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A Note on Persistency of Excitation

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***Abstract*— We prove that if a component of the response signal of a controllable linear time-invariant system is persistently exciting of sufficiently high order, then the windows of the signal span the full system behavior. This is then applied to obtain conditions under which the state trajectory of a state representation spans the whole state space.**

***Index Terms*— Behavioral systems, persistency of excitation, lags, annihilators.**

I. Introduction

Persistency of excitation of an input or a noise signal is of importance in system identification and adaptive control, see, for example, [2], [4], [5], [3], [1]. In this paper, we examine consequences of persistency of excitation using the behavioral language.

The problem studied may be posed as follows. Assume that a response

$$\tilde{w}(1), \tilde{w}(2), \dots, \tilde{w}(T)$$

of a linear time-invariant system is observed. Now consider, for some L, $1 \leq L \leq T$, the 'windows' of length L:

$$\begin{matrix} \big[\tilde{w}(1), \tilde{w}(2), \dots, \tilde{w}(L)\big] \\ \big[\tilde{w}(2), \tilde{w}(3), \dots, \tilde{w}(L+1)\big] \\ \vdots \\ \big[\tilde{w}(T-L+1), \tilde{w}(T-L+2), \dots, \tilde{w}(T)\big]. \end{matrix} \tag{1}$$

Under which conditions do these windows span the whole space of all possible windows of length L which the system can produce?

We will show that a sufficient condition for this is that a component (typically the input component) of the observed signal is persistently exciting of order $L+\mathtt{n}$, where $\mathtt{n}$ equals the dimension of the state space of the system.

II. Linear time-invariant systems

Let $\mathfrak{B} \in \mathfrak{L}^{\bullet}$. Denote, for $T \in \mathbb{N}$, by $\mathfrak{B}|_{[1,T]}$ the $w \in \mathfrak{B}$ restricted to $[1,T]$, *i.e.*,

$$\mathfrak{B}|_{[1,T]} := \{ w : [1,T] \to \mathbb{R}^{\mathtt{w}} \mid \exists\ v \in \mathfrak{B}\ : w(t) = v(t) \text{ for } 1 \leq t \leq T \}.$$

Closely associated with a kernel representation $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is the module of *annihilators* of $\mathfrak{B}$, $\mathfrak{N}_{\mathfrak{B}} \subseteq \mathbb{R}^{\mathtt{w}}[\xi]$, defined by

$$\mathfrak{N}_{\mathfrak{B}} := \{ n \in \mathbb{R}^{\mathtt{w}}[\xi] \mid n^{\top}(\sigma)\mathfrak{B} = 0 \}.$$

This paper is a short version of [9]. Please refer to [9] for formal definitions and complete proofs.

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It is easy to see that $\mathfrak{N}_{\mathfrak{B}}$ is a submodule of $\mathbb{R}^{\mathtt{w}}[\xi]$. $\mathfrak{N}_{\ker(R(\sigma))}$ is, in fact, the submodule generated by the rows of R. Consider also for nonnegative integers $\Delta \in \mathbb{Z}_+$ the annihilators of degree less than Δ,

$$\mathfrak{N}^{\Delta}_{\mathfrak{B}} := \{ n \in \mathbb{R}^{\mathtt{w}}[\xi] \mid \text{each element of } n \text{ is of degree} \leq \Delta \}.$$

Observe that there holds (with apologies for the slight abuse of notation), for $L \in \mathbb{Z}_+$,

$$\mathfrak{N}^{L-1}_{\mathfrak{B}} = \ker(\mathfrak{B}_{[1,L]}).$$

There are a number of important 'integer invariants' associated with $\mathfrak{L}^{\bullet}$. The following are of interest to us in this paper:

- $\mathbf{w} : \mathfrak{L}^{\bullet} \to \mathbb{Z}_+$, the *variable cardinality*,
- $\mathbf{m} : \mathfrak{L}^{\bullet} \to \mathbb{Z}_+$, the *input cardinality*,
- $\mathbf{p} : \mathfrak{L}^{\bullet} \to \mathbb{Z}_+$, the *output cardinality*,
- $\mathbf{n} : \mathfrak{L}^{\bullet} \to \mathbb{Z}_+$, the *state cardinality*,
- $\mathbf{L} : \mathfrak{L}^{\bullet} \to \mathbb{Z}_+$, the *lag*,
- $\mathbf{l} : \mathfrak{L}^{\bullet} \to \mathbb{Z}_+$, the *shortest lag*.

Denote by $\mathfrak{L}^{\bullet}_{\text{controllable}}$, respectively $\mathfrak{L}^{\mathtt{w}}_{\text{controllable}}$, the controllable elements of $\mathfrak{L}^{\bullet}$, respectively $\mathfrak{L}^{\mathtt{w}}$.

III. Sequences with spanning windows

Let $\mathfrak{B} \in \mathfrak{L}^{\bullet}$, and assume that a finite trajectory $\tilde{w} \in \mathfrak{B}|_{[1,T]}$ is 'observed'. *Under which conditions it is possible to recover from $\tilde{w}$ the laws of the system $\mathfrak{B}$ that generated $\tilde{w}$?* This question is closely related to the question asked in the introduction: *Under which conditions do the observed windows of length L span the space of all possible windows of length L which the system can produce?*

Define the Hankel matrix of *depth* L associated with the vector signal $f(1), f(2), \dots, f(T)$ by

$$\mathfrak{H}_L(f) := \begin{bmatrix} f(1) & f(2) & \cdots & f(T-L+1) \\ f(2) & f(3) & \cdots & f(T-L+2) \\ \vdots & \vdots & & \vdots \\ f(L) & f(L+1) & \cdots & f(T) \end{bmatrix},$$

Note that the columns of the Hankel matrix $\mathfrak{H}_L(\tilde{w})$ correspond to the windows of $\tilde{w}$ displayed in the introduction. Of course, since $\tilde{w} \in \mathfrak{B}|_{[1,T]}$, any $n \in \mathfrak{N}^{L-1}_{\mathfrak{B}}$, $n(\xi) = n_0 + n_1\xi + \cdots + n_{L-1}\xi^{L-1}$, is such that

$$\begin{bmatrix} n_0^{\top} & n_1^{\top} & \cdots & n_{L-1}^{\top} \end{bmatrix} \mathfrak{H}_L(\tilde{w}) = 0.$$

Therefore, the left kernel of $\mathfrak{H}_L(\tilde{w})$ contains the vectors generated by the elements of $\mathfrak{N}^{L-1}_{\mathfrak{B}}$. The question is: *When are there no other annihilators?* Equivalently (with a very slight abuse of notation): When is $\text{leftkernel}\big(\mathfrak{H}_L(\tilde{w})\big) = \mathfrak{N}^{L-1}_{\mathfrak{B}}$, equivalently, $\text{colspan}\big(\mathfrak{H}_L(\tilde{w})\big) = \mathfrak{B}|_{[1,L]}$?

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Crucial in our result is the persistency of excitation of a component (typically, the input component) of $\tilde{w}$. The signal $f = [1,T] \cap \mathbb{N} \to \mathbb{R}^{\mathbf{f}}$ is said to be *persistently exciting of order* $L :\Leftrightarrow \text{rank}(\mathfrak{H}_L(f)) = L\mathbf{f}$, *i.e.*, if there exist no non-trivial linear relations of order L among the $f(t)$'s. In other words, there are no $a_1, a_2, \ldots, a_L \in \mathbb{R}^{\mathbf{f}}$, not all zero, such that $a_1^\top f(t) + a_2^\top f(t+1) + \cdots + a_L^\top f(t+L) = 0$, for $t = 1,2,\ldots,T-L$.

The following is the main result of the paper.

Theorem 1: Consider $\mathfrak{B} \in \mathfrak{L}^{\mathbf{w}}_{\text{controllable}}$. Let $\tilde{u} : [1,T] \to \mathbb{R}^{\mathbf{m}(\mathfrak{B})}$, $\tilde{y} : [1,T] \to \mathbb{R}^{\mathbf{p}(\mathfrak{B})}$, and $\tilde{w} = (\tilde{u}, \tilde{y})$. Assume that $\tilde{w} \in \mathfrak{B}|_{[1,T]}$. Then, if $\tilde{u}$ is persistently exciting of order $L + \mathbf{n}(\mathfrak{B})$,

$$\text{leftkernel}\big(\mathfrak{H}_L(\tilde{w})\big) = \mathfrak{N}^L_{\mathfrak{B}}, \tag{K}$$

and

$$\text{colspan}\big(\mathfrak{H}_L(\tilde{w})\big) = \mathfrak{B}|_{[1,L]}. \tag{I}$$

IV. Comments and corollaries

1. Note that, we have to assume a 'deeper' persistency of excitation on $\tilde{u}$ than the width of the windows of $(\tilde{u}, \tilde{y})$ which are considered.
2. An interesting special case is when $\mathfrak{B}$ is the usual state space system $\sigma x = Ax + Bu$. Note that for this system, $\mathbf{L}(\mathfrak{B}) = 1$. Theorem 1 yields the following corollary.

Corollary 2: Assume that $\sigma x = Ax + Bu$ is controllable. Consider a trajectory $\tilde{u}(1), \tilde{u}(2), \ldots, \tilde{u}(T)$; $\tilde{x}(1), \tilde{x}(2), \ldots, \tilde{x}(T)$ of this system. Then

(i) If $\tilde{u}$ is persistently exciting of order $\dim(x)$, then $\text{rank}\begin{bmatrix}\tilde{x}(1) & \tilde{x}(2) & \cdots & \tilde{x}(T)\end{bmatrix} = \dim(x)$

(ii) If $\tilde{u}$ is persistently exciting of order $\dim(x)+1$, then $\text{rank}\begin{bmatrix}\tilde{u}(1) & \cdots & \tilde{u}(T) \\ \tilde{x}(1) & \cdots & \tilde{x}(T)\end{bmatrix} = \dim(x) + \dim(u)$.

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