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Consistent estimation of an ellipsoid with known center [1]

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Abstract

A consistent estimator is derived for the parameter of an implicit quadratic measurement error model. The true model describes a centered ellipsoid and the true values of the measurement points are on the boundary of the ellipsoid. The estimation procedure is applicable for ellipsoid fitting with known center. Due to the nonlinearity of the model the ordinary least squares and the orthogonal regression estimators are inconsistent, *i.e.*, their estimates does not converge to the true value of the parameter as the sample size tends to infinity. The proposed consistent estimator is derived by correcting the ordinary least squares cost function. The correction is explicitly given in terms of the measurement error variance. If the variance is unknown an estimator for this parameter is proposed as well. The measurement errors are assumed to form an i.i.d. sequence of normal random variables. The derived estimator is asymptotically normal.

CONSISTENT ESTIMATION OF AN ELLIPSOID WITH KNOWN CENTER

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1 Introduction

A parameter estimation problem occurs when the relation among some observed variables $x_1, \ldots, x_n$ is described by a parameterized model. The parameters identify a unique model in a given *model class*, and the problem is to choose a model from the model class, given a set of observations $\{x^{(l)}\}_{l=1}^m$, where $x^{(l)} := [x_1^{(l)} \cdots x_n^{(l)}]^\top$ is the l-th observed vector of variables. The model is selected according to certain performance criteria, specified later.

We consider a *quadratic form model*, $x^\top Ax = 1$, $A > 0$ relating the variables $x := [x_1 \cdots x_n]^\top$. The parameter of the model is the positive definite matrix A. The set of points $\mathcal{E}(A) := \{\, x \in \mathbb{R}^n : x^\top Ax = 1 \,\}$ satisfying the quadratic form model is a centered hyper ellipsoid.

The vector of variables x is observed with additive error $\tilde{x} = [\tilde{x}_1 \cdots \tilde{x}_n]^\top$ and the error is described stochastically. The true value $\bar{x} = [\bar{x}_1 \cdots \bar{x}_n]^\top$ of the measured variables is assumed to satisfy the model for some unknown true value $\bar{A} > 0$ of the parameter. This assumption defines a *true model* in the model class. Models in which the variables are measured with additive noise $x = \bar{x} + \tilde{x}$ are called *measurement error models*.

The quadratic form model is linear in the parameters, so that the linear least squares technique can be applied. This corresponds to estimation criterion: $\min_A \sum_{l=1}^m (x^{(l)T} Ax^{(l)} - 1)^2$. We will call the resulting estimator the *ordinary least squares* (OLS) estimator, in order to distinguish it from the adjusted least squares estimator, introduced later. The presence of mea-

surement errors in all the covariates makes the OLS estimator biased, see, *e.g.*, [1].

Another approach for estimating the parameter of the quadratic from model from the observed data points is the *orthogonal regression* estimation. Let $\operatorname{dist}(x, \mathcal{E})$ be the Euclidean distance from the point x to the set $\mathcal{E}$. The orthogonal regression estimator is defined as a global solution of the following optimization problem: $\min_A \sum_{l=1}^m \operatorname{dist}(x^{(l)}, \mathcal{E}(A))^2$. The nonlinearity of the model with respect to the measurements, implies the inconsistency of this estimator as well, see [8] and the discussion in [3, p. 250].

We assume that the measurement errors $\tilde{x}^{(1)}, \ldots, \tilde{x}^{(m)}$ are centered, independent among the measurement, and normally distributed, $\tilde{x}^{(l)} \sim \mathrm{N}(0, \bar{\sigma}^2 I)$ for all l, with noise variance $\bar{\sigma}^2 I$. We consider both cases, when $\bar{\sigma}^2$ is given, and when $\bar{\sigma}^2$ is unknown. The stochastic description of the measurement errors can be viewed as a model with parameter σ^2 (a nuisance parameter of the model).

Using the noise model assumptions, we apply an adjustment procedure that takes into account the quadratic structure of the model and corrects the OLS estimate appropriately. The resulting estimator, called an *adjusted least squares* (ALS) estimator, is consistent.

A nice feature of the ALS estimator is that its computation, as the computation of the OLS estimator requires solving a linear system of equations. If $\bar{\sigma}^2$ is a priori known, we give the corrected system in terms of $\bar{\sigma}^2$. If however, $\bar{\sigma}^2$ is unknown, then it has to be estimated together with the model parameters. We propose a consistent procedure to estimate the unknown measurement error variance.

The theory presented in the paper is generalized for the ellipsoid estimation problem with *unknown* center in [5]. The computational aspect of the ellipsoid estimation method with unknown center is treated in [7]. Due to space limitation the proofs of the statements are not given. They can be found in the technical report [4], available from

`ftp://ftp.esat.kuleuven.ac.be/pub/SISTA/markovsky/`

2 Model $x^\top A x = 1$

Let $x^{(l)} \in \mathbb{R}^{n\times 1}$, $l = 1, \ldots, m$, represent observations of boundary points of an ellipsoid. The model is

$$x^{(l)T} A x^{(l)} \approx 1, \quad \text{for} \quad l = 1, \ldots, m, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a positive semidefinite symmetric matrix to be estimated. We suppose that the vectors $x^{(l)}$ are observed with errors,

$$x^{(l)} = \bar{x}^{(l)} + \tilde{x}^{(l)}, \quad \text{for} \quad l = 1, \ldots, m, \tag{2}$$

and that there exists a symmetric positive semidefinite matrix $\bar{A} \in \mathbb{R}^{n\times n}$, such that $\bar{x}^{(l)T} \bar{A} \bar{x}^{(l)} = 1$, for $l = 1, \ldots, m$. The matrix $\bar{A}$ is the true value of

the parameter A in the model (1). The vectors $\bar{x}^{(l)}$ are the true values of the measurements $x^{(l)}$, and $\tilde{x}^{(l)}$ represent the measurement errors.

We derive a strongly consistent estimator for $\bar{A}$, based on the approach proposed in [6]. We suppose that the error terms $\{\tilde{x}^{(l)}, l \geq 1\}$ form an i.i.d. sequence and the distribution of $\tilde{x}^{(l)}$, denoted as $\mathcal{L}(\tilde{x}^{(l)})$, is normal $N(0, \sigma^2)$, with $\sigma^2 > 0$.

3 The adjusted least squares estimator

We start with the OLS objective function

$$q_{\rm ls}(A, x) := (x^\top A x - 1)^2, \qquad A \in \mathbb{S}, \quad x \in \mathbb{R}^{n\times 1},$$

where $\mathbb{S}$ denotes the space of the $n \times n$ symmetric matrices. In the space of the square matrices $\mathbb{R}^{n\times n}$, we introduce a scalar product $\langle T, S\rangle := \mathrm{tr}(TS^\top)$, for $T, S \in \mathbb{R}^{n\times n}$. The derivative $\partial q_{\rm ls}/\partial A$ is a linear functional on $\mathbb{R}^{n\times n}$. It acts on $H \in \mathbb{R}^{n\times n}$ according to the rule

$$\frac{1}{2}\frac{\partial q_{\rm ls}}{\partial A}(H) = (x^\top A x - 1)x^\top H x = \ \langle (x^\top A x - 1)xx^\top,\ H\rangle. \tag{3}$$

We can identify the derivative $\partial q_{\rm ls}/\partial A$ with a matrix, which represents it in the equality (3). Thus

$$\frac{1}{2}\frac{\partial q_{\rm ls}}{\partial A} = (x^\top A x)xx^\top - xx^\top := \psi_{\rm ls}(A, x).$$

In this section, we suppose that σ^2 is known, therefore the measurement error distribution is known.

We are looking for a corrected score function ψ, such that

$$\mathbf{E}\,\psi(A, \bar{x} + \tilde{x}) = \psi_{\rm ls}(A, \bar{x}), \qquad \text{for all } \bar{x} \in \mathbb{R}^{n\times 1}. \tag{4}$$

Here $\mathcal{L}(\tilde{x}) = \mathcal{L}(\tilde{x}^{(l)}) = N(0, \sigma^2)$. We have

$$\psi_{\rm ls}(A, x) = \psi_{{\rm ls},1}(A, x) - \psi_{{\rm ls},2}(A, x), \tag{5}$$

with $\psi_{{\rm ls},1}(A, x) := (x^\top A x)xx^\top$, $\psi_{{\rm ls},2}(A, x) := xx^\top$. Using (5), we split the problem (4) into two problems. Thus we look for the auxiliary functions $\psi_s(A, x)$, $s = 1, 2$, such that

$$\mathbf{E}\,\psi_s(A, \bar{x} + \tilde{x}) = \psi_{{\rm ls},s}(A, \bar{x}), \quad \text{for} \quad s = 1, 2. \tag{6}$$

Then $\psi_2(A, x) = xx^\top - \sigma^2 I_n$, where I_n is the $n \times n$ identity matrix. We have for the (p, q)-th entry of the matrix $\psi_{{\rm ls},1}$, $[\psi_{{\rm ls},1}(A, x)]_{pq} = \sum_{i,j=1}^n a_{ij}x_i x_j x_p x_q$.

Let f_{ijpq} be such a polynomial function of x, of order 4, that

$$\mathbf{E}\,f_{ijpq}(\bar{x} + \tilde{x}) = \bar{x}_i\bar{x}_j\bar{x}_p\bar{x}_q, \qquad \text{for all } \bar{x} \in \mathbb{R}^{n\times 1}. \tag{7}$$

Then the (p,q)-th entry of ψ_1 equals $[\psi_1(A,x)]_{pq} = \sum_{i,j=1}^{n} a_{ij} f_{ijpq}(x)$.

Therefore the (p,q)-th entry of ψ is

$$[\psi(A,x)]_{pq} = \sum_{i,j=1}^{n} a_{ij} f_{ijpq}(x) - x_p x_q + \sigma^2 \delta_{pq}, \tag{8}$$

where δ_{pq} is the Kronecker symbol. Now, consider the function (7). Let t_s be such a polynomial that $\mathbf{E}\, t_s(\xi + \sigma N(0,1)) = \xi^s$, $s = 1, 2, \ldots$ Then, see, *e.g.*, [2], $t_1(\xi) = \xi$, $t_2(\xi) = \xi^2 - \sigma^2$, $t_3(\xi) = \xi^3 - 3\xi\sigma^2$, $t_4(\xi) = \xi^4 - 6\xi^2\sigma^2 + 3\sigma^4$. We have the following cases.

a). All i, j, p, q are different. Then $f_{ijpq}(x) = x_i x_j x_p x_q$.

b). $i = j = p$, $q \neq i$ (with permutations). Then $f_{iiiq}(x) = x_q t_3(x_i)$.

c). $i = j = p = q$. Then $f_{iiii}(x) = t_4(x_i)$.

d). $i = j$, $p = q$, $i \neq p$. Then $f_{iipp}(x) = t_2(x_i) t_2(x_p)$.

e). $i = j$ and i, p, q are different. Then $f_{iipq}(x) = x_p x_q t_2(x_i)$.

The preliminary adjusted least squares (ALS) estimator $\hat{A}_1$ is defined from (8) by the observations (2) as a solution of the set of equations

$$\sum_{i,j=1}^{n} a_{ij} \cdot \frac{1}{m} \sum_{l=1}^{m} f_{ijpq}(x^{(l)}) = \frac{1}{m} \sum_{l=1}^{m} x_p^{(l)} x_q^{(l)} - \sigma^2 \delta_{pq}, \qquad 1 \le p \le q \le n. \tag{9}$$

Since $a_{ij} = a_{ji}$, the values a_{ij}, $1 \le i \le j \le n$ identify the matrix $A \in \mathbb{S}$. Here $x_p^{(l)}$ is the p-th coordinate of the vector $x^{(l)} \in \mathbb{R}^{n \times 1}$. Later on we will show that under appropriate conditions the system of equations (9) has a unique solution $(\hat{a}_{ij},\ 1 \le i \le j \le n)$, for all $m \ge \bar{m}(\omega)$, a.s. The estimator $\hat{A}_1$ is defined as a random symmetric matrix, of which the entries coincide with the unique solution of (9), provided (9) actually has a unique solution. At the second stage, the ALS estimator $\hat{A}$ is defined as the projection of $\hat{A}_1$ on the subspace of the symmetric positive semidefinite matrices.

4 Consistency

We need two assumptions.

(i). There exist a constant $c_1 > 0$ and a number $\bar{m}$, such that for each $m \ge \bar{m}$ and for each $H \in \mathbb{S}$, $\frac{1}{m} \sum_{l=1}^{m} \left(\bar{x}^{(l)T} H \bar{x}^{(l)} \right)^2 \ge c_1 ||H||_F^2$, where $|| \bullet ||_F$ denotes the Frobenius norm.

(ii). There exists a constant $c_2 > 0$ and a number $\gamma \in [0,1)$, such that $\frac{1}{m} \sum_{l=1}^{m} ||\bar{x}^{(l)}||^6 \le c_2 m^{\gamma}$, for all $m \ge 1$, where $||\bullet||$ denotes the Euclidean norm.

The assumption (i) is a contrast condition, see the discussion in [6]. A necessary, but not sufficient condition for (i) is that $\mathrm{span}\{\bar{x}^{(l)}\bar{x}^{(l)T},\ l \geq 1\} = \mathbb{S}$. Assumption (i) means that the true points $\bar{x}^{(l)}$, $l \geq 1$ are nicely dispersed on the boundary of the ellipsoid.

The assumption (ii) is a restriction from above. The $\bar{x}^{(l)}$'s are not allowed to grow quickly. This condition will be used in the corresponding strong law of large numbers. If $\bar{A}$ is positive definite then the ellipsoid is a bounded set, and (ii) holds with $\gamma = 0$.

Theorem 1 (Strong consistency) *Assume that conditions (i) and (ii) hold. Then* $||\hat{A} - \bar{A}||_F \to 0$, *as* $m \to \infty$, *a.s.*

5 Asymptotic normality

Again we suppose that σ^2 is known, and under stronger assumptions derive the asymptotic normality of the estimator $\hat{A}_1$. If $\bar{A}$ is positive definite, this implies the asymptotic normality of $\hat{A}$. First we strengthen the assumptions (i) and (ii).

(iii). The sequence of operators $L_m A := \frac{1}{m}\sum_{l=1}^{m}\left(\bar{x}^{(l)T}A\bar{x}^{(l)}\right)\bar{x}^{(l)}\bar{x}^{(l)T}$ converges to an operator L, which is positive definite on $\mathbb{S}$.

(iv). The sequence $\{\bar{x}^{(l)},\ l \geq 1\}$ is bounded.

Assumption (iii) implies assumption (i), and assumption (iv) implies assumption (ii) with $\gamma = 0$. Recall that assumption (iv) holds if $\bar{A}$ is positive definite. The next assumption is about the stability of empirical moments of $\bar{x}^{(l)}$ up to the sixth order.

(v). For each $s = 1, 2, \ldots, 6$ and for each $\{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}$, there exists a finite limit $\min_{m\to\infty} \frac{1}{m}\sum_{l=1}^{m} \bar{x}_{i_1}^{(l)}\bar{x}_{i_2}^{(l)}\cdots\bar{x}_{i_s}^{(l)}$.

Theorem 2 (Asymptotic normality)

a). Assume that the conditions (iii) to (v) hold. Then $\sqrt{m}\cdot\mathrm{vec}(\hat{A}_1 - \bar{A}) \xrightarrow{d} N(0, \Sigma)$, *where* Σ *is a positive semidefinite covariance matrix.*

b). If $\bar{A}$ *is positive definite then the conditions (iii) and (v) imply the convergence* $\sqrt{m}\cdot\mathrm{vec}(\hat{A} - \bar{A}) \xrightarrow{d} N(0, \Sigma)$.

6 The case of unknown noise variance

We start with the following observation: under σ^2 known and using the ALS estimator, we can re-estimate σ^2.

Lemma 1 *Assume that conditions (i) and (ii) hold and, in addition, that* $\frac{1}{m}\sum_{l=1}^{m}||\bar{x}^{(l)}||^2 = \mathrm{O}(1)$. *Then the ALS estimator* $\hat{A}$ *satisfies the relation*

$$\varphi(\hat{A}) := \frac{1}{\mathrm{tr}(\hat{A})}\cdot\frac{1}{m}\sum_{l=1}^{m}\left(x^{(l)T}\hat{A}x^{(l)} - 1\right) \to \sigma^2, \quad as \quad m \to \infty,\ a.s. \qquad (10)$$

Lemma 1 shows that if σ^2 is indeed the true value of the variance and based on this number we construct $\hat{A}$, then $\varphi(\hat{A})$ is close to σ^2. We propose the following empirical criterion: select σ^2 to minimize $|\varphi(\hat{A}_{\sigma^2}) - \sigma^2|$, where $\hat{A}_{\sigma^2}$ is the ALS estimator, determined by taking the number σ^2 as the true value of the variance. This gives the following iterative algorithm:

1. Given $\{x^{(l)}\}_{l=1}^m$ and a convergence tolerance ε.
2. Select an initial guess $\sigma^2(0) > 0$ for $\bar{\sigma}^2$, *e.g.*, $\sigma^2(0) = \frac{\lambda}{nm} \sum_{l=1}^m ||x^{(l)}||^2$, for some $0 < \lambda < 1/2$ and let $k := 0$.
3. `until` $|\sigma^2(k-1) - \sigma^2(k)| \geq \varepsilon$ `do`
   3.1. Calculate $\hat{A}_{\sigma^2(k)}$.
   3.2. Let $\sigma^2(k+1) := \max\{0, \varphi(\hat{A}_{\sigma^2(k)})\}$ and $k := k+1$.
4. If $|\sigma^2(k-1) - \sigma^2(k)| \geq \varepsilon$, then we select $\hat{\sigma}^2$, as $\sigma^2(s)$ for which
$$|\sigma^2(s-1) - \sigma^2(s)| = \min_{1 \leq k \leq N} |\sigma^2(k-1) - \sigma^2(k)|.$$
   Otherwise we select $\hat{\sigma}^2 = \sigma^2(k)$.
5. Return the noise variance estimate $\hat{\sigma}^2$ and the parameter estimate $\hat{A}_{\hat{\sigma}^2}$.

7 Simulation examples

In this section, we show simulation examples, comparing the ALS estimators with the OLS estimator, and the orthogonal regression (OR) estimator. In all experiments, the true value of the parameter is

$$\bar{A} = \begin{bmatrix} 1 & -0.75 \\ -0.75 & 1 \end{bmatrix}.$$

First, we consider an estimation with sample size $m = 100$ data points and noise with standard deviation $\sigma = 0.3$. The true values of the data points are equidistantly distributed on the boundary of the ellipsoid. For a particular noise realization, the relative errors of estimation $e := ||\hat{A} - \bar{A}||_F / ||\bar{A}||_F$, for the different estimates are: $e_{\text{ols}} = 0.4966$, $e_{\text{als1}} = 0.0534$, $e_{\text{als2}} = 0.0350$, $e_{\text{orth}} = 0.0357$, where "ALS1" stands for the ALS estimation with known noise variance and "ALS2" stands for the ALS estimation with unknown noise variance. Figure 1, left, shows the data points used for the estimation and Figure 1, right, shows the estimated ellipses and the true ellipse. Note that the ALS2 estimator is better than the ALS1 estimator. This relation also holds when the relative error is averaged over many repetitions of the experiment.

Figure 2 shows the averaged relative error of estimation $\bar{e} = \frac{1}{N} \sum_{k=1}^N e^{(k)}$ as a function of the sample size m. Here $\hat{A}^{(k)}$ is the estimate obtained on the k-th experiment and in total $N = 500$ repetitions of the estimation, with different noise realizations, are used. The noise standard deviation is $\sigma = 0.1$.

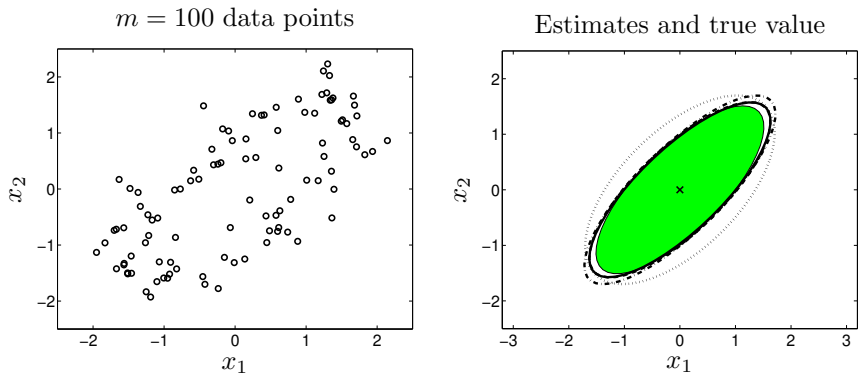

Figure 1: True ellipse—shaded, OLS—dotted, ALS1—solid, ALS2—dashed (coincides with ALS1), OR—dashed-dotted.

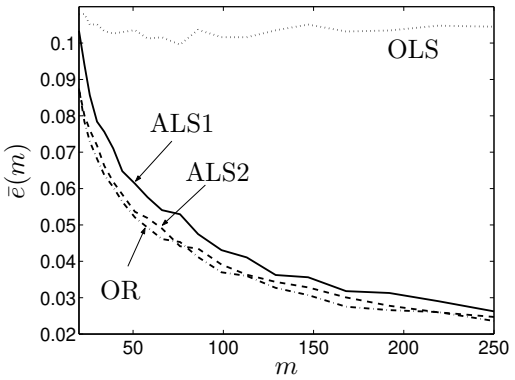

Figure 2: Relative error of estimation as a function of the sample size.

8 Conclusion

We considered a problem of estimating an ellipsoid with known center from observations of points on its boundary. We assumed that the measurement errors are normal. In the case of known variance, we constructed the ALS estimator, starting with the OLS objective function and using the adjustment for the measurement errors. Under appropriate conditions, we proved the strong consistency and asymptotic normality of the estimator. Though we did not present an approximate asymptotic covariance matrix, it can be obtained using the empirical mixed moments of the data up to the sixth order. In the case of unknown variance, we proposed an iterative procedure that estimates simultaneously the noise variance and the model parameters.

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