On the a-posteriori SNR of the speech-distortion weighted Wiener filter for single-channel noise reduction

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1 Introduction

Wiener filtering is a commonly applied technique for noise reduction in single- and multi-channel signals [1], e.g. in speech enhancement applications [2, 3, 4, 5, 6, 7]. The standard Wiener filter (WF) minimises the mean square error (MSE) between the output signal and the speech component of the input signal. Hence, the error signal typically consists of a term related to noise reduction and a term related to speech distortion. While the standard Wiener filter assigns equal importance to both terms, a generalised version of the Wiener filter, the so-called speech-distortion weighted Wiener filter (SDW-WF) [4, 6], provides a trade-off between the noise reduction and the speech distortion term.
In this report we prove that *the a-posteriori SNR after noise reduction with the single-channel (speech-distortion weighted) Wiener filter is always larger than or equal to the a-priori SNR.* Although this may seem a trivial result, we have not found any proof in the literature. Strictly speaking, it is actually not so obvious that minimisation of the MSE leads to an SNR improvement, since no direct relationship exists between MSE and SNR.

2 Single-channel (SDW) Wiener filter

2.1 Wiener filter

Consider a noisy microphone signal $y(n)$ at time n, consisting of a zero-mean clean speech signal $x(n)$ and a zero-mean (white or coloured) noise signal $v(n)$, which is assumed to be uncorrelated with $x(n)$,

$$y(n) = x(n) + v(n) \; . \tag{2.1}$$

The goal of the Wiener filter $\mathbf{h}_o$ is to provide an estimate of the clean speech signal $x(n)$, by minimising the MSE between the clean speech signal and the output signal,

$$\mathbf{h}_o = \underset{\mathbf{h}}{\operatorname{argmin}} \; J_x(\mathbf{h}) = \underset{\mathbf{h}}{\operatorname{argmin}} \; \mathcal{E}\big\{ \left[x(n) - \mathbf{h}^T \mathbf{y}(n)\right]^2 \big\} \; , \tag{2.2}$$

where ^{T} denotes transpose of a vector or a matrix, $\mathbf{h}$ is an FIR filter of length L and $\mathbf{y}(n)$ is an L-dimensional data vector,

$$\mathbf{h} = \left[\begin{array}{cccc} h_0 & h_1 & \ldots & h_{L-1} \end{array} \right]^T \tag{2.3}$$

$$\mathbf{y}(n) = \left[\begin{array}{cccc} y(n) & y(n-1) & \ldots & y(n-L+1) \end{array} \right]^T \; . \tag{2.4}$$

Solving (2.2) and using the assumption that speech and noise are uncorrelated, the Wiener filter can be written as

$$\mathbf{h}_o = \mathbf{R}_y^{-1} \mathbf{R}_x \mathbf{u}_1 = \left[\mathbf{R}_x + \mathbf{R}_v\right]^{-1} \mathbf{R}_x \mathbf{u}_1 \; , \tag{2.5}$$

with

$$\mathbf{R}_y = \mathcal{E}\{\mathbf{y}(n)\mathbf{y}^T(n)\} \tag{2.6}$$

$$\mathbf{R}_x = \mathcal{E}\{\mathbf{x}(n)\mathbf{x}^T(n)\} \tag{2.7}$$

$$\mathbf{R}_v = \mathcal{E}\{\mathbf{v}(n)\mathbf{v}^T(n)\} \tag{2.8}$$

the $L \times L$-dimensional symmetric correlation matrices of the noisy signal, the clean speech signal and the noise signal, respectively, and

$$\mathbf{u}_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T . \tag{2.9}$$

In addition, we define the normalised speech and noise correlation matrices

$$\tilde{\mathbf{R}}_x = \frac{\mathbf{R}_x}{\sigma_x^2} \tag{2.10}$$

$$\tilde{\mathbf{R}}_v = \frac{\mathbf{R}_v}{\sigma_v^2} , \tag{2.11}$$

with σ_x^2 and σ_v^2 the power of the speech and the noise signal and

$$\text{SNR} = \frac{\sigma_x^2}{\sigma_v^2} \tag{2.12}$$

the a-priori signal-to-noise ratio (SNR). Note that the diagonal elements of the normalised correlation matrices $\tilde{\mathbf{R}}_x$ and $\tilde{\mathbf{R}}_v$ are equal to 1. Using (2.10), (2.11) and (2.12), the Wiener filter can be written as

$$\boxed{\mathbf{h}_o = \left[\tilde{\mathbf{R}}_x + \frac{\tilde{\mathbf{R}}_v}{\text{SNR}}\right]^{-1} \tilde{\mathbf{R}}_x \mathbf{u}_1} \tag{2.13}$$

2.2 SDW Wiener filter

The MSE cost function $J_x(\mathbf{h})$ in (2.2) can be written as

$$J_x(\mathbf{h}) = \underbrace{\mathcal{E}\{ \left[x(n) - \mathbf{h}^T\mathbf{x}(n)\right]^2 \}}_{\epsilon_x^2(n)} + \underbrace{\mathcal{E}\{ \left[\mathbf{h}^T\mathbf{v}(n)\right]^2 \}}_{\epsilon_v^2(n)} , \tag{2.14}$$

consisting of a term $\epsilon_x^2(n)$ related to speech distortion and a term $\epsilon_v^2(n)$ related to noise reduction. While the standard Wiener filter assigns equal importance to both terms, a generalised version of the Wiener filter, the so-called speech-distortion weighted Wiener filter (SDW-WF), provides a trade-off between the noise reduction and the speech distortion term [4, 6]. This generalised cost function can be written as

$$J_\mu(\mathbf{h}) = \mathcal{E}\{ \left[x(n) - \mathbf{h}^T\mathbf{x}(n)\right]^2 \} + \mu\, \mathcal{E}\{ \left[\mathbf{h}^T\mathbf{v}(n)\right]^2 \} , \tag{2.15}$$

where $\mu \geq 0$ is the trade-off parameter between noise reduction and speech distortion. The SDW-WF $\mathbf{h}_\mu$ minimising this generalised cost function is equal to

$$\boxed{\mathbf{h}_\mu = \left[\mathbf{R}_x + \mu\mathbf{R}_v\right]^{-1} \mathbf{R}_x\mathbf{u}_1 = \left[\tilde{\mathbf{R}}_x + \frac{\mu\,\tilde{\mathbf{R}}_v}{\text{SNR}}\right]^{-1} \tilde{\mathbf{R}}_x\mathbf{u}_1} \tag{2.16}$$

From this formula one can see that the SDW-WF for a noisy signal with a certain a-priori signal-to-noise ratio SNR is equal to the WF for the same signal with a-priori signal-to-noise ratio SNR/μ. If $\mu > 1$, the noise level is assumed to be higher than the actual level, such that the residual noise level is reduced at the expense of increased signal distortion. On the contrary, if $\mu < 1$, signal distortion is reduced at the expense of decreased noise reduction [8].

2.3 A-posteriori SNR

The a-posteriori signal-to-noise ratio SNR_o after noise reduction with the SDW-WF $\mathbf{h}_\mu$ is equal to

$$\boxed{\text{SNR}_o = \frac{\mathbf{h}_\mu^T \mathbf{R}_x \mathbf{h}_\mu}{\mathbf{h}_\mu^T \mathbf{R}_v \mathbf{h}_\mu} = \text{SNR}\, \frac{\mathbf{h}_\mu^T \tilde{\mathbf{R}}_x \mathbf{h}_\mu}{\mathbf{h}_\mu^T \tilde{\mathbf{R}}_v \mathbf{h}_\mu}} \tag{2.17}$$

Hence, the ratio between the a-posteriori and the a-priori SNR is equal to

$$\frac{\text{SNR}_o}{\text{SNR}} = \frac{\mathbf{u}_1^T \tilde{\mathbf{R}}_x \left[\tilde{\mathbf{R}}_x + \frac{\mu \tilde{\mathbf{R}}_v}{\text{SNR}}\right]^{-1} \tilde{\mathbf{R}}_x \left[\tilde{\mathbf{R}}_x + \frac{\mu \tilde{\mathbf{R}}_v}{\text{SNR}}\right]^{-1} \tilde{\mathbf{R}}_x \mathbf{u}_1}{\mathbf{u}_1^T \tilde{\mathbf{R}}_x \left[\tilde{\mathbf{R}}_x + \frac{\mu \tilde{\mathbf{R}}_v}{\text{SNR}}\right]^{-1} \tilde{\mathbf{R}}_v \left[\tilde{\mathbf{R}}_x + \frac{\mu \tilde{\mathbf{R}}_v}{\text{SNR}}\right]^{-1} \tilde{\mathbf{R}}_x \mathbf{u}_1} \,. \tag{2.18}$$

The goal of this report is to prove that for every possible $\tilde{\mathbf{R}}_x$, $\tilde{\mathbf{R}}_v$, SNR and μ this ratio is larger than or equal to 1.

2.4 Generalised eigenvalue decomposition

The generalised eigenvalue decomposition of the normalised correlation matrices $\tilde{\mathbf{R}}_x$ and $\tilde{\mathbf{R}}_v$ is defined as [9]

$$\left\{ \begin{array}{rcl} \tilde{\mathbf{R}}_x & = & \mathbf{Q}\, \mathbf{\Lambda}\, \mathbf{Q}^T \\ \tilde{\mathbf{R}}_v & = & \mathbf{Q}\, \mathbf{Q}^T \,, \end{array} \right. \tag{2.19}$$

with $\mathbf{Q}$ an $L \times L$-dimensional invertible, but not necessarily orthogonal, matrix and $\mathbf{\Lambda}$ an $L \times L$-dimensional diagonal matrix, containing the generalised eigenvalues λ_i, $i = 1 \ldots L$ [1]. Using (2.19), the SDW-WF in (2.16) can be written as

$$\mathbf{h}_\mu = \mathbf{Q}^{-T} \left[\mathbf{\Lambda} + \delta \mathbf{I}\right]^{-1} \mathbf{\Lambda} \mathbf{q}_1 \,, \tag{2.20}$$

with $\mathbf{I}$ the $L \times L$-dimensional identity matrix, $\delta = \mu/\text{SNR} \geq 0$ and $\mathbf{q}_1 = \mathbf{Q}^T \mathbf{u}_1$. The elements of the L-dimensional vector $\mathbf{q}_1$ will be denoted by

$$\mathbf{q}_1 = \left[\begin{array}{cccc} \alpha_1 & \alpha_2 & \ldots & \alpha_L \end{array} \right]^T \,. \tag{2.21}$$

The ratio between the a-posteriori and the a-priori SNR can now be written as

$$\frac{\text{SNR}_o}{\text{SNR}} = \frac{\mathbf{q}_1^T \left[\mathbf{\Lambda} + \delta \mathbf{I}\right]^{-2} \mathbf{\Lambda}^3 \mathbf{q}_1}{\mathbf{q}_1^T \left[\mathbf{\Lambda} + \delta \mathbf{I}\right]^{-2} \mathbf{\Lambda}^2 \mathbf{q}_1} = \frac{\sum_{i=1}^{L} \frac{\lambda_i^3}{(\lambda_i + \delta)^2}\, \alpha_i^2}{\sum_{i=1}^{L} \frac{\lambda_i^2}{(\lambda_i + \delta)^2}\, \alpha_i^2} \,. \tag{2.22}$$

Since the diagonal elements of $\tilde{\mathbf{R}}_v$ and $\tilde{\mathbf{R}}_x$ are equal to 1, it can be easily shown that

$$\mathbf{q}_1^T \mathbf{q}_1 = \mathbf{u}_1^T\, \mathbf{Q}\mathbf{Q}^T\, \mathbf{u}_1 = \mathbf{u}_1^T \tilde{\mathbf{R}}_v \mathbf{u}_1 = 1 \tag{2.23}$$

$$\mathbf{q}_1^T \mathbf{\Lambda} \mathbf{q}_1 = \mathbf{u}_1^T\, \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T\, \mathbf{u}_1 = \mathbf{u}_1^T \tilde{\mathbf{R}}_x \mathbf{u}_1 = 1 \,, \tag{2.24}$$

[1] The generalised eigenvalues λ_i are always real and $\lambda_i \geq 0$, since $\tilde{\mathbf{R}}_x$ and $\tilde{\mathbf{R}}_v$ are positive (semi-)definite.

such that

$$\sum_{i=1}^{L} \alpha_i^2 = 1 \qquad \sum_{i=1}^{L} \lambda_i \alpha_i^2 = 1 \; . \tag{2.25}$$

From these constraints it can be easily seen that

$$0 \leq \alpha_i^2 \leq 1, \quad 0 \leq \lambda_i \alpha_i^2 \leq 1, \quad i = 1 \ldots L \; . \tag{2.26}$$

3 Proof that $\mathbf{SNR}_o \geq \mathbf{SNR}$

Using (2.22), proving that the a-posteriori SNR is larger than or equal to the a-priori SNR is equivalent to proving that

$$\sum_{i=1}^{L} \frac{\lambda_i^2(\lambda_i - 1)}{(\lambda_i + \delta)^2} \, \alpha_i^2 \geq 0 \; . \tag{3.1}$$

Incorporating the constraints formulated in (2.25), this comes down to proving that the solution of the constrained minimisation problem

$$\boxed{\min \sum_{i=1}^{L} \frac{\lambda_i^2(\lambda_i - 1)}{(\lambda_i + \delta)^2} \, \alpha_i^2, \quad \text{subject to} \quad \sum_{i=1}^{L} \alpha_i^2 = 1, \quad \sum_{i=1}^{L} \lambda_i \alpha_i^2 = 1} \tag{3.2}$$

is equal to 0.

CASE 1: $\delta = 0$ ($\mu = 0$ and/or SNR $= \infty$)
In this case we can easily prove that

$$\sum_{i=1}^{L} \frac{\lambda_i^2(\lambda_i - 1)}{(\lambda_i + \delta)^2} \, \alpha_i^2 = \sum_{i=1}^{L} (\lambda_i - 1) \, \alpha_i^2 = 1 - 1 = 0 \; . \tag{3.3}$$

CASE 2: $\delta > 0$
The constrained optimisation problem (3.2) can be solved by introducing the Lagrange-multipliers β and γ, defining the cost function

$$J(\beta, \gamma) = \sum_{i=1}^{L} \frac{\lambda_i^2(\lambda_i - 1)}{(\lambda_i + \delta)^2} \, \alpha_i^2 + \beta \left[\sum_{i=1}^{L} \alpha_i^2 - 1 \right] + \gamma \left[\sum_{i=1}^{L} \lambda_i \alpha_i^2 - 1 \right] \; , \tag{3.4}$$

and setting the $(2L + 2)$ partial derivatives equal to zero, i.e.

$$\frac{\partial J(\beta, \gamma)}{\partial \lambda_i} = \frac{\lambda_i(\lambda_i^2 + 3\delta\lambda_i - 2\delta)}{(\lambda_i + \delta)^3} \alpha_i^2 + \gamma \alpha_i^2 = 0, \quad i = 1 \ldots L \tag{3.5}$$

$$\frac{\partial J(\beta, \gamma)}{\partial \alpha_i} = \frac{2\lambda_i^2(\lambda_i - 1)}{(\lambda_i + \delta)^2} \alpha_i + 2\beta\alpha_i + 2\gamma\lambda_i\alpha_i = 0, \quad i = 1 \ldots L \tag{3.6}$$

$$\frac{\partial J(\beta, \gamma)}{\partial \beta} = \sum_{i=1}^{L} \alpha_i^2 - 1 = 0 \tag{3.7}$$

$$\frac{\partial J(\beta, \gamma)}{\partial \gamma} = \sum_{i=1}^{L} \lambda_i \alpha_i^2 - 1 = 0 \; . \tag{3.8}$$

The minimum cost function is found by solving this set of equations for the $(2L+2)$ variables λ_i $(i = 1 \dots L)$, α_i $(i = 1 \dots L)$, β and γ. For any solution (provided that more than one solution exists), one can immediately verify that by multiplying (3.6) by $\alpha_i/2$ and summing over all i, one obtains

$$\sum_{i=1}^{L} \frac{\lambda_i^2(\lambda_i - 1)}{(\lambda_i + \delta)^2}\, \alpha_i^2 + \beta \sum_{i=1}^{L} \alpha_i^2 + \gamma \sum_{i=1}^{L} \lambda_i \alpha_i^2 = 0\ , \tag{3.9}$$

such that the minimum cost function J_{min} is equal to

$$J_{min} = \sum_{i=1}^{L} \frac{\lambda_i^2(\lambda_i - 1)}{(\lambda_i + \delta)^2}\, \alpha_i^2 = -\beta - \gamma\ . \tag{3.10}$$

We will now simultaneously solve the set of equations (3.5)-(3.8). Since $\delta > 0$, equations (3.5) and (3.6) can be rewritten as

$$\alpha_i^2 \underbrace{\left[\lambda_i(\lambda_i^2 + 3\delta\lambda_i - 2\delta) + \gamma(\lambda_i + \delta)^3\right]}_{f(\lambda_i,\gamma,\delta)} = 0, \quad i = 1 \dots L \tag{3.11}$$

$$2\alpha_i \underbrace{\left[\lambda_i^2(\lambda_i - 1) + \beta(\lambda_i + \delta)^2 + \gamma\lambda_i(\lambda_i + \delta)^2\right]}_{g(\lambda_i,\beta,\gamma,\delta)} = 0, \quad i = 1 \dots L \tag{3.12}$$

where $f(\lambda, \gamma, \delta)$ and $g(\lambda, \beta, \gamma, \delta)$ are 3th-order polynomials in the variable λ. From (3.11) and (3.12), one can verify that the variables $\{\lambda_i, \alpha_i\}$, $i = 1 \dots L$, either satisfy

$$(A) \qquad \alpha_i = 0, \quad \lambda_i \geq 0 \tag{3.13}$$

$$(B) \qquad 0 < \alpha_i^2 \leq 1, \quad f(\lambda_i, \gamma, \delta) = 0 \ \text{ and } \ g(\lambda_i, \beta, \gamma, \delta) = 0\ . \tag{3.14}$$

It is not possible for all solutions $\{\lambda_i, \alpha_i\}$ to belong to (A), since in that case $\sum_{i=1}^{L} \alpha_i^2 = 0$. The variables λ_i, $i \in B$, can only take on a *finite* number of possible values, since for a certain value of β, γ, and δ, the polynomials $f(\lambda, \gamma, \delta)$ and $g(\lambda, \beta, \gamma, \delta)$ have a finite number of common roots. In Section 4 it is shown that the polynomials $f(\lambda, \gamma, \delta)$ and $g(\lambda, \beta, \gamma, \delta)$ maximally have 2 common roots, which we will represent by λ_{B1} and λ_{B2}. Moreover, it is shown that λ_{B1} and λ_{B2} satisfy one of the following conditions:

$$\begin{cases} (a) & \lambda_{B1}, \lambda_{B2} \ \text{ are complex} \\ (b) & 0 < \lambda_{B1} < \frac{2}{3},\ 0 < \lambda_{B2} \leq \frac{1}{3} \\ (c) & \lambda_{B1} > 0,\ \lambda_{B2} \leq 0 \\ (d) & \lambda_{B1} < 0,\ \lambda_{B2} < 0 \end{cases} \tag{3.15}$$

Since λ_{B1} and λ_{B2} represent real and positive generalised eigenvalues, conditions (a) and (d) are impossible. Condition (b) corresponds to 2 possible positive solutions $\lambda_{B1} < \frac{2}{3},\ \lambda_{B2} \leq \frac{1}{3}$, while condition (c) corresponds to 1 possible solution[2] $\lambda_{B1} > 0$.

[2]The case $\lambda_{B2} = 0$ is not feasible, since in order for $f(\lambda, \gamma, \delta)$ and $g(\lambda, \beta, \gamma, \delta)$ to share a common root $\lambda = 0$, one should have $f(0, \gamma, \delta) = \gamma\delta^3 = 0$ and $g(0, \beta, \gamma, \delta) = \beta\delta^2 = 0$, hence $\beta = 0$ and $\gamma = 0$ (since $\delta > 0$). If $\beta = 0$ and $\gamma = 0$, the variables $\{\lambda_i, \alpha_i\}$, $i = 1 \dots L$, either satisfy (A) $\alpha_i = 0$, $\lambda_i \geq 0$, (B) $0 < \alpha_i^2 \leq 1$, $\lambda_i = 0$, or (C) $\lambda_i^2 + 3\delta\lambda_i - 2\delta = 0$, $\lambda_i - 1 = 0$ (which would imply $\delta = 0$). Moreover, it is not possible for all solutions $\{\lambda_i, \alpha_i\}$ to belong to (A) and (B), since in that case $\sum_{i=1}^{L} \lambda_i \alpha_i^2 = 0$.

In addition to satisfying (3.11) and (3.12), the variables $\{\lambda_i, \alpha_i\}$, $i = 1 \ldots L$, should also satisfy the constraints (3.7) and (3.8), i.e.

$$\sum_{i=1}^{L} \alpha_i^2 = \sum_{i\in A} \alpha_i^2 + \sum_{i\in B} \alpha_i^2 = \sum_{i\in B} \alpha_i^2 = 1 \tag{3.16}$$

$$\sum_{i=1}^{L} \lambda_i \alpha_i^2 = \sum_{i\in A} \lambda_i \alpha_i^2 + \sum_{i\in B} \lambda_i \alpha_i^2 = \sum_{i\in B} \lambda_i \alpha_i^2 = 1 \; . \tag{3.17}$$

However, using (3.16), one can show that for condition (b)

$$\sum_{i=1}^{L} \lambda_i \alpha_i^2 = \lambda_{B1} \sum_{i\in B1} \alpha_i^2 + \lambda_{B2} \sum_{i\in B2} \alpha_i^2 < \sum_{i\in B1} \alpha_i^2 + \sum_{i\in B2} \alpha_i^2 = \sum_{i\in B} \alpha_i^2 = 1 \; , \tag{3.18}$$

which is in contradiction with (3.17). Hence, condition (b) is also impossible, leaving condition (c) as the only possibility. Using (3.16), one can show that for condition (c)

$$\sum_{i=1}^{L} \lambda_i \alpha_i^2 = \lambda_{B1} \sum_{i\in B} \alpha_i^2 = \lambda_{B1} \; , \tag{3.19}$$

such that λ_{B1} should be equal to 1 in order to satisfy (3.17). Hence, solving $f(\lambda_i, \gamma, \delta) = 0$ and $g(\lambda_i, \beta, \gamma, \delta) = 0$ for β and γ with $\lambda_i = \lambda_{B1} = 1$ leads to

$$\gamma = -\frac{\delta}{(1+\delta)^3}, \quad \beta = \frac{\delta}{(1+\delta)^3} \; , \tag{3.20}$$

such that, using (3.10),

$$\boxed{J_{min} = -\beta - \gamma = 0} \tag{3.21}$$

4 Solutions for $f(\lambda, \gamma, \delta) = 0$ and $g(\lambda, \beta, \gamma, \delta) = 0$

The 3-th order polynomials $f(\lambda, \gamma, \delta)$ and $g(\lambda, \beta, \gamma, \delta)$, defined in (3.11) and (3.12), can be written as

$$f(\lambda, \gamma, \delta) = (1+\gamma)\lambda^3 + 3\delta(1+\gamma)\lambda^2 + \delta(3\gamma\delta - 2)\lambda + \gamma\delta^3 \tag{4.1}$$

$$g(\lambda, \beta, \gamma, \delta) = (1+\gamma)\lambda^3 + (2\gamma\delta + \beta - 1)\lambda^2 + \delta(2\beta + \gamma\delta)\lambda + \beta\delta^2 \tag{4.2}$$

For any λ for which $f(\lambda, \gamma, \delta) = 0$ and $g(\lambda, \beta, \gamma, \delta) = 0$, it also holds that

$$f(\lambda, \gamma, \delta) - g(\lambda, \beta, \gamma, \delta) = (3\delta + \gamma\delta - \beta + 1)\lambda^2 + 2\delta(\gamma\delta - \beta - 1)\lambda + \delta^2(\gamma\delta - \beta) = 0 \; , \tag{4.3}$$

such that the polynomials $f(\lambda, \gamma, \delta)$ and $g(\lambda, \beta, \gamma, \delta)$ can maximally have 2 common roots, which we will represent by λ_{B1} and λ_{B2},

$$\lambda_{B1} = \frac{\left[\rho + 1 + \sqrt{1 + 3(\delta+1)\rho}\right]\delta}{3\delta + 1 - \rho} \tag{4.4}$$

$$\lambda_{B2} = \frac{\left[\rho + 1 - \sqrt{1 + 3(\delta+1)\rho}\right]\delta}{3\delta + 1 - \rho} \; , \tag{4.5}$$

with $\rho = \beta - \gamma\delta$.

Depending on the value of ρ, different cases can be distinguished:

1. $\underline{\rho < -1/3(\delta+1)}$: both λ_{B1} and λ_{B2} are complex
2. $\underline{\rho = -1/3(\delta+1)}$: $\lambda_{B1} = \lambda_{B2} = \frac{\delta}{3\delta+2}$ (for all possible δ : $0 < \lambda_{B1} = \lambda_{B2} \leq \frac{1}{3}$)
3. $\underline{-1/3(\delta+1) < \rho < 0}$: $\frac{\delta}{3\delta+2} < \lambda_{B1} < \frac{2\delta}{3\delta+1}$, $0 < \lambda_{B2} < \frac{\delta}{3\delta+2}$ (for all possible δ : $0 < \lambda_{B1} < \frac{2}{3}$, $0 < \lambda_{B2} < \frac{1}{3}$)
4. $\underline{\rho = 0}$: $\lambda_{B1} = \frac{2\delta}{3\delta+1}$, $\lambda_{B2} = 0$ (for all possible δ : $0 < \lambda_{B1} < \frac{2}{3}$)
5. $\underline{0 < \rho < 3\delta+1}$: $\lambda_{B1} > \frac{2\delta}{3\delta+1}$, $-\frac{(3\delta+1)\delta}{2(3\delta+2)} < \lambda_{B2} < 0$ (for all possible δ : $\lambda_{B1} > 0$)
6. $\underline{\rho = 3\delta+1}$: (4.3) reduces to a linear equation with one solution $\lambda_{B2} = -\frac{(3\delta+1)\delta}{2(3\delta+2)}$
7. $\underline{\rho > 3\delta+1}$: $\lambda_{B1} < -\delta$, $-\delta < \lambda_{B2} < -\frac{(3\delta+1)\delta}{2(3\delta+2)}$

These cases can be summarised as follows:

$$\begin{cases} (a) & \lambda_{B1}, \lambda_{B2} \text{ are complex} \\ (b) & 0 < \lambda_{B1} < \frac{2}{3},\ 0 < \lambda_{B2} \leq \frac{1}{3} \\ (c) & \lambda_{B1} > 0,\ \lambda_{B2} \leq 0 \\ (d) & \lambda_{B1} < 0,\ \lambda_{B2} < 0 \end{cases} \tag{4.6}$$

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