an MEP into a joint generalized eigenvalue problem (GEP) or a coupled square multiparameter eigenvalue system, respectively. These transformations allow using the existing (efficient) numerical algorithms to obtain the eigenvalues. An implementation of both approaches can be found in the MultiParEig toolbox [36]. In our work [7, 8, 37–39], we have introduced the block Macaulay matrix, which also retrieves the solutions of MEPs via numerical linear algebra techniques. Different block Macaulay matrix algorithms have been implemented in the MacaulayLab toolbox [40].

These recent algorithmic developments have mainly been driven by the fact that various applications can be reformulated as an MEP. For instance, the higher-order generalization of the Heine–Stieltjes problem corresponds to a MEP [4, 32]. The MEP also pops up in cryptanalysis with the minimum rank problem [5]. Two emerging application areas are globally optimal least-squares identification of dynamical systems [6–8] and globally optimal $\mathcal{H}_2$-norm model order reduction problem [10–12].

[2] Prior to this, a more general problem involving $k \times l$ matrices, where $k \geq l$, appeared under the name *eigentuple-eigenvector problem* in [24, 25] and as a rank-reducing perturbation problem in [26].

2 Spectrum of a multivariate polynomial matrix

The set of eigenvalues $\boldsymbol{\lambda}^*$ of the polynomial matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ is called the spectrum.

Definition 2. *The **(affine) spectrum** of a polynomial matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ is defined as the geometric locus of the points $\boldsymbol{\lambda}^* \in \mathbb{C}^m$ that satisfy $\rho^* < \rho$, where ρ^* and ρ are the rank and normal rank of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$, respectively.*

In this section, we start by connecting the points $\boldsymbol{\lambda}^*$ of the (affine) spectrum with the solutions of the characteristic equations (Section 2.1). This connection allows us to discuss the multiplicity (structure) of the solutions (Section 2.2).

2.1 Minors and characteristic equations

A minor of order ρ of the polynomial matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ is the determinant of a $\rho \times \rho$ submatrix obtained by deleting $k-\rho$ rows and $l-\rho$ columns of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$. Let

$$\begin{aligned} \mathcal{I} &= \{i_1, \ldots, i_\rho : 1 \leq i_1 \leq \cdots \leq i_\rho \leq k\}, \\ \mathcal{J} &= \{j_1, \ldots, j_\rho : 1 \leq j_1 \leq \cdots \leq j_\rho \leq l\}, \end{aligned} \tag{12}$$

be ordered sequences of indices. The different minors of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ are defined as

$$\det(\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}))_{\mathcal{I},\mathcal{J}}, \tag{13}$$

where the subscripts indicate the ρ rows and ρ columns selected in the determinant. There are $\binom{k}{\rho}\binom{l}{\rho}$ minors of order ρ. Since we only consider problems with full column rank (see Remark 1), the normal rank ρ is equal to the number of columns l. Consequently, a polynomial matrix as in Definition 1 has $L = \binom{k}{l}$ different minors, which correspond to row selections. We denote these row selections by $\mathcal{I}_1, \ldots, \mathcal{I}_L$ and define the resulting characteristic equations.

Definition 3. *Let $\mathcal{I}_1, \ldots, \mathcal{I}_L$ denote the $L = \binom{k}{l}$ selections of l rows from a set of k rows. If we consider a polynomial matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ as defined in Definition 1, then the **characteristic equations** are given by*

$$\chi_i(\boldsymbol{\lambda}) = \det(\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}))_{\mathcal{I}_i} = 0, \quad i = 1, \ldots, L. \tag{14}$$

Theorem 1. *The solutions of the system of characteristic equations in (14) correspond to the solutions of the MEP in Definition 1. The (affine) spectrum of the polynomial matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ is given by this solution set.*

Proof. A necessary and sufficient condition for an $k \times l$ matrix with $k \geq l$ to be rank deficient is that all minors, $\det(\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}))_{\mathcal{I},\mathcal{J}}$, are equal to zero [41, Theorem 4.1]. Since the normal rank of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ is equal to l, the eigenvalues $\boldsymbol{\lambda}^*$ that make these minors vanish are the solutions of the system of characteristic equations. □

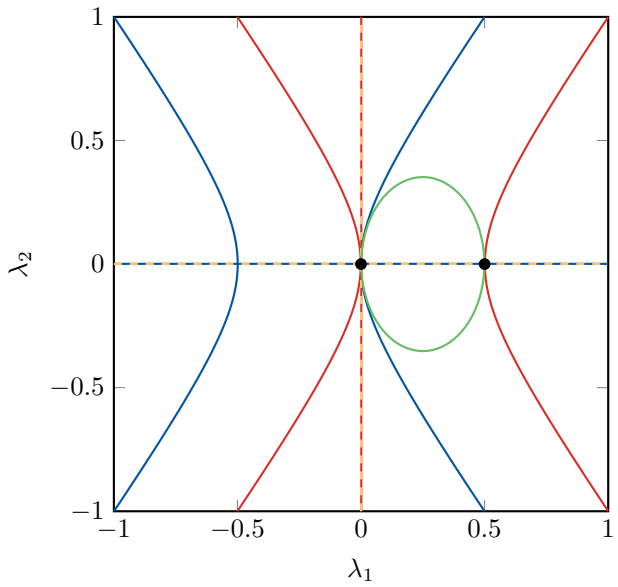

Figure 2 The secular equations of the linear two-parameter eigenvalue problem in (10) are the multivariate polynomials derived from the minors of the polynomial matrix in Example 4: $\chi_1(\boldsymbol{\lambda})$ (——), $\chi_2(\boldsymbol{\lambda})$ (——), $\chi_3(\boldsymbol{\lambda})$ (- - -), and $\chi_4(\boldsymbol{\lambda})$ (——). The intersection points (●) of the secular equations correspond to the eigenvalues of (10).

Example 4. *When looking at the linear two-parameter eigenvalue problem in* (10)*, the system of characteristic equations (visualized in Fig. 2) is*

$$\begin{cases} \chi_1(\boldsymbol{\lambda}) = -2\lambda_1^2\lambda_2 - \lambda_1\lambda_2 + \lambda_2^3 = 0, \\ \chi_2(\boldsymbol{\lambda}) = 2\lambda_1^3 - \lambda_1^2 - \lambda_1\lambda_2^2 = 0, \\ \chi_3(\boldsymbol{\lambda}) = 2\lambda_1\lambda_2 = 0, \\ \chi_4(\boldsymbol{\lambda}) = -2\lambda_1^2 + \lambda_1 - \lambda_2^2 = 0. \end{cases} \tag{15}$$

This overdetermined multivariate polynomial system has a triple affine solution in the origin and a single solution in $(\frac{1}{2}, 0)$*, which corresponds to the eigenvalues* $\boldsymbol{\lambda}^*$ *of* (10)*.*

The (affine) spectrum of a polynomial matrix is the collection of $(m - r)$-dimensional solutions of the MEP, with $1 \leq r \leq m$. It consists of regular solutions $(r = 1)$ and/or singular solutions $(r > 1)$:

- The set of $(m - 1)$-dimensional solutions of (14) correspond to the regular (affine) part of the spectrum of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$. These solutions can be described via so-called *eigenpolynomials* of the form $\varphi(\boldsymbol{\lambda})$, which are the common (non-constant) divisors of the characteristic polynomials. The (affine) spectrum's regular part is uniquely determined by the set of irreducible eigenpolynomials. Moreover, it can be described completely by the greatest common divisor of the characteristic polynomials in (14) [42]. There exist algorithms to determine these eigenpolynomials and the greatest common divisor, see for example, [42].

- The other solutions, which are lower-dimensional, form the singular (affine) part of the spectrum of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$.

The most interesting solutions in applications are, typically, the eigenvalues that are zero-dimensional ($r = m$). These isolated points are discussed in Section 4.

Example 5. *Consider, for example the matrix polynomial*

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}) = \begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 \\ \lambda_2 & \lambda_1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \tag{16}$$

which has characteristic polynomials

$$\begin{cases} \chi_1(\boldsymbol{\lambda}) = \lambda_2^2 - \lambda_1^2, \\ \chi_2(\boldsymbol{\lambda}) = \lambda_1^2\lambda_2 - \lambda_1^2\lambda_2, . \\ \chi_3(\boldsymbol{\lambda}) = \lambda_1^3 - \lambda_1\lambda_2^2, \end{cases} \tag{17}$$

The eigenpolynomials are $\varphi_1(\boldsymbol{\lambda}) = \lambda_1 - \lambda_2$ and $\varphi_2(\boldsymbol{\lambda}) = \lambda_1 + \lambda_2$. Their greatest common divisor, $\varphi_{GCD}(\boldsymbol{\lambda}) = \lambda_1^2 - \lambda_2^2$, describes the regular (affine) spectrum of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$.

2.2 Multiplicity structure

The algebraic multiplicity of a point $\boldsymbol{\lambda}^*$ of the (affine) spectrum is the multiplicity of this point as a common root of the characteristic polynomial system defined in (14). Therefore, the algebraic multiplicity can not exceed $\rho\sigma$. Because of the link with the characteristic polynomials, we first need to introduce two concepts from algebraic geometry: dual spaces and differential functionals.

Definition 4. *The **dual space** $W = V^*$ of a vector space V over $\mathbb{C}$ is the vector space of all linear functionals $l : V \to \mathbb{C}$. For a given polynomial ideal $I \subset \mathcal{P}_m$ with quotient ring $R[I] = \mathcal{P}_m \setminus I$ (where $\mathcal{P}_m$ is the ring of complex polynomials in m variables), this dual space is the set of linear functionals in $(R[I])^*$ that vanish on the ideal, i.e.,*

$$\{l \in \mathcal{P}_m^* | l(a) = 0, \forall a \in I\}\,. \tag{18}$$

Definition 5. *For a multi-index $\boldsymbol{\tau} \in \mathbb{N}^m$ and a point $\boldsymbol{\lambda}^* \in \mathbb{C}^m$, the **differential functional** $\partial_{\boldsymbol{\tau}}[\boldsymbol{\lambda}^*] \in \mathcal{P}_m^*$ is defined as*

$$\partial_{\boldsymbol{\tau}}[\boldsymbol{\lambda}^*](a) = \frac{1}{\tau_1! \cdots \tau_m!} \frac{\partial^{|\boldsymbol{\tau}|}}{\partial\lambda_1^{\tau_1} \cdots \partial\lambda_m^{\tau_m}} [a(\boldsymbol{\lambda})]_{\boldsymbol{\lambda}^*}\,, \tag{19}$$

for any polynomial $a(\boldsymbol{\lambda}) \in \mathcal{P}_m$ in the ring of complex polynomials in m variables.

The definition of algebraic multiplicity now follows from applying [43, Definition 8.20] to the characteristic polynomial system. It generalizes the notion of counting how many derivatives of the characteristic polynomial of a one-parameter eigenvalue problem vanish.

Definition 6. *An eigenvalue* $\boldsymbol{\lambda}^*$ *has* ***algebraic multiplicity*** μ_a *if there exists a closed set of* μ_a *linearly independent linear combinations of differential functionals,* $\sum_{\{\boldsymbol{\tau}\}} c_{\boldsymbol{\tau}} \partial_{\boldsymbol{\tau}} [\boldsymbol{\lambda}^*]$*, in the dual space of the ideal generated by the characteristic equations* $\chi_i(\boldsymbol{\lambda})$ *in* (14)*, i.e.,* $\langle \chi_1(\boldsymbol{\lambda}), \ldots, \chi_L(\boldsymbol{\lambda}) \rangle$*.*

The different linear combinations of differential functionals needed for a basis of the dual space form the multiplicity structure of the eigenvalue as a common root of the characteristic polynomial system. Finding this multiplicity structure (and thus the algebraic multiplicity) is a nontrivial task, but some approaches have been proposed in the literature [44, 45]. Eigenvalues with an algebraic multiplicity equal to one are called simple eigenvalues.

The geometric multiplicity of a point $\boldsymbol{\lambda}^*$ of the (affine) spectrum is related to the rank drop in this point, which corresponds to the number of linearly independent eigenvectors associated with that eigenvalue.

Definition 7. *The* ***geometric multiplicity*** μ_g *of an eigenvalue* $\boldsymbol{\lambda}^*$ *is equal to* $\mu_g = \rho - \rho^*$*, where* $\rho = \operatorname{nrank}(\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}))$ *and* $\rho^* = \operatorname{rank}(\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}^*))$*.*

As in the one-parameter case, the geometric multiplicity never exceeds the algebraic multiplicity.

Lemma 1. *The geometric multiplicity* μ_g *of an eigenvalue* $\boldsymbol{\lambda}^*$ *does not exceed its algebraic multiplicity* μ_a*, i.e.,* $\mu_g \leq \mu_a$*.*

Proof. The partial derivatives of order $\tau \geq 1$ of each of the characteristic equations $\chi_i(\boldsymbol{\lambda})$ of $\rho \times \rho$ submatrices can be represented as a sum of determinants of order $\rho - \tau$. Therefore, if all the $(\rho^* + 1) \times (\rho^* + 1)$ minors vanish in the point $\boldsymbol{\lambda}^*$, which is the case for a point of geometric multiplicity equal to $\rho - \rho^*$, then every partial derivative of order not exceeding $\rho - \rho^*$ also vanishes at $\boldsymbol{\lambda}^*$. □

Example 6. *One of the eigenvalues of* (10) *is the point* $\boldsymbol{\lambda}^* = (0, 0)$*. The algebraic multiplicity* μ_a *of this point is equal to three, since there are three linear combinations of differential functionals in the basis of the dual space of the characteristic polynomial system (*$\partial_{00}[\boldsymbol{\lambda}^*]$*,* $\partial_{01}[\boldsymbol{\lambda}^*]$*, and* $\partial_{02}[\boldsymbol{\lambda}^*] + 2\partial_{10}[\boldsymbol{\lambda}^*]$*). The geometric multiplicity* μ_g *of* $(0, 0)$ *is equal to one because the rank drops at this point with* $\rho - \rho^* = 3 - 2 = 1$*. The geometric multiplicity is thus smaller than the algebraic multiplicity at* $(0, 0)$*.*

3 Intermezzo about the homogeneous problem setting

Due to the singularity of their coefficient matrices, some MEPs have solutions at infinity. We move from the m-dimensional complex space $\mathbb{C}^m$ to the m-dimensional projective space $\mathbb{P}^m$ (Section 3.1), which allows us to consider the points with infinite coordinates in the (complete) spectrum (Section 3.2).

3.1 Projective space and homogeneous polynomials

In the projective space, points are specified by sets of $m+1$ homogeneous coordinates.

Definition 8. *The* m*-dimensional* ***projective space*** $\mathbb{P}^m$ *is the set of equivalence classes on* $\mathbb{C}_0^{m+1}$*. Each nonzero* $(m+1)$*-tuple* $\widetilde{\boldsymbol{\lambda}} = (\lambda_0, \lambda_1, \ldots, \lambda_m)$ *defines a point in*

the projective space $\mathbb{P}^m$*. The above-mentioned equivalence relation* $\sim$ *on the nonzero points of* $\mathbb{C}^{m+1}$ *is defined by the setting*

$$(\lambda'_0, \lambda'_1, \ldots, \lambda'_m) \sim (\lambda_0, \lambda_1, \ldots, \lambda_m) \tag{20}$$

if there exists a nonzero scalar $\delta \in \mathbb{C}$ *such that*

$$(\lambda'_0, \lambda'_1, \ldots, \lambda'_m) = (\delta\lambda_0, \delta\lambda_1, \ldots, \delta\lambda_m). \tag{21}$$

In words: two points $\widetilde{\boldsymbol{\lambda}}'$ and $\widetilde{\boldsymbol{\lambda}}$ are considered equal if and only if all their coordinates are proportional. The origin $\mathbf{0} \in \mathbb{C}^{m+1}$ is not a point in the projective space $\mathbb{P}^m$, because the equivalence relation $\sim$ would associate an infinite number of projective points with $\mathbf{0}$. The affine space $\mathbb{C}^m$ can be retrieved as a "slice" of this projective space:

$$\mathbb{C}^m = \{(1, \lambda_1, \ldots, \lambda_m) \in \mathbb{P}^m\}\,. \tag{22}$$

The projective points for which $\lambda_0 = 0$ are called the points at infinity. If $a(\boldsymbol{\lambda})$ is a polynomial of degree σ, then the corresponding polynomial in the projective space is defined as

$$a^\pi(\widetilde{\boldsymbol{\lambda}}) = \lambda_0^\sigma\, a\left(\frac{\lambda_1}{\lambda_0}, \ldots, \frac{\lambda_k}{\lambda_0}\right) \tag{23}$$

The polynomial $a^\pi(\widetilde{\boldsymbol{\lambda}})$ is a homogeneous polynomial: every monomial of $a^\pi(\widetilde{\boldsymbol{\lambda}})$ has total degree equal to σ. More information about the projective space and homogeneous polynomials can be found in, for example, [46, Chapter 8].

3.2 Homogeneous eigenvalue problem formulation

The MEP can be considered in the projective space by making the polynomial matrix homogeneous. For an MEP of degree σ, we have that

$$\boldsymbol{\mathcal{A}}^\pi(\widetilde{\boldsymbol{\lambda}}) = \lambda_0^\sigma \boldsymbol{\mathcal{A}}\left(\frac{\lambda_1}{\lambda_0}, \ldots, \frac{\lambda_k}{\lambda_0}\right) \tag{24}$$

is the homogeneous equivalent of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$. Every entry $a_{ij}^\pi(\widetilde{\boldsymbol{\lambda}})$ of the polynomial matrix $\boldsymbol{\mathcal{A}}^\pi(\widetilde{\boldsymbol{\lambda}})$ is a homogeneous polynomial of degree σ. From that homogeneous polynomial matrix, we can compute the homogeneous characteristic equations by taking all the minors of order ρ, i.e.,

$$\chi_i^\pi(\widetilde{\boldsymbol{\lambda}}) = \det\left(\boldsymbol{\mathcal{A}}^\pi(\widetilde{\boldsymbol{\lambda}})\right)_{\mathcal{I}_i} = 0, \quad i = 1, \ldots, L, \tag{25}$$

where the row selections $\mathcal{I}_i$ are identical to those in Definition 3. Notice that $a_{ij}^\pi(\widetilde{\boldsymbol{\lambda}})$ is not necessarily equal to the homogenization of $a_{ij}(\boldsymbol{\lambda})$ because the total degree of that polynomial may differ from σ. Similarly, the homogeneous characteristic equations differ from the homogenization of polynomials in (14) with factors of the form $\lambda_0^{\sigma_i}$.

Example 7. *For the MEP in (11), the homogenization of* $a_{22}(\boldsymbol{\lambda}) = \lambda_2 + 1$ *is equal to* $\lambda_2 + \lambda_0$*, while the* $a^{\pi}_{22}(\widetilde{\boldsymbol{\lambda}})$ *is equal to* $\lambda_2\lambda_0 + \lambda_0^2$*.*

Definition 9. *The* ***(complete) spectrum*** *of a homogeneous polynomial matrix* $\boldsymbol{\mathcal{A}}^{\pi}(\widetilde{\boldsymbol{\lambda}})$ *is defined as the geometric locus of the points* $\widetilde{\boldsymbol{\lambda}}^* \in \mathbb{P}^m$ *that satisfy* $\rho^* = \operatorname{rank}\left(\boldsymbol{\mathcal{A}}^{\pi}(\widetilde{\boldsymbol{\lambda}}^*)\right) < \rho$*, where* ρ *is the normal rank of* $\boldsymbol{\mathcal{A}}^{\pi}(\widetilde{\boldsymbol{\lambda}})$*.*

These are, equivalently, also the points that solve the homogeneous characteristic equations in (25).

Corollary 1. *The affine part of the complete spectrum coincides with the affine spectrum defined by the characteristic equations in (14).*

Example 8. *The homogeneous characteristic equations for the homogenization of (10) are given by*

$$\begin{cases} \chi^{\pi}_1(\widetilde{\boldsymbol{\lambda}}) = -2\lambda_1^2\lambda_2 + \lambda_0\lambda_1\lambda_2 + \lambda_2^3 = 0, \\ \chi^{\pi}_2(\widetilde{\boldsymbol{\lambda}}) = 2\lambda_1^3 - \lambda_0\lambda_1^2 - \lambda_1\lambda_2^2 = 0, \\ \chi^{\pi}_3(\widetilde{\boldsymbol{\lambda}}) = 2\lambda_0\lambda_1\lambda_2 = 0, \\ \chi^{\pi}_4(\widetilde{\boldsymbol{\lambda}}) = -2\lambda_0\lambda_1^2 + \lambda_0^2\lambda_1 - \lambda_0\lambda_2^2 = 0, \end{cases} \tag{26}$$

which has two solutions for which $\lambda_0 = 0$*. Setting* $\lambda_0 = 1$ *results again in (14).*

4 Zero-dimensional solutions in the isolated spectrum

The isolated part of the spectrum that contains the zero-dimensional solutions is important in many applications. Zero-dimensional solutions often represent specific configurations or equilibria, for example, the stationary points in optimization problems.

In this section, we give conditions for the existence of an isolated spectrum (Section 4.1) and provide an expression for the number of affine zero-dimensional solutions of a generic MEP (Section 4.2).

4.1 Conditions for an isolated spectrum

A necessary (but not sufficient) condition for the m-parameter eigenvalue problem to have an isolated spectrum is that the coefficient matrices have the correct size. There are k equations and one non-triviality constraint on the eigenvector $\boldsymbol{z}$ (e.g., $\|\boldsymbol{z}\|_2 = 1$) for $m + l$ unknowns (m eigenvalues and l elements of the eigenvector). Thus, the (rectangular) coefficient matrices must satisfy $k + 1 \geq m + l$ for a well-determined MEP with isolated spectrum. A sufficient condition for linear MEPs in the case that $k + 1 = m + l$ is given in the following lemma [32, Lemma 1].

Lemma 2 (Shapiro–Shapiro)**.** *Let* $\boldsymbol{A}_{10\cdots0}, \ldots, \boldsymbol{A}_{0\cdots01} \in \mathbb{C}^{k\times l}$ *with* $k = l + m - 1$*. If*

$$\operatorname{rank}(\boldsymbol{A}_{10\cdots0}\lambda_1 + \cdots + \boldsymbol{A}_{0\cdots01}\lambda_m) = l \tag{27}$$

for all $\boldsymbol{\lambda} \neq \mathbf{0}$*, then for each* $\boldsymbol{A}_{0\cdots0} \in \mathbb{C}^{k\times l}$ *the linear MEP constructed from these coefficient matrices,*

$$(\boldsymbol{A}_{10\cdots0}\lambda_1 + \cdots + \boldsymbol{A}_{0\cdots01}\lambda_m + \boldsymbol{A}_{0\cdots0})\,\boldsymbol{z} = \mathbf{0}, \tag{28}$$

has exactly

$$\binom{l+m-1}{m} \tag{29}$$

zero-dimensional solutions, counted with algebraic multiplicities.

Proof. A proof of this lemma can be found in [32]. □

4.2 Generic number of zero-dimensional solutions

There are two theorems available in the literature about the number of solutions in the isolated spectrum. The first formula was given by Shapiro and Shapiro (see Lemma 2) and only considers linear MEPs, while the second theorem by Hochstenbach et al. [34, Theorem 11] specifies the number of solutions for a generic polynomial MEP.

Theorem 2 (Hochstenbach–Košir–Plestenjak)**.** *Consider a generic polynomial* m*-parameter eigenvalue problem with* $(l+m-1)\times l$ *coefficient matrices. This problem has exactly*

$$\sigma^m \binom{l+m-1}{m} \tag{30}$$

affine eigenvalues.

Proof. A proof of this theorem can be found in [34]. □

Example 9. *The quadratic two-parameter eigenvalue problem in* (11) *has* 12 *affine solutions, which agrees with the expression in Theorem 2:*

$$2^2\binom{2+2-1}{2} = 4\cdot 3 = 12. \tag{31}$$

Example 10. *Notice that Theorem 2 only provides an expression for the number of affine solutions of generic MEPs. Consider the quadratic two-parameter eigenvalue problem in* (2)*, which is expected to have* 12 *affine solutions according to* (30)*. However, due to some missing coefficient matrices, the problem is non-generic and it only has nine affine solutions.*

5 Algorithms to compute the isolated spectrum

We can distinguish the numerical algorithms to obtain the points of the isolated spectrum into two distinct classes: optimization-based approaches, which are iterative algorithms like gradient descent that focus on one solution, and matrix-based approaches, which rely on (structured) matrices and efficient standard/generalized eigenvalue solvers to find all solutions. While the former were dominant in the past, recent advances have made the latter more suitable for solving MEPs.

Table 1 Overview of the main properties of the numerical algorithms in Section 5.1. The nonlinear optimization-based approach is heuristic in the sense that its result depends on the initial guess.

algorithm	types	# solutions	heuristic	toolbox
nonlinear optimization	polynomial	one	yes	not available
operator determinants	linear	all	no	MultiParEig
block Macaulay matrices	polynomial	all	no	MacaulayLab
randomized sketching	polynomial[†]	all + additional	no	MultiParEig

[†]The type of problems that the randomized sketching approach can solve depends on the specific algorithm used to tackle the obtained coupled square multiparameter eigenvalue system.

After computing the eigenvalues, it is in most situations straightforward to compute the eigenvectors by determining the null space of the matrix obtained by evaluating the matrix polynomial $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$ in the eigenvalue $\boldsymbol{\lambda}^*$.

Example 11. *We discuss the different numerical algorithms by solving a linear three-parameter eigenvalue problem with* $(l+2) \times l$ *coefficient matrices* $\boldsymbol{A}_{\boldsymbol{\kappa}}$*,*

$$\left(\boldsymbol{A}_{100}\lambda_1 + \boldsymbol{A}_{010}\lambda_2 + \boldsymbol{A}_{001}\lambda_3 + \boldsymbol{A}_{000}\right)\boldsymbol{z} = \boldsymbol{0}. \tag{32}$$

We present four different approaches for solving MEPs (Section 5.1) with the help of the above-mentioned three-parameter eigenvalue problem. Afterwards, we compare implementations of these approaches for solving (32) with an increasing number of columns (Section 5.2).

5.1 Explanation of four different approaches to solve MEPs

We present four different numerical algorithms. An overview of their main properties is given in Table 1 While the first approach is based on nonlinear optimization (Section 5.1.1), the other approaches rely on structured matrices to find the eigenvalues (Sections 5.1.2 to 5.1.4).

5.1.1 Nonlinear optimization

Historically, the first algorithmic approaches to solve MEPs relied on nonlinear optimization approaches, by minimizing a cost function related to the matrix polynomial and approximation one solution at a time. Blum and co-authors [24, 25, 33] explored different iterative optimization algorithms to obtain the eigenvalues, including gradient descent techniques [24, 33, 47], minimal residual quotient iterations [25], and Newton-based methods [48]. Khazanov [30] also explored several iterative methods to find the solutions.

These nonlinear optimization algorithms intend to solve the matrix equation $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z} = \boldsymbol{0}$, under the constraint $\langle \boldsymbol{z}, \boldsymbol{w} \rangle = 1$ where $\boldsymbol{w} \in \mathbb{C}_0^l$ is a particular constraint vector. A common constraint is to choose $\boldsymbol{w} = \boldsymbol{z}$ and use a normalization of the eigenvector via $\langle \boldsymbol{z}, \boldsymbol{z} \rangle = 1$. The basic idea is to solve the nonlinear system

$$\boldsymbol{\mathcal{F}}(\boldsymbol{\lambda}, \boldsymbol{z}) = \begin{bmatrix} \boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z} \\ \boldsymbol{z}^{\mathrm{H}}\boldsymbol{w} - 1 \end{bmatrix} = 0, \tag{33}$$

by minimizing $\boldsymbol{\mathcal{F}}$. Obtaining the solution of this nonlinear system coincides with the minimization of the residual vector

$$f(\boldsymbol{\lambda}, \boldsymbol{z}) = \|\boldsymbol{\mathcal{F}}(\boldsymbol{\lambda}, \boldsymbol{z})\| = \|\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}, \boldsymbol{z})\boldsymbol{z}\| + \left|\boldsymbol{z}^{\mathrm{H}}\boldsymbol{w} - 1\right|. \tag{34}$$

Several nonlinear optimization algorithms could be used to minimize the residual vector, either by directly minimizing $\boldsymbol{\mathcal{F}}(\boldsymbol{\lambda}, \boldsymbol{z})$ or by tackling $f(\boldsymbol{\lambda}, \boldsymbol{z})$. In any case, both approaches are iterative and improve a given initial guess $(\boldsymbol{\lambda}^{(0)}, \boldsymbol{z}^{(0)})$.

For the linear three-parameter eigenvalue problem in (32) and eigenvector normalization, a nonlinear iterative method can minimize the nonlinear least-squares problem

$$\boldsymbol{\mathcal{F}}(\boldsymbol{\lambda}, \boldsymbol{z}) = \begin{bmatrix} (\boldsymbol{A}_{100}\lambda_1 + \boldsymbol{A}_{010}\lambda_2 + \boldsymbol{A}_{001}\lambda_3 + \boldsymbol{A}_{000})\,\boldsymbol{z} \\ \boldsymbol{z}^{\mathrm{H}}\boldsymbol{z} - 1 \end{bmatrix} \tag{35}$$

or minimize the scalar function

$$f(\boldsymbol{\lambda}, \boldsymbol{z}) = \|(\boldsymbol{A}_{100}\lambda_1 + \boldsymbol{A}_{010}\lambda_2 + \boldsymbol{A}_{001}\lambda_3 + \boldsymbol{A}_{000})\,\boldsymbol{z}\| + \left|\boldsymbol{z}^{\mathrm{H}}\boldsymbol{x} - 1\right|. \tag{36}$$

Surprisingly, there are no implementations of iterative optimization algorithms available for MEPs. We had to create our own functions for the comparison in Section 5.2, and in doing so we noticed that most of the above-listed references restrict their derivations and analysis to square coefficient matrices.

5.1.2 Operator determinants

Operator determinants have been used for solving coupled square multiparameter eigenvalue systems since Atkinson's work in the 1960s [21], which linked the problem to a joint generalized eigenvalue problem using these structured matrices. The idea is reminiscent of Cramer's rule for linear systems, where the solution of a system of linear equations is expressed in terms of determinants. Only recently, however, operator determinants have been applied to solve (rectangular) MEPs [12, 34, 35]. This approach only works for linear MEPs, requiring an additional linearization step when the problem is polynomial [34].

Operator determinants for linear MEPs are constructed from structured Kronecker products of the coefficient matrices as

$$\boldsymbol{\Delta}_0 = \begin{vmatrix} \boldsymbol{A}_{\boldsymbol{\kappa}(1)} & \cdots & \boldsymbol{A}_{\boldsymbol{\kappa}(m)} \\ \vdots & \ddots & \vdots \\ \boldsymbol{A}_{\boldsymbol{\kappa}(1)} & \cdots & \boldsymbol{A}_{\boldsymbol{\kappa}(m)} \end{vmatrix}_{\otimes} \quad \text{and} \tag{37}$$

$$\boldsymbol{\Delta}_i = -\begin{vmatrix} \boldsymbol{A}_{\boldsymbol{\kappa}(1)} & \cdots & \boldsymbol{A}_{\boldsymbol{\kappa}(i-1)} & \boldsymbol{A}_{\boldsymbol{\kappa}(0)} & \boldsymbol{A}_{\boldsymbol{\kappa}(i+1)} & \cdots & \boldsymbol{A}_{\boldsymbol{\kappa}(m)} \\ \boldsymbol{A}_{\boldsymbol{\kappa}(1)} & \cdots & \boldsymbol{A}_{\boldsymbol{\kappa}(i-1)} & \boldsymbol{A}_{\boldsymbol{\kappa}(0)} & \boldsymbol{A}_{\boldsymbol{\kappa}(i+1)} & \cdots & \boldsymbol{A}_{\boldsymbol{\kappa}(m)} \\ \boldsymbol{A}_{\boldsymbol{\kappa}(1)} & \cdots & \boldsymbol{A}_{\boldsymbol{\kappa}(i-1)} & \boldsymbol{A}_{\boldsymbol{\kappa}(0)} & \boldsymbol{A}_{\boldsymbol{\kappa}(i+1)} & \cdots & \boldsymbol{A}_{\boldsymbol{\kappa}(m)} \end{vmatrix}_{\otimes}, \quad i = 0, \ldots, m, \tag{38}$$

where $|\cdot|_{\otimes}$ denotes the operator determinant, which considers the coefficient matrices[3] $\boldsymbol{A}_{\boldsymbol{\kappa}(i)}$ as elements of a large matrix and takes the determinant of that matrix

[3]The subscript of the coefficient matrices, $\boldsymbol{\kappa}(i)$, denotes the ith coefficient matrix of the support $\mathcal{K}$.

while replacing the ordinary product by the Kronecker product [22]. For example, the operator determinant $\boldsymbol{\Delta}_2$ of the three-parameter eigenvalue problem in (32) is given by

$$\boldsymbol{\Delta}_2 = \begin{vmatrix} \boldsymbol{A}_{100} & \boldsymbol{A}_{000} & \boldsymbol{A}_{001} \\ \boldsymbol{A}_{100} & \boldsymbol{A}_{000} & \boldsymbol{A}_{001} \\ \boldsymbol{A}_{100} & \boldsymbol{A}_{000} & \boldsymbol{A}_{001} \end{vmatrix}_{\otimes}. \tag{39}$$

Via these operator determinants, it is possible to transform the MEP into a joint GEP, with one GEP per parameter,

$$\boldsymbol{\Delta}_i \boldsymbol{w} = \lambda_i \boldsymbol{\Delta}_0 \boldsymbol{w}, \quad i = 1, \ldots, m. \tag{40}$$

The joint GEP in (40) has the solutions of the MEP in its solution set [12, 34].

However, the joint GEP in (40) has l^m solutions, while the linear MEP has maximum only $\binom{k}{l}$ solutions (use Theorem 2 with $\sigma = 1$). The additional solutions are due to permutations in the Kronecker structure, but can be removed by multiplying the operator determinants $\boldsymbol{\Delta}_i$ by compression matrices $\boldsymbol{L} \in \mathbb{C}^{\binom{k}{l} \times l^m}$ and $\boldsymbol{T} \in \mathbb{C}^{l^m \times \binom{k}{l}}$ [12]. To understand the permutations, consider the vector $\boldsymbol{w}$ for the linear three-parameter eigenvalue problem in (32). It is constructed as $\boldsymbol{w} = \boldsymbol{z} \otimes \boldsymbol{z} \otimes \boldsymbol{z}$, meaning that $w_i = z_1 z_2 z_3$ is equal to $w_j = z_2 z_1 z_3$ or $w_k = z_3 z_1 z_2$, while $i \neq j \neq k$. The compression step (proposed in [12]),

$$\widehat{\boldsymbol{\Delta}}_i = \boldsymbol{L} \boldsymbol{\Delta}_i \boldsymbol{T} \in \mathbb{C}^{\binom{k}{l} \times \binom{k}{l}}, \tag{41}$$

removes the permutations and results in a compressed joint GEP

$$\widehat{\boldsymbol{\Delta}}_i \widehat{\boldsymbol{w}} = \lambda_i \widehat{\boldsymbol{\Delta}}_0 \widehat{\boldsymbol{w}}, \tag{42}$$

which has the same solutions as the linear MEP and can be used to solve it effectively.

The Kronecker matrices in the operator determinants $\boldsymbol{\Delta}_i$ can become large, especially for a higher number of parameters. In an efficient implementation of this transformation, it is therefore recommended to exploit the sparsity of the compression matrices or even to avoid explicitly building most of these matrices altogether ([12] explains such a sparse approach). The operator determinant approach has been implemented in the MultiParEig toolbox [36], but without exploiting this sparsity and structure. It uses MATLAB's QZ algorithm to solve the joint GEPs. Note that there exist also other algorithms for solving the joint GEPs, for example simultaneous triangularization via the QZ algorithm [49–51] or randomized algorithms [52].

5.1.3 Block Macaulay matrix

Another numerical linear algebra approach for solving MEPs involves organizing the coefficient matrices into a structured matrix known as the block Macaulay matrix. This approach was introduced by the authors in [6–8, 38] and it is inspired by the (scalar) Macaulay matrix from algebraic geometry. Via one or multiple multidimensional realization problems in the null space of the block Macaulay matrix constructed from the coefficient matrices, this approach retrieves all solutions of the MEP.

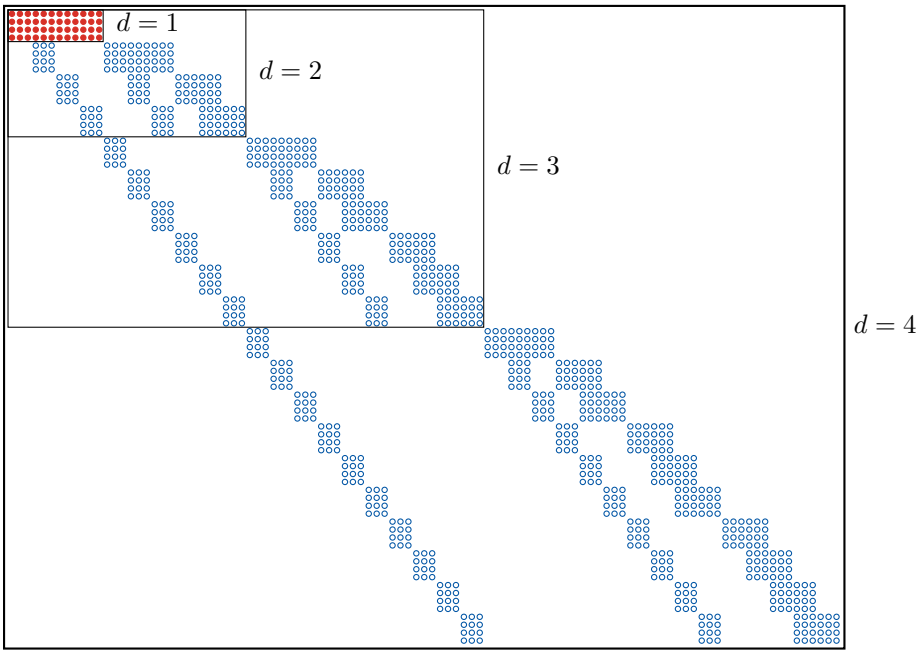

Figure 3 Spy plot of a block Macaulay matrix. The three-parameter eigenvalue problem in (32) with 4×3 coefficient matrices (•) generates a family of block Macaulay matrices parametrized by the degree $d = 1, 2, 3, \ldots$ The repeated coefficient matrices (◦) are arranged according to order of the associated monomials. These block Macaulay matrices are sparse and structured, but quickly become very large.

The coefficient matrices of the MEP are arranged and repeated within the structured block Macaulay matrix according to a specific pattern that resembles the support of the problem. The block columns correspond to the monomials $\boldsymbol{\lambda}^{\boldsymbol{\kappa}}$ that these coefficient matrices $\boldsymbol{A}_{\boldsymbol{\kappa}}$ are associated with. The block Macaulay matrix forms a family of matrices parameterized by a degree, which corresponds to the largest degree of the included monomials. For example, block Macaulay matrices of different subsequent degrees for the three-parameter eigenvalue problem in (32) are visualized in Fig. 3. If the nullity coincides with the number of solutions, then the null space has a shift-invariant structure. This shift-structure gives raise to one or multiple multidimensional realization problems, and we can retrieve a joint GEP from it,

$$\boldsymbol{Z}_i \boldsymbol{w} = \lambda_i \boldsymbol{Z}_0 \boldsymbol{w}, \quad i = 1, \ldots, m, \tag{43}$$

where the matrices $\boldsymbol{Z}_0$ and $\boldsymbol{Z}_i$ correspond to a selection of specific rows from a numerical basis matrix $\boldsymbol{Z}$ of the null space before and after the shifts in the multidimensional realization problem.

One limiting factor of the algorithm is that the block Macaulay matrix can become large before the shift-invariant structure is available in the null space. However, since this matrix is sparse and structured, the algorithm has been adapted to avoid the explicit construction of this block Macaulay matrix [53] and to exploit the iterative

nature of the algorithm [39]. This approach has been implemented in the MacaulayLab toolbox [40].

5.1.4 Randomized sketching approach

Randomized sketching, a well-known technique from numerical linear algebra, can also be used to transform the MEP into a coupled square multiparameter eigenvalue system. The core idea behind randomized sketching is both simple and powerful: using random projections or samples, it compresses large matrices into smaller "sketches" that (approximately) preserve the properties of the larger problem. The idea of applying random projections or row selections was used for the first time in the MEP setting by Hochstenbach et al. [34] This transformation makes it possible to compute the eigenvalues of the original MEP via the existing numerical algorithms for solving coupled square multiparameter eigenvalue systems. Examples of such algorithms are homotopy continuation algorithms [54–56] and subspace approaches [51, 57–60], next to the standard simultaneous triangularization of the associated joint GEPs.

Sketching of an m-parameter eigenvalue problem results in m square matrix equations via m (random) projections:

$$\boldsymbol{P}_i \left(\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\right) \boldsymbol{z}_i = \boldsymbol{0}, \quad i = 1, \ldots, m, \tag{44}$$

where the (random) projection matrices $\boldsymbol{P}_i \in \mathbb{C}^{l \times k}$ create a coupled square multiparameter eigenvalue system. For the three-parameter eigenvalue problem in Example 11, this results in the following system:

$$\begin{cases} \boldsymbol{P}_1 \left(\boldsymbol{A}_{100}\lambda_1 + \boldsymbol{A}_{010}\lambda_2 + \boldsymbol{A}_{001}\lambda_3 + \boldsymbol{A}_{000}\right) \boldsymbol{z}_1 = \boldsymbol{0}, \\ \boldsymbol{P}_2 \left(\boldsymbol{A}_{100}\lambda_1 + \boldsymbol{A}_{010}\lambda_2 + \boldsymbol{A}_{001}\lambda_3 + \boldsymbol{A}_{000}\right) \boldsymbol{z}_2 = \boldsymbol{0}, \\ \boldsymbol{P}_3 \left(\boldsymbol{A}_{100}\lambda_1 + \boldsymbol{A}_{010}\lambda_2 + \boldsymbol{A}_{001}\lambda_3 + \boldsymbol{A}_{000}\right) \boldsymbol{z}_3 = \boldsymbol{0}, \end{cases} \tag{45}$$

where the projection matrices $\boldsymbol{P}_i \in \mathbb{R}^{l \times (l+2)}$ are random matrices (e.g., obtained via randn in MATLAB). The coupled square multiparameter eigenvalue system obtained via (44) has more solutions than the original MEP, due to the fact that the vectors $\boldsymbol{z}_i$ are not necessarily collinear (i.e., they do not correspond to the same eigenvector $\boldsymbol{z}$). For a large m, only a modest portion of the obtained solutions is also a solution to the original MEP (see, for example, the numerical experiment in [34, Table 1]). So, the obtained solution candidates need to be pruned by, for example, computing their residuals in the original problem.

Note that a lot of the sparsity gets lost when using random sketching matrices $\boldsymbol{P}_i$ and the sketching procedure ends up with more coefficient matrices than in the original MEP.

The randomized sketching approach is also implemented in the MultiParEig toolbox [36]. While the default implementation solves the joint GEP related to the obtained coupled square multiparameter eigenvalue system, other efficient approaches from that toolbox can also be used to tackle the obtained matrix equations (44).

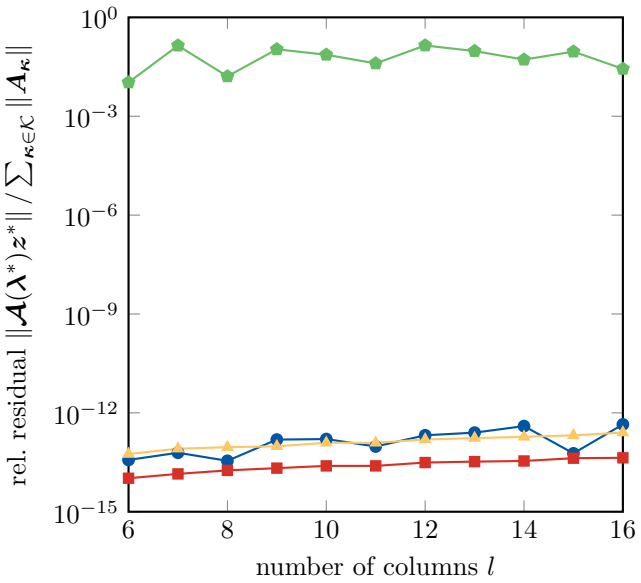

Figure 4 Mean relative residuals for the algorithms discussed in Section 5 when applied to three-parameter eigenvalue problems of increasing size (Example 11). The matrix-based approaches, which are the operator determinants (—■—), block Macaulay matrix (—●—), and randomized sketching (—▲—), find the eigenvalues with good precision. The results of the nonlinear optimization-based approach, Gauss–Newton (—⬟—) in this numerical example, depends on the initial guess and can fail completely, leading to the large mean residuals for these two approaches.

5.2 Comparison of the numerical algorithms

We reconsider Example 11 and compare[4] the different numerical algorithms by solving linear three-parameter eigenvalue problems (32) for increasing sizes of coefficient matrices.

Example 11 (Continued)**.** *We solve the three-parameter eigenvalue problem in* (32) *with real random coefficient matrices* $\boldsymbol{A}_{\boldsymbol{\kappa}} \in \mathbb{C}^{(l+2)\times l}$ *and increase* l*. The results are averaged over* 30 *experiments. The relative residuals (i.e., residuals* $\|\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}^*)\boldsymbol{z}^*\|_2$ *normalized by the norm of the coefficient matrices* $\sum_{\boldsymbol{\kappa}\in\mathcal{K}} \|\boldsymbol{A}_{\boldsymbol{\kappa}}\|_2$*) and computation time of the different solution approaches are given for increasing sizes of coefficient matrices in Figs. 4 and 5.*

The nonlinear optimization method is heuristic: One particular solution, depending on the initial guess, is obtained in every run. It converges to a stationary point of the optimization problem (36), which could be one of the solutions of the MEP. However, it can go terribly wrong, as visible in Fig. 4. The mean relative residual

[4]The numerical experiment is ran in MATLAB version 2024b on a MacBook Pro with M1 CPU working at 2.4 GHz and 16 GB RAM. We use a self-implemented Gauss–Newton method with a dogleg trust region and random initial guess for the nonlinear optimization approach. For the operator determinants and randomized sketching approach, the default implementations from the MultiParEig [36] toolbox are used. Both approaches use MATLAB's QZ algorithm to solve the obtained joint GEP or coupled square multiparameter eigenvalue system. The block Macaulay matrix approach in this comparison is the default version from the MacaulayLab toolbox [40], which reduces the computation time by not explicitly constructing the block Macaulay matrix (Section 5.1.3).

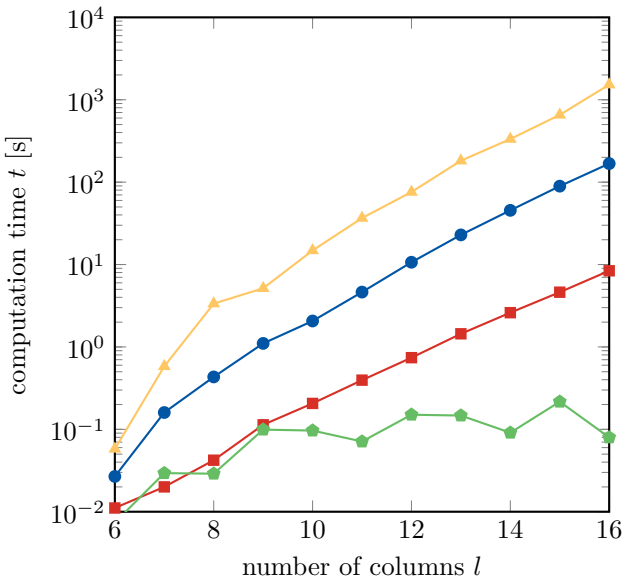

Figure 5 Comparison of computation time for the algorithms discussed in Section 5 when applied to three-parameter eigenvalue problems of increasing size (Example 11). The nonlinear optimization-based approach, Gauss–Newton (—⬟—) in this numerical example, is faster than the matrix-based methods. However, it only computes one solution per run and depends on an initial guess. The matrix-based approaches, which are the operator determinants (—■—), block Macaulay matrix (—●—), and randomized sketching (—▲—), are have a similar complexity w.r.t. the number of columns. The operator determinants approach is the fastest, since it works on substantially smaller matrices to find the eigenvalues.

of the nonlinear optimization approach is quite large, since some of the obtained points are not eigenvalues. Its main advantage is the efficiency, clearly outperforming the matrix-based approaches in Fig. 5. The matrix-based approaches (i.e., operator determinants, block Macaulay matrix, and randomized sketching) find all the eigenvalues with a small relative residual, as visible in Fig. 4. The computation time of the three matrix-based approaches seems to have a similar complexity with respect to the number of columns, but with different factors. The computation time of the block Macaulay matrix approach is a direct consequence of the above-mentioned adaptations (Section 5.1.3), providing a more efficient implementation than a naive implementation. For three-parameter eigenvalue problems, the operator determinant approach has the lowest computation time among the matrix-based algorithms, which can be explained by the fact that it operates on substantially smaller matrices than the other two approaches. Note that it is possible to influence the obtained results of the randomized sketching approach by choosing different solvers for the coupled square multiparameter eigenvalue system (Section 5.1.4).

Although this numerical example allows us to compare the different numerical algorithms, it only represents one part of the overall picture. For example, the number of spectral parameters has a major influence on the computation time and influences the numerical algorithms differently. Figure 6 shows how the relative ranking of the

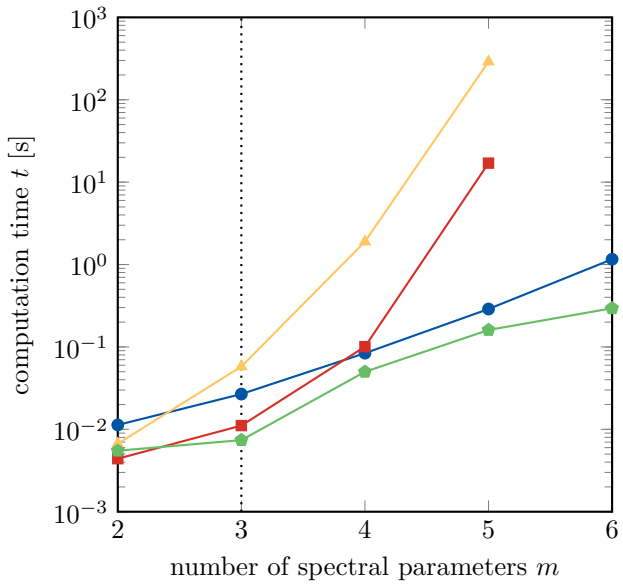

Figure 6 Influence of the number of spectral parameters on the computation time for the different algorithms discussed in Section 5: the block Macaulay matrix (—●—), operator determinants (—■—), randomized sketching (—▲—), and Gauss–Newton (—⬟—) are used to solve a linear m-parameter eigenvalue problem with $(m+5)\times 6$ coefficient matrices. The dotted vertical line (·····) indicates the situation described in Fig. 5, where we compare the same four algorithms for three parameters with increasing sizes. The situation clearly changes when the number of parameters increases.

numerical algorithms changes when increasing the number of spectral parameters for a fixed number of columns. The operator determinants and randomized sketching approach suffer from the blow-up in size of the Kronecker matrices.

6 Some examples of application domains

The MEP can be applied to a wide variety of applications. In this section, we highlight four examples: a higher-order problem in Heine–Stieltjes theory (Section 6.1), a cryptography problem that looks for the minimum rank of a rectangular matrix (Section 6.2), a system identification problem (Section 6.3), and a model order reduction problem (Section 6.4).

6.1 Multiparameter problems in Heine–Stieltjes theory

The higher-order Heine–Stieltjes problem [4] consists in finding, for each positive integer μ, all the polynomials $V(z)$ of degree at most r such that the equation

$$\delta(z)S(z) + V(z)S(z) = 0 \tag{46}$$

has a polynomial solution $S(z)$ of degree μ. The polynomials $V(z)$ and $S(z)$ are called higher-order Van Vleck and Stieltjes polynomials, respectively. A higher-order Lamé

operator is a linear ordinary differential operator $\delta(z)$, i.e.,

$$\delta(z) = \sum_{i=1}^{\nu} Q_i(z) \frac{\mathrm{d}^i}{\mathrm{d}z^i}. \tag{47}$$

The number $r = \max_{i=1,\ldots,\nu} \deg\left(Q_i(z) - i\right)$ is the so-called Fuchs index of $\delta(z)$. It is possible to show that, under mild non-degeneracy assumptions, there exist exactly $\binom{\mu+r}{\mu}$ such polynomials $V(z)$ whose corresponding eigenpolynomials $S(z)$ are of degree μ [4, Theorem 5].

How does the MEP enters the picture when tackling the higher-order Heine–Stieltjes problem? Given a non-degenerate operator $\delta(z)$, in order to find all of its Van Vleck polynomials $V(z)$ having at least one Stieltjes polynomial $S(z)$ of degree at most μ, we need to study the action of $\delta(z)$ on the linear space of all univariate polynomials of degree at most μ. Using the standard monomial basis, the action of $\delta(z)$ is represented by a triangular banded matrix $\boldsymbol{A}$ of size $(\mu + r + 1) \times (\mu + 1)$. Adding an arbitrary polynomial $V(z) = \lambda_1 z^{m-1} + \lambda_2 z^{m-2} + \cdots + \lambda_m$ with degree $r = m - 1$ to $\delta(z)$ corresponds to adding the linear combination of $\boldsymbol{B}_1\lambda_1 + \boldsymbol{B}_2\lambda_2 + \cdots + \boldsymbol{B}_m\lambda_m$ to the matrix $\boldsymbol{A}$, where $\boldsymbol{B}_i$ is the $(\mu + r + 1) \times (\mu + 1)$ matrix whose entries are given by $b_{ij} = 0$ if $j - i \neq s$ and 1 otherwise. The existence of a non-trivial Stieltjes polynomial $S(z)$ of degree at most μ is equivalent to the question whether the linear matrix equation

$$\boldsymbol{A} + \boldsymbol{B}_1\lambda_1 + \boldsymbol{B}_2\lambda_2 + \cdots + \boldsymbol{B}_m\lambda_m \tag{48}$$

has a non-trivial null space. Thus, for a given non-degenerate operator $\delta(z)$, the problem of solving the higher-order Heine–Stieltjes problem is exactly the same as determining the solutions of the MEP given by the matrices in (48).

6.2 Cryptographic minimum rank problem

The minimum rank problem (often written as *MinRank*) was originally introduced in [61] as a theoretical linear algebra question and, later on, restated in [62] within a cryptographic setting. It was shown to be related to several multivariate public key cryptosystems [63–65] and as the basis of an efficient zero-knowledge authentication scheme [62]. The minimum rank problem consists in finding the eigenvalues that satisfy a rank drop, i.e.,

$$\operatorname{rank}\left(\sum_{i=1}^{m} \boldsymbol{B}_i\lambda_i - \boldsymbol{B}_0\right) \leq \rho, \tag{49}$$

where ρ is the normal rank. The problem setting is similar to Definition 1, but the numbers under consideration are typically not complex numbers, but elements of another field $\mathbb{K}$ (for example, a finite field). The matrices $\boldsymbol{B}_i \in \mathbb{K}^{k\times l}$ encode public keys and the strategy to recover a secret key corresponds to finding a linear combination of the matrices that has a rank smaller than the normal rank.

There exist two main techniques for solving the minimum rank problem. One technique, called the *kernel attack*, consist in guessing vectors of the null space of the polynomial matrix and then solving the resulting linear system [62, 65]. Another technique consists in modeling the problem as a system of multivariate polynomial

equations [62, 63], for example, by considering the characteristic equations of the MEP. For complex numbers, the numerical algorithms from Section 5 could also be applied when the rank drops for isolated values.

6.3 Least-squares identification of misfit-versus-latency models

System identification is a discipline that aims at constructing models for dynamical systems from measured data [66–68]. The constructed model tries to capture the relations between input, output, and noise. It depends on a set of model parameters, which are selected to best fit the measured data. While a lot of heuristic approaches are used in practice, the identification procedure can be formulated as an MEP for some model structures. This approach has already been applied to different model structures, e.g., for the least-squares realization problem [7, 8] and the autoregressive moving-average model [6] (cf., the numerical example in Fig. 1).

For example, consider a given data sequence $\boldsymbol{y} \in \mathbb{R}^{N\times 1}$ and try to adapt it such that it can be described by a linear time-invariant system. The least-squares realization problem is to find a model that generates an adapted data sequence $\widehat{\boldsymbol{y}} \in \mathbb{R}^{N\times 1}$ that is the output of this linear time-invariant system, i.e.,

$$\widehat{y}_k = \boldsymbol{c}\boldsymbol{A}^k\boldsymbol{x}_0, \tag{50}$$

where $\boldsymbol{c} \in \mathbb{R}^{1\times n}$ is the output vector, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is the system matrix, $\boldsymbol{x}_0 \in \mathbb{R}^{n\times 1}$ is the initial state vector, and n is the order of the system. This system can also be described by its difference equation:

$$\widehat{y}_k + \alpha_1\widehat{y}_{k-1} + \cdots + \alpha_n\widehat{y}_{k-n} = 0. \tag{51}$$

One way of solving the least-squares realization problem is by rephrasing the underlying optimization problem,

$$\begin{aligned} &\min_{\boldsymbol{\alpha},\widehat{\boldsymbol{y}}} \left\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\right\|_2, \\ &\text{subject to } \widehat{y}_k = \boldsymbol{c}\boldsymbol{A}^k\boldsymbol{x}_0 \quad k = 1,\ldots,N, \end{aligned} \tag{52}$$

as a quadratic n-parameter eigenvalue problem that has the unknown model parameters as its eigenvalues (the exact coefficient matrices of which can be found in [7, 8]). The stationary points are characterized by the solutions of the MEP. At least one of the eigenvalues corresponds to the global minimum and thus yields the globally optimal parameters of the model in (50).

6.4 Model order reduction for transfer functions

In model order reduction, we start from a higher-order model of a dynamical system,

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \tag{53}$$

and we want to reduce the complexity by approximating it by a reduced-order model,

$$\widehat{H}(s) = \frac{\widehat{b}(s)}{\widehat{a}(s)} = \frac{\widehat{b}_{r-1}s^{r-1} + \cdots + \widehat{b}_1 s + \widehat{b}_0}{s^r + \widehat{a}_{r-1}s^{r-1} + \cdots + \widehat{a}_1 s + \widehat{a}_0}, \tag{54}$$

with $r \ll n$ [69]. Large models may be too complicated for simulation or for control system design. The approximation problem can, in certain norms, be rephrased as a multivariate polynomial optimization problem [10, 11], where the stationary points are candidates for model parameters of the reduced-order models. For example, in the $\mathcal{H}_2$-norm,

$$\min_{\widehat{\boldsymbol{a}},\widehat{\boldsymbol{b}}} \left\| H(s) - \widehat{H}(s) \right\|_{\mathcal{H}_2}, \tag{55}$$

which corresponds to

$$\min_{\widehat{\boldsymbol{a}},\widehat{\boldsymbol{b}}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| H(j\omega) - \widehat{H}(j\omega) \right|^2 \mathrm{d}\omega. \tag{56}$$

The underlying optimization problem is non-convex, implying that there may exist many local minima and that obtaining the global minimizers is known to be a very challenging task. State-of-the-art solvers provide a heuristic approach to the optimization problem. These algorithms are not guaranteed to converge to the globally optimal solution, despite the use of several heuristic rules during their initialization.

Another angle of attack is to obtain the globally optimal solution of the $\mathcal{H}_2$-norm model order reduction problem via a reformulation of the problem as an MEP, characterizing the stationary points as the solutions of the MEP. Every combination of eigenvalues and eigenvectors corresponds to a reduced-order transfer function, one of which is the globally optimal solution of the $\mathcal{H}_2$-norm model order reduction problem. The MEP can be obtained in different ways, the computational properties of which may vary a lot, depending on the specific orders of the original high-order and required reduced-order model. Where [12] has followed an approach inspired by iterative rational Krylov methods and [11] have applied Walsh's theorem in the frequency domain to obtain an MEP, [10] exploit the Lyapunov equation to rewrite the cost function of the underlying optimization problem. The result in all three cases is an MEP in terms of the unknown parameters of the reduced-order transfer function.

7 Conclusion and future challenges

In this survey on the rectangular multiparameter eigenvalue problem, typically referred to as the MEP, we reviewed its theoretical properties, including the spectral characterization, homogeneous interpretation, and the number of (isolated) solutions. We used didactic examples to illustrate these properties and provided a uniform explanation, including some new minor results to create a coherent story. Furthermore, we also discussed different solution approaches and various application areas, two aspects of the problem that gained additional attention during the past decade. A comparison of the available numerical algorithms to find the isolated spectrum of an MEP

was made by solving three-parameter eigenvalue problems, allowing us to draw some conclusions about the performance of the different approaches.

Although MEPs are being used in a growing list of interesting applications, many challenges in both the theoretical and algorithmic domain lie ahead. Expressions or bounds on the number of solutions for non-generic problems are still missing. They can help us to understand the structure of the problem, develop algorithms for non-generic problems, and provide insights into specific applications. The development of more efficient algorithms will be crucial to tackle larger problems (for example, different model classes in system identification or large model orders in model order reduction) and penetrate new areas in science and engineering. It will be important to exploit the problem structure, for example, the support of the polynomial matrix. The use of different number fields can enable problems from other domains, like cryptography.

References

[1] Tisseur, F., Meerbergen, K.: The quadratic eigenvalue problem. SIAM Review **43**(2), 235–286 (2001)

[2] Golub, G.H., Vorst, H.A.: Eigenvalue computation in the 20th century. Journal of Computational and Applied Mathematics **123**(1–2), 35–65 (2000)

[3] Mackey, S.D., Mackey, N., Tisseur, F.: Polynomial eigenvalue problems: Theory, computation, and structure. In: Benner, P., Bollhöfer, M., Kressner, D., Mehl, C., Stykel, T. (eds.) Numerical Algebra, Matrix Theory, Differential-Algebraic Equations and Control Theory, pp. 319–348. Springer, Cham, Switzerland (2015)

[4] Shapiro, B.: Algebro-geometric aspects of Heine–Stieltjes theory. Journal of the London Mathematical Society **83**(1), 36–56 (2010)

[5] Faugère, J.-C., Levy-dit-Vehel, F., Perret, L.: Cryptanalysis of MinRank. In: Proc. of the 28th Annual International Cryptology Conference, Santa Barbara, CA, USA, pp. 280–296 (2008)

[6] Vermeersch, C., De Moor, B.: Globally optimal least-squares ARMA model identification is an eigenvalue problem. IEEE Control Systems Letters **3**(4), 1062–1067 (2019)

[7] De Moor, B.: Least squares realization of LTI models is an eigenvalue problem. In: Proc. of the 18th European Control Conference (ECC), Naples, Italy, pp. 2270–2275 (2019)

[8] De Moor, B.: Least squares optimal realisation of autonomous LTI systems is an eigenvalue problem. Communications in Information and Systems **20**(2), 163–207 (2020)

[9] Lagauw, S., Vanpoucke, L., De Moor, B.: Exact characterization of the global optima of least squares realization of autonomous LTI models as a multiparameter

eigenvalue problem. In: Proc. of the 22nd European Control Conference (ECC), Stockholm, Sweden, pp. 3446–3451 (2024)

[10] Agudelo, O.M., Vermeersch, C., De Moor, B.: Globally optimal $\mathcal{H}_2$-norm model reduction: A numerical linear algebra approach. IFAC-PapersOnLine **54**(9), 564–571 (2021). Part of special issue: 24th International Symposium on Mathematical Theory of Networks and Systems (MTNS)

[11] Lagauw, S., Agudelo, O.M., De Moor, B.: Globally optimal SISO h_2-norm model reduction using Walsh's theorem. IEEE Control System Letters **7**, 1670–1675 (2023)

[12] Alsubaie, F.F.F.: $\mathcal{H}_2$ optimal model reduction for linear dynamic systems and the solution of multiparameter matrix pencil problems. PhD thesis, Imperial College London, London, UK (2019)

[13] Sleeman, B.D.: Multiparameter spectral theory and separation of variables. Journal of Physics A: Mathematical and Theoretical **41**(1), 1–20 (2007)

[14] Atkinson, F.V., Mingarelli, A.B.: Multiparameter Eigenvalue Problems: Sturm–Liouville Theory. CRC Press, Boca Raton, FL, USA (2010)

[15] Plestenjak, B., Gheorghiu, C.I., Hochstenbach, M.E.: Spectral collocation for multiparameter eigenvalue problems arising from separable boundary value problems. Journal of Computational Physics **298**, 585–601 (2015)

[16] Gheorghiu, C.I., Hochstenbach, M.E., Plestenjak, B., Rommes, J.: Spectral collocation solutions to multiparameter Mathieu's system. Applied Mathematics and Computation **218**(24), 11990–12000 (2012)

[17] Carmichael, R.D.: Boundary value and expansion problems: Algebraic basis of the theory. American Journal of Mathematics **43**(2), 69–101 (1921)

[18] Carmichael, R.D.: Boundary value and expansion problems: Formulation of various transcedental problems. American Journal of Mathematics **43**(4), 232–270 (1921)

[19] Carmichael, R.D.: Boundary value and expansion problems: Oscillation, comparison and expansion theorems. American Journal of Mathematics **44**(2), 129–152 (1922)

[20] Volkmer, H.: Multiparameter Eigenvalue Problems and Expansion Theorems. Lecture Notes in Mathematics, vol. 1356. Springer, Berlin, Germany (1988)

[21] Atkinson, F.V.: Multiparameter spectral theory. Bulletin of the American Mathematical Society **74**(1), 1–27 (1968)

[22] Atkinson, F.V.: Multiparameter Eigenvalue Problems. Mathematics in Science

and Engineering, vol. 82. Academic Press, New York, NY, USA (1972)

[23] Khazanov, V.B.: Generating eigenvectors of a multiparameter polynomial matrix. Journal of Mathematical Sciences **101**(4), 3326–3337 (2000)

[24] Blum, E.K., Curtis, A.R.: A convergent gradient method for matrix eigenvector-eigentuple problems. Numerische Mathematik **31**(3), 247–263 (1978)

[25] Blum, E.K., Chang, A.-F.: A numerical method for the solution of the double eigenvalue problem. IMA Journal of Applied Mathematics **22**(1), 29–42 (1978)

[26] Wicks, M.A., DeCarlo, R.A.: Computing most nearly rank-reducing structured matrix perturbations. SIAM Journal on Matrix Analyis and Applications **16**(1), 123–137 (1995)

[27] Khazanov, V.B.: The multiparameter eigenvalue problem: Jordan vector semilattices. Journal of Mathematical Sciences **86**(4), 2944–2949 (1997)

[28] Khazanov, V.B.: On spectral properties of multiparameter polynomial matrices. Journal of Mathematical Sciences **89**(6), 1775–1800 (1998)

[29] Kublanovskaya, V.N.: On some factorizations of two-parameter polynomial matrices. Journal of Mathematical Sciences **86**(4), 2866–2879 (1997)

[30] Khazanov, V.B.: Methods for solving spectral problems for multiparameter matrix pencils. Journal of Mathematical Sciences **127**, 2033–2050 (2005)

[31] Khazanov, V.B.: To solving spectral problems for multiparameter polynomial matrices. Journal of Mathematical Sciences **141**(6), 1690–1700 (2007)

[32] Shapiro, B., Shapiro, M.: On eigenvalues of rectangular matrices. Proc. of the Steklov Institute of Mathematics **267**(1), 248–255 (2009)

[33] Blum, E.K., Geltner, P.B.: Numerical solution of eigentuple-eigenvector problems in Hilbert spaces by a gradient method. Numerische Mathematik **31**(3), 231–246 (1978)

[34] Hochstenbach, M.E., Košir, T., Plestenjak, B.: Numerical methods for rectangular multiparameter eigenvalue problems, with applications to finding optimal ARMA and LTI models. Numerical Linear Algebra with Applications **31**(2), 2540–123 (2024)

[35] Gungah, S.K., Alsubaie, F.F.F., Jaimoukha, I.M.: On the two-parameter matrix pencil problem. SIAM Journal on Matrix Analysis and Applications **45**(3), 1287–1309 (2024)

[36] Plestenjak, B.: MultiParEig: Toolbox for Multiparameter Eigenvalue Problems. https://mathworks.com/matlabcentral/fileexchange/47844-multipareig.

(Visited on July 8, 2023) (2023)

[37] De Cock, K., De Moor, B.: Multiparameter eigenvalue problems and shift-invariance. IFAC-PapersOnLine **54**(9), 159–165 (2021). Part of special issue: 24th International Symposium on Mathematical Theory of Networks and Systems (MTNS)

[38] Vermeersch, C., De Moor, B.: Two complementary block Macaulay matrix algorithms to solve multiparameter eigenvalue problems. Linear Algebra and its Applications **654**, 177–209 (2022)

[39] Vermeersch, C., De Moor, B.: Two double recursive block Macaulay matrix algorithms to solve multiparameter eigenvalue problems. IEEE Control Systems Letters **7**, 319–324 (2023)

[40] Vermeersch, C., De Moor, B.: Solving multivariate polynomial systems and rectangular multiparameter eigenvalue problems with MacaulayLab. Technical report, KU Leuven, Leuven, Belgium (2025)

[41] Joshi, V.N.: A determinant for rectangular matrices. Bulletin for the Australian Mathematical Society **21**(1), 137–146 (1980)

[42] Khazanov, V.B.: Computation of characteristics of the regular finite spectrum of a singular multiparameter polynomial matrix. Journal of Mathematical Sciences **249**(2), 281–289 (2020)

[43] Stetter, H.J.: Numerical Polynomial Algebra. SIAM, Philadelphia, PA, USA (2004)

[44] Dayton, B.H., Li, T.-Y., Zeng, Z.: Multiple zeros of nonlinear systems. Mathematics of Computation **80**(276), 2143–2168 (2011)

[45] Zeng, Z.: The closedness subspace method for computing the multiplicity structure of a polynomial system. In: Bates, D.J., Besana, G.M., Di Rocco, S., Wampler, C.W. II (eds.) Contemporary Mathematics: Interactions of Classical and Numerical Algebraic Geometry, vol. 496, pp. 347–361 (2009)

[46] Cox, D.A., Little, J.B., O'Shea, D.: Ideals, Varieties, and Algorithms, 4th edn. Undergraduate Texts in Mathematics. Springer, New York, NY, USA (2015)

[47] Browne, P.J., Sleeman, B.D.: A numerical technique for multiparameter eigenvalue problems. IMA Journal of Numerical Analysis **2**(4), 451–457 (1982)

[48] Bohte, Z.: Numerical solution of some two-parameter eigenvalue problems. In: Anton Kuhelj Memorial Volume, pp. 17–28. Slovenian Academy of Science and Art, Ljubljana, Slovenia (1982). [citation only]

[49] Ji, X.: Numerical solution of joint eigenpairs of a family of commutative matrices.

Applied Mathematics Letters **4**(3), 57–60 (1991)

[50] Slivnik, T., Tomšič, G.: A numerical method for the solution of two-parameter eigenvalue problems. Journal of Computational and Applied Mathematics **15**(1), 109–115 (1986)

[51] Hochstenbach, M.E., Košir, T., Plestenjak, B.: A Jacobi–Davidson type method for the two-parameter eigenvalue problem. SIAM Journal on Matrix Analysis and Applications **26**(2), 477–497 (2004)

[52] He, H., Kressner, D., Plestenjak, B.: Randomized methods for computing joint eigenvalues, with applications to multiparameter eigenvalue problems and root finding. Numerical Algorithms, 1–32 (2024)

[53] Vermeersch, C., De Moor, B.: Recursive algorithms to update a numerical basis matrix of the null space of the block row, (banded) block Toeplitz, and block Macaulay matrix. SIAM Journal on Scientific Computing (SISC) **45**(2), 596–620 (2023)

[54] Plestenjak, B.: A continuation method for a right definite two-parameter eigenvalue problem. SIAM Journal on Matrix Analysis and Applications **21**(4), 1163–1184 (2000)

[55] Dong, B., Yu, B., Yu, Y.: A homotopy method for finding all solutions of a multiparameter eigenvalue problem. SIAM Journal on Matrix Analysis and Applications **37**(2), 550–571 (2016)

[56] Rodriguez, J.I., Du, J.-H., You, Y., Lim, L.-H.: Fiber product homotopy method for multiparameter eigenvalue problems. Numerische Mathematik **148**(4), 853–888 (2021)

[57] Hochstenbach, M.E., Meerbergen, K., Mengi, E., Plestenjak, B.: Subspace methods for three-parameter eigenvalue problems. Numerical Linear Algebra with Applications **26**(4), 1–22 (2019)

[58] Hochstenbach, M.E., Plestenjak, B.: A Jacobi–Davidson type method for a right definite two-parameter eigenvalue problem. SIAM Journal on Matrix Analysis and Applications **24**(2), 392–410 (2002)

[59] Hochstenbach, M.E., Plestenjak, B.: Harmonic Rayleigh–Ritz extraction for the multiparameter eigenvalue problem. Electronic Transactions on Numerical Analysis **29**, 81–96 (2008)

[60] Ruymbeek, K., Meerbergen, K., Michiels, W.: Subspace method for multiparameter-eigenvalue problems based on tensor-train representations. Numerical Linear Algebra with Applications **29**(5), 1–202439 (2022)

[61] Buss, J.F., Frandsen, G.S., Shallit, J.O.: The computational complexity of some problems of linear algebra. Journal of Computer and System Sciences **58**(3), 572–596 (1999)

[62] Courtois, N.T.: Efficient zero-knowledge authentication based on a linear algebra problem MinRank. In: Proc. of the 7th International Conference on the Theory and Application of Cryptology and Information Security, Gold Coast, Australia, pp. 401–421 (2001)

[63] Kipnis, A., Shamir, A.: Cryptanalysis of the HFE public key cryptosystem by relinearization. In: Proc. of the 19th Annual International Cryptology Conference, Santa Barbara, CA, USA, pp. 19–30 (1999)

[64] Patarin, J.: Hidden fields equations (HFE) and isomorphisms of polynomials (IP): Two new families of asymmetric algorithms. In: Proc. of the 15th International Conference on the Theory and Application of Cryptographic Techniques, Saragossa, Spain, pp. 33–48 (1996)

[65] Goubin, L., Courtois, N.T.: Cryptanalysis of the TTM cryptosystem. In: Proc. of the 6th International Conference on the Theory and Application of Cryptology and Information Security, Kyoto, Japan, pp. 44–57 (2000)

[66] Ljung, L.: System Identification: Theory for the User, 2nd edn. Prentice Hall Information and System Science Series. Prentice Hall PTR, Upper Saddle River, NJ, USA (1999)

[67] Box, G.E.P., Jenkins, G.M.: Time Series Analysis: Forecasting and Control, Revised edn. Holden-Day Series in Time Series Analysis. Holden Day, Oakland, CA, USA (1976)

[68] Van Overschee, P., De Moor, B.: Subspace Identification for Linear Systems: Theory — Implementation — Applications. Kluwer Academic Publishers, Dordrecht, The Netherlands (1996)

[69] Antoulas, A.C.: Approximation of Large-Scale Dynamical Systems. Advances in Design and Control. SIAM, Philadelphia, PA, USA (2005)