MIMO Broadcast Channel Optimal Spectrum Balancing under Per-Modem Total Power Constraints and Non-Zero Signal to Noise Ratio Gap

Vincent Le Nir, Marc Moonen, Jochen Maes, Mamoun Guenach

Abstract— **In Multiple Input Multiple Output Broadcast Channels (MIMO BC), finding the capacity region has been a challenging task that has received some attention during the past few years. It has been shown that by exploiting the duality between the MIMO Multiple Access Channel (MAC) and the MIMO BC, it is possible to achieve the same set of rates in both domains under the same total power constraint and under a zero Signal to Noise Ratio (SNR) gap [1]. However, more practical scenarios for the MIMO BC consider per-modem total power constraints as well as a non-zero SNR gap. In this paper, we extend the work done in [2], [3], [4] and we derive a unified framework for the MIMO BC that can cope with a total power constraint as well as with per-modem total power constraints and non-zero SNR gap, which is relevant in xDSL and wireless contexts. More specifically we devise an algorithm called MIMO BC Optimal Spectral Balancing (MIMO BC OSB) which finds the data rate region for per-modem total power constraints and non-zero SNR gap scenarios. All other configurations can easily be derived from this algorithm and for all SNR operating points. By properly incorporating the SNR gap in the duality formulas [1] (derived initially for zero SNR gap and the same total power constraint) and by extending the work of [3] to multiple receivers (MIMO) and non-zero SNR gap, the high SNR assumption of [4] can be relaxed and per-modem total power constraints can be met. Simulation results are given for VDSL2 channels and for independent complex Gaussian channels between the tones, as well as for a bonded VDSL2 scenario exploiting Phantom Mode (PM) transmission with external noise coming from VDSL2 disturbing lines.**

I. Introduction

The MIMO BC has attracted a lot of attention during the past few years due to the lack of a simple characterization of its capacity region. However, owing to the duality with the MIMO MAC, the capacity region of the MIMO BC can be analysed based on its dual MIMO MAC, under the same total power constraint [1]. The optimal transmission structure that achieves the MIMO BC capacity region is based on Minimum Mean Square Error Dirty Paper Coding (MMSE DPC) and basically consists of a Matched Filter (MF) and a Decision Feedback Equalizer (DFE) whose filters are similar to those of the MIMO MAC MMSE-DFE [5], [6]. Therefore, it is possible to achieve the same set of rates in both the MIMO MAC and the MIMO BC under the same total power constraint and with the same feedforward and feedback filters.

However, more practical scenarios for the MIMO BC consider per-modem total power constraints as well as a non-zero so-called SNR gap. Indeed, in the xDSL context for instance, it is more relevant to consider a constraint on the total power used by each modem separately instead of a constraint on the total power used by all modems together. The capacity region of the MIMO BC under per-antenna total power constraints (in the wireless context) has been shown to be the dual of the MIMO MAC by incorporating an uncertain noise matrix [2]. This leads to a minmax optimization of the weighted rate sum over transmit covariance matrices and the uncertain noise matrix. A less complex algorithm driven only by the maximization on the input covariance matrices called BC-Optimal Spectral Balancing (BC OSB) has been proposed for optimal power allocation under per-modem total power constraints in the xDSL context where the SNR gap is assumed to be zero [3]. Considering a non-zero SNR gap, it has been shown that it is still possible to obtain the same set of rates by the duality between the MIMO MAC and the MIMO BC under the same total power constraint in the context of a practical implementation with a Zero Forcing Tomlinson Harashima Precoder (ZF-THP) but with a high SNR assumption [4].

In this paper, we extend the work done in [2], [3], [4] and we derive a unified framework that can cope with a total power constraint as well as with per-modem total power constraints and a non-zero SNR gap, which is relevant in xDSL and wireless contexts. We will focus on the per-modem total power constraints and non-zero SNR gap as all other scenarios (e.g. a total power constraint with zero SNR gap) can easily be derived from this case. We devise an algorithm called MIMO BC OSB able to cope with these different configurations and for all SNR operating points. We generalize the work done in [3] by two ways: 1) We incorporate the non-zero SNR gap in the duality formulas and 2) We consider the general case where each user has multiple receivers (MIMO transmission) so that application to MIMO wireless sensors is possible. By properly incorporating the SNR gap in the duality formulas [1] derived initially for zero SNR gap and

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total power constraint and by extending the work of [3] to multiple receivers (MIMO), the high SNR assumption can be relaxed and the per-modem total power constraints can be met. In fact, the primal MIMO BC optimization problem subject to per-modem total power constraints and a non-zero SNR gap is transformed into a collection of per-tone unconstrained MIMO MAC optimization problems using Lagrangian parameters. By the combination of a MIMO MAC algorithm in [7], [8] and the duality between the MIMO MAC and the MIMO BC with modified formulas for the case of non-zero SNR gap, we can compute the optimal transmit covariance matrices for the MIMO BC under per-modem total power constraints achieving the same set of rates as the MIMO MAC.

In section II, we first devise the MIMO MAC and MIMO BC duality transformation formulas under a non-zero SNR gap. Then, in section III, we present the MIMO BC OSB algorithm which provides the optimal power allocation for the MIMO BC under per-modem total power constraints and with non-zero SNR gap. All other configurations (single receiver, zero-SNR gap, total power constraint) can easily be derived from this general framework. In section IV, simulation results are given for VDSL2 channels and for independent complex Gaussian channels between the tones, as well as for a bonded VDSL2 scenario exploiting Phantom Mode (PM) transmission with external noise coming from VDSL2 disturbing lines.

II. Duality under non-zero SNR gap

In this section, we extend the concept of duality between the MIMO MAC and the MIMO BC with zero SNR gap initially presented in [1] to the more general duality between the MIMO MAC and the MIMO BC with non-zero SNR gap. We consider a MIMO BC with T transmitters serving K users, each user k having M_k receivers, and using Discrete Multi-Tone (DMT) modulation with a cyclic prefix longer than the maximum delay spread of the channels. This model (depicted in **Figure 1**) can represent on the one hand a wireless downlink system where the base station is equipped with T antennas serving K users, each user k having M_k antennas. On the other hand this model can represent an x-DSL downlink system where the Central Office/Remote Terminal (CO/RT) is equipped with T modems serving K users, each user k having M_k modems. The transmission on tone i can then be modelled as:

$$\mathbf{y}_i = \mathbf{H}_i\mathbf{x}_i + \mathbf{n}_i \quad \text{where} \quad \mathbf{H}_i = \begin{bmatrix} \mathbf{H}_{i1} \\ \vdots \\ \mathbf{H}_{iK} \end{bmatrix} \quad i = 1 \dots N_c \tag{1}$$

- N_c is the number of subcarriers
- $\mathbf{x}_i = \sum_{j=1}^{K} \mathbf{q}_{ij}$ is the $T \times 1$ transmitted vector on tone i
- $\mathbf{q}_{ij}$ is the $T \times 1$ vector of data from the j^{th} user
- $\mathbf{y}_i$ the received signal vector of length $\left(\sum_{k=1}^{K} M_k\right) \times 1$
- $\mathbf{H}_i$ the $(\sum_{k=1}^{K} M_k) \times T$ MIMO channel matrix

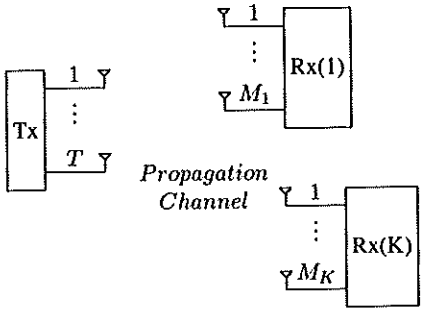


Fig. 1. MIMO BC with T transmitters serving K users, each user k having M_k receivers

- $\mathbf{H}_{ik}$ the $M_k \times T$ MIMO channel matrix for the k^{th} user
- $\mathbf{n}_i$ the vector containing Additive White Gaussian Noise (AWGN)

The dual MIMO Multiple Access Channel (MAC) can be written as:

$$\mathbf{v}_i = \mathbf{H}_i^H\mathbf{u}_i + \mathbf{w}_i \quad \text{where} \quad \mathbf{H}_i^H = \begin{bmatrix} \mathbf{H}_{i1}^H & \dots & \mathbf{H}_{iK}^H \end{bmatrix} \tag{2}$$

where $\mathbf{u}_i = [\mathbf{u}_{i1}^T \dots \mathbf{u}_{iK}^T]^T$ and $\mathbf{u}_{ik}$ is the $M_k \times 1$ vector transmitted on tone i by the k^{th} user. Here $(.)^H$ denotes the hermitian transpose. In this paper, we assume $E[\mathbf{w}_i\mathbf{w}_i^H] = E[\mathbf{n}_i\mathbf{n}_i^H] = \mathbf{I}$, which can always be obtained after a whitening step. In this paper, we consider the maximization of the weighted rate sum for queue stability and fair scheduling between users [8]. The weighted rate sum function for the MIMO MAC with a given decoding order 1,...,K-1,K (i.e. user K is decoded last) and with non-zero SNR gap (assuming the same SNR gap for all the different users) is given by:

$$R^{MAC} = \sum_{k=1}^{K} w_k \sum_{i=1}^{N_c} log_2\left[det\left(\mathbf{I} + \frac{1}{\Gamma_M}\mathbf{B}_{ik}^{-1}\mathbf{H}_{ik}^H\Phi_{ik}\mathbf{H}_{ik}\right)\right] \tag{3}$$

- Γ_M the MIMO MAC SNR gap required to achieve a target probability of error for a given modulation and coding scheme
- $E[\mathbf{u}_{ik}\mathbf{u}_{ik}^H] = \Phi_{ik}$ is the transmit covariance matrix for user k and tone i
- $\mathbf{B}_{ik} = \mathbf{I} + \sum_{j=k+1}^{K} \mathbf{H}_{ij}^H\Phi_{ij}\mathbf{H}_{ij}$

The w_k's are the weights assigned to the different users. In the MAC, the optimal detection order is defined by the weights, i.e. the user with the largest weight is decoded last [7], [8]. Without loss of generality, we will assume $w_1 < \dots < w_K$. The MAC-BC duality dictates a reversal of the decoding/encoding order, hence in the BC the user with the largest weight has to be encoded first. The weighted rate sum function for the MIMO BC with a given encoding order K,K-1,...,1 (i.e. user K is encoded first) and with non-zero SNR

gap is then given by:

$$R^{BC} = \sum_{k=1}^{K} w_k \sum_{i=1}^{N_c} log_2 \left[det\left(\mathbf{I} + \frac{1}{\Gamma_B} \mathbf{A}_{ik}^{-1} \mathbf{H}_{ik} \mathbf{Q}_{ik} \mathbf{H}_{ik}^H \right) \right] \quad (4)$$

- Γ_B the MIMO BC SNR gap required to achieve a target probability of error for a given modulation and coding scheme
- $E[\mathbf{q}_{ik}\mathbf{q}_{ik}^H] = \mathbf{Q}_{ik}$, $E[\mathbf{x}_i\mathbf{x}_i^H] = \sum_{k=1}^{K} \mathbf{Q}_{ik}$ the sum of K transmit covariance matrices for the different users over tone i
- $\mathbf{A}_{ik} = \mathbf{I} + \sum_{j=1}^{k-1} \mathbf{H}_{ik} \mathbf{Q}_{ij} \mathbf{H}_{ik}^H$

As $det(\mathbf{I} + \mathbf{AB}) = det(\mathbf{I} + \mathbf{BA})$, we can rewrite the weighted rate sum for the MIMO BC as:

$$R^{BC} = \sum_{k=1}^{K} w_k \sum_{i=1}^{N_c} log_2 \left[det\left(\mathbf{I} + \frac{1}{\sqrt{\Gamma_B}} \mathbf{A}_{ik}^{-1/2} \mathbf{H}_{ik} \mathbf{Q}_{ik} \mathbf{H}_{ik}^H \mathbf{A}_{ik}^{-1/2} \frac{1}{\sqrt{\Gamma_B}} \right) \right] \quad (5)$$

with $(.)^{-1/2}$ the inverse square root. We can also rewrite the weighted rate sum for the MIMO MAC as:

$$R^{MAC} = \sum_{k=1}^{K} w_k \sum_{i=1}^{N_c} log_2 \left[det\left(\mathbf{I} + \frac{1}{\sqrt{\Gamma_M \Gamma_B}} \mathbf{B}_{ik}^{-1/2} \mathbf{H}_{ik}^H \mathbf{A}_{ik}^{-1/2} \sqrt{\Gamma_B} \mathbf{A}_{ik}^{1/2} \mathbf{\Phi}_{ik} \mathbf{A}_{ik}^{1/2} \sqrt{\Gamma_B} \mathbf{A}_{ik}^{-1/2} \mathbf{H}_{ik} \mathbf{B}_{ik}^{-1/2} \frac{1}{\sqrt{\Gamma_M \Gamma_B}} \right) \right] \quad (6)$$

We consider now $\mathbf{\Sigma}$ as a transmit covariance matrix and $\mathbf{H}$ as an effective channel matrix such that $\mathbf{H} = \mathbf{FLG}^H$ with $\mathbf{L}$ the matrix containing the singular values and $\mathbf{F}$ and $\mathbf{G}$ the unitary matrices containing the singular vectors. We define a transmit covariance matrix $\overline{\mathbf{\Sigma}} = \mathbf{F}\mathbf{G}^H\mathbf{\Sigma}\mathbf{G}\mathbf{F}^H$. Then it can be shown that the following equality holds $det(\mathbf{I} + \mathbf{H}^H\mathbf{\Sigma}\mathbf{H}) = det(\mathbf{I} + \mathbf{H}\overline{\mathbf{\Sigma}}\mathbf{H}^H)$ [1]. By defining the effective channel matrix $\mathbf{H}^H$ for each tone i as $\frac{1}{\sqrt{\Gamma_M \Gamma_B}} \mathbf{B}_{ik}^{-1/2} \mathbf{H}_{ik}^H \mathbf{A}_{ik}^{-1/2} = \mathbf{F}_{ik}\mathbf{L}_{ik}\mathbf{G}_{ik}^H$ with $\mathbf{L}_{ik}$ the matrix containing the singular values and $\mathbf{F}_{ik}$ and $\mathbf{G}_{ik}$ the unitary matrices containing the singular vectors, we use the previous equality in order to obtain the MIMO MAC formula for the flipped channel [1]:

$$R^{MAC} = \sum_{k=1}^{K} w_k \sum_{i=1}^{N_c} log_2 \left[det\left(\mathbf{I} + \frac{1}{\sqrt{\Gamma_M \Gamma_B}} \mathbf{A}_{ik}^{-1/2} \mathbf{H}_{ik} \mathbf{B}_{ik}^{-1/2} \overline{\sqrt{\Gamma_B} \mathbf{A}_{ik}^{1/2} \mathbf{\Phi}_{ik} \mathbf{A}_{ik}^{1/2} \sqrt{\Gamma_B}} \frac{1}{\sqrt{\Gamma_M \Gamma_B}} \mathbf{B}_{ik}^{-1/2} \mathbf{H}_{ik}^H \mathbf{A}_{ik}^{-1/2} \right) \right] \quad (7)$$

with $\overline{\sqrt{\Gamma_B} \mathbf{A}_{ik}^{1/2} \mathbf{\Phi}_{ik} \mathbf{A}_{ik}^{1/2} \sqrt{\Gamma_B}}$ the transmit covariance matrix for the flipped channel. Equating term by term the contents of the determinants in (5) and (7), one can compute the optimal transmit covariance matrices for the MIMO BC as:

$$\mathbf{Q}_{ik} = \mathbf{B}_{ik}^{-1/2} \frac{1}{\sqrt{\Gamma_M}} \mathbf{F}_{ik} \mathbf{G}_{ik}^H \sqrt{\Gamma_B} \mathbf{A}_{ik}^{1/2} \mathbf{\Phi}_{ik} \mathbf{A}_{ik}^{1/2} \sqrt{\Gamma_B} \mathbf{G}_{ik} \mathbf{F}_{ik}^H \frac{1}{\sqrt{\Gamma_M}} \mathbf{B}_{ik}^{-1/2} \quad (8)$$

Therefore the equivalence between the MIMO MAC and the MIMO BC weighted rate sums as well as the equivalence between the total power constraint in the MIMO MAC and the MIMO BC still holds under a non-zero SNR gap if $\Gamma_M = \Gamma_B$. This duality result will be used in the following section to convert optimal transmit covariance matrices in the MIMO MAC to their dual optimal transmit covariance matrices in the MIMO BC[1].

III. MIMO BC OSB UNDER PER-MODEM TOTAL POWER CONSTRAINTS AND WITH NON-ZERO SNR GAP

In this section, we devise the optimal power allocation for the MIMO BC under per-modem total power constraints and with non-zero SNR gap, leading to the MIMO BC OSB algorithm. The primal problem of finding optimal transmit vector covariance matrices in the MIMO BC under a per-modem total power constraint P_t^{tot} for each modem is:

$$\begin{aligned} &\max_{(\mathbf{Q}_{ik})_{i=1\ldots N_c}^{k=1\ldots K}} R^{BC} \\ &\text{subject to } \sum_{k=1}^{K} \sum_{i=1}^{N_c} [\mathbf{Q}_{ik}]_{tt} \leq P_t^{tot} \ \forall t \\ &\mathbf{Q}_{ik} \succeq 0, i = 1 \ldots N_c, k = 1 \ldots K \end{aligned} \quad (9)$$

where $[\mathbf{Q}_{ik}]_{tt}$ is the t^{th} diagonal element of the transmit vector covariance matrix for user k over tone i, and R^{BC} is the weighted rate sum function for the MIMO BC given by equation (4). The idea of dual decomposition is to solve (9) via its Lagrangian. The Lagrangian decouples into a set of N_c smaller problem, thus reducing the complexity of equation (4). The BC dual objective function is:

$$F^{BC}(\mathbf{\Lambda}) = \max_{(\mathbf{Q}_{ik})_{i=1\ldots N_c}^{k=1\ldots K}} \mathcal{L}^{BC}(\mathbf{\Lambda}, (\mathbf{Q}_{ik})_{i=1\ldots N_c}^{k=1\ldots K}) \quad (10)$$

with $\mathbf{\Lambda} = diag(\lambda_1, \ldots, \lambda_T)$ the Lagrange multipliers associated with the MIMO BC per-modem total power constraints and

$$\begin{aligned} \mathcal{L}^{BC}(\mathbf{\Lambda}, (\mathbf{Q}_{ik})_{i=1\ldots N_c}^{k=1\ldots K}) = &\sum_{i=1}^{N_c} \left(\sum_{k=1}^{K} w_k log_2 \left[det\left(1 + \frac{\mathbf{H}_{ik}\mathbf{Q}_{ik}\mathbf{H}_{ik}^H}{\Gamma_B(\mathbf{I} + \mathbf{H}_{ik}(\sum_{j=1}^{k-1} \mathbf{Q}_{ij})\mathbf{H}_{ik}^H)} \right) \right] - \sum_{k=1}^{K} Trace(\mathbf{\Lambda}\mathbf{Q}_{ik}) \right) \\ &+ Trace\left(\mathbf{\Lambda} diag(P_1^{tot}, \ldots, P_T^{tot})\right) \end{aligned} \quad (11)$$

The dual optimization problem is:

$$\begin{aligned} &\text{minimize} \quad F^{BC}(\mathbf{\Lambda}) \\ &\text{subject to} \quad \mathbf{\Lambda} \geq \mathbf{0} \end{aligned} \quad (12)$$

[1]Note that from an implementation point of view, once the optimal covariance matrices $(\mathbf{Q}_{ik})_{i=1\ldots N_c, k=1\ldots K}$ are determined, the transmitted data symbols $\mathbf{x}_i$ can be constructed as follows:

1) The $T \times 1$ vector of the M-QAM data symbols $\mathbf{s}_{ik}$ is precoded using the $T \times T$ $\mathbf{L}_{ik}$ matrix, i.e. $\mathbf{q}_{ik} = \mathbf{L}_{ik}\mathbf{s}_{ik}$, such that $E[\mathbf{q}_{ik}\mathbf{q}_{ik}^H] = \mathbf{Q}_{ik} = \mathbf{L}_{ik}\mathbf{L}_{ik}^H$ with $\mathbf{L}_{ik}$ the Cholesky function (note that $\mathbf{Q}_{ik}$ is a positive semi-definite matrix).

2) Then $\mathbf{x}_i = \sum_{k=1}^{K} \mathbf{q}_{ik}$ will be sent on the T antennas or lines (the t^{th} element of $\mathbf{x}_i$ will be sent on the t^{th} antenna or line).

The dual objective function is neither convex nor concave [1]. Therefore finding the optimal transmit covariance matrices in the MIMO BC is a difficult task. Fortunately, the duality between the MIMO MAC and the MIMO BC says that it is possible to achieve the same rates in both domains under the same total power constraint. We will use this argument to find the optimal MIMO BC covariance matrices under per-modem total power constraints. The crucial step is then to rescale the channel matrix by the inverse square root of the Lagrange multiplier matrix as in [3] :

$$\mathbf{y}_i = \overbrace{\mathbf{H}_i\mathbf{\Lambda}^{-1/2}}^{\mathbf{H}'_i}\mathbf{x}'_i + \mathbf{n}_i \tag{13}$$

where $\mathbf{x}'_i = \mathbf{\Lambda}^{1/2}\mathbf{x}_i$. For this rescaled channel, the dual objective function in the BC becomes:

$$\mathcal{L}^{BC}(\mathbf{\Lambda}, (\mathbf{Q}'_{ik})_{i=1\ldots N_c}^{k=1\ldots K}) = \sum_{i=1}^{N_c}\left(\sum_{k=1}^{K} w_k log_2\Big[det\Big(1+\frac{\mathbf{H}'_{ik}\mathbf{Q}'_{ij}\mathbf{H}'^{H}_{ik}}{\Gamma_B(\mathbf{I}+\mathbf{H}'_{ik}(\sum_{j=1}^{k-1}\mathbf{Q}'_{ij})\mathbf{H}'^{H}_{ik})}\Big)\Big] - \sum_{k=1}^{K} Trace(\mathbf{Q}'_{ik})\right) + Trace\left(diag(P'^{tot}_1,\ldots,P'^{tot}_T)\right) \tag{14}$$

where $\mathbf{Q}'_{ik} = \mathbf{\Lambda}^{1/2}\mathbf{Q}_{ik}\mathbf{\Lambda}^{1/2}$ and $diag(P'^{tot}_1,\ldots,P'^{tot}_T) = \mathbf{\Lambda}^{1/2}diag(P^{tot}_1,\ldots,P^{tot}_T)\mathbf{\Lambda}^{1/2}$. For a given set of Lagrange multipliers, the precoder matrix $\mathbf{\Lambda}^{1/2}$ then effectively transforms the per-modem total power constraints into a total power $Trace\left(diag(P'^{tot}_1,\ldots,P'^{tot}_T)\right)$, by hiding the Lagrange multipliers into the rescaled channels $\mathbf{H}'_{ik}$ and the new covariance matrices $\mathbf{Q}'_{ik}$. The duality between the MIMO MAC and the MIMO BC then says that it is possible to achieve the same set of rates in both domains under the same total power constraint. As the optimal power allocation in the MIMO MAC is tractable [7], [8], one can calculate the optimal transmit covariance matrices in the MIMO MAC and then transform these into optimal transmit covariance matrices for the MIMO BC. The dual MIMO MAC, corresponding to the rescaled MIMO BC, can be written as:

$$\mathbf{v}_i = \mathbf{\Lambda}^{-1/2}\mathbf{H}_i^H\mathbf{u}'_i + \mathbf{w}_i \tag{15}$$

The dual objective function in the MIMO MAC then becomes:

$$F^{MAC}(\mathbf{\Lambda}) = \max_{(\mathbf{\Phi}'_{ik})_{i=1\ldots N_c}^{k=1\ldots K}} \mathcal{L}^{MAC}(\mathbf{\Lambda}, (\mathbf{\Phi}'_{ik})_{i=1\ldots N_c}^{k=1\ldots K}) \tag{16}$$

with

$$\mathcal{L}^{MAC}(\mathbf{\Lambda}, (\mathbf{\Phi}'_{ik})_{i=1\ldots N_c}^{k=1\ldots K}) = \sum_{i=1}^{N_c}\left(\sum_{k=1}^{K} w_k log_2\Big[det\Big(\mathbf{I}+\frac{\mathbf{H}'^{H}_{ik}\mathbf{\Phi}'_{ik}\mathbf{H}'_{ik}}{\Gamma_M(\mathbf{I}+\sum_{j=k+1}^{K}\mathbf{H}'^{H}_{ij}\mathbf{\Phi}'_{ij}\mathbf{H}'_{ij})}\Big)\Big] - \sum_{k=1}^{K} Trace(\mathbf{\Phi}'_{ik})\right) + Trace\left(diag(P'^{tot}_1,\ldots,P'^{tot}_T)\right) \tag{17}$$

The search for the optimal $\mathbf{\Lambda}$ in (12) involves evaluations of the dual objective function (16), i.e. maximizations of the Lagrangian $\mathcal{L}^{MAC}$, which is decoupled over the tones for a given set of λ_i's. The Lagrangian is maximized over the covariance matrices in the MIMO MAC leading to the MIMO MAC-OSB algorithm which is similar to the MAC-OSB algorithm of [7]. By setting $\Gamma_B = \Gamma_M$ and for a given $\mathbf{\Lambda}$, we can thus compute the optimal power allocation in the MIMO MAC and use the duality formulas given in section II to obtain the optimal transmit covariance matrices in the MIMO BC that achieve the same rates as in the MIMO MAC and under same total power constraint. The Lagrange multipliers are then adjusted so that the per-modem total power constraints are not violated in the MIMO BC. We define ϵ_t as the tolerance between the actual per-modem total power constraint $\sum_{i=1}^{N_c}[\mathbf{\Lambda}^{-1/2}(\sum_{k=1}^{K}\mathbf{Q}'_{ik})\mathbf{\Lambda}^{-1/2}]_{tt}$ and the target per-modem total power constraint P_t^{tot} for the t^{th} modem. The algorithm for adjusting the Lagrangian multiplier $\mathbf{\Lambda}$ is given in the Annex as **Algorithm 1** and leads to the optimal transmit vector covariance matrices in the MIMO BC under per-modem total power constraints. This algorithm will be referred to as MIMO BC OSB. One can also use an iterative search (or other convex programming methods) for the maximizations of the Lagrangian (line 8 in the **Algorithm 1**), which also lead to optimal transmit covariance matrices due to the convexity of the MIMO MAC problem [8].

The MIMO BC OSB algorithm under a total power constraint and non-zero SNR gap is a particular case of the MIMO BC OSB algorithm under per-modem total power constraints and non-zero SNR gap. In this case $\mathbf{\Lambda} = \lambda\mathbf{I}$. The same derivation as in the previous section can be given. The different formulas and the algorithm description can be easily modified accordingly.

IV. Results

1) The first set of simulation results are obtained for VDSL2 measured channels with 2 lines of 400 meters and 800 meters respectively. In this particular case, we have the number of transmitters $T = 2$, the number of users $K = 2$ and the number of receivers $M_1 = M_2 = 1$. Spectral masks for VDSL2 Fiber To The exchange (FTTex) are used [9], with SNR gap Γ=12.33 dB, an AWGN of -140 dBm/Hz and maximum transmit power P_j^{tot}=14.5 dBm per line. The frequency range is from 0 to 12 MHz with 4.3125 kHz spacing between subcarriers and 4 kHz symbol rate. The FDD band plan of VDSL2 corresponds to 2 frequency bands in the downlink scenario which are 138kHz-3.75MHz and 5.2MHz-8.5MHz.

Table 1 shows the rates obtained in the BC for the 2 users under a Total Power Constraint (TPC) P^{tot}=29 dBm, and for different sets of weights $(w_1, w_2) = (w_1, 1-w_1)$. The first line corresponds to the rates R1 and R2 when user 1 is encoded first in the BC domain. The difference between the possible detection orders is negligible owing to the diagonal dominance of the channel matrix. **Table 2** shows the rates obtained in the BC for the 2 users under Per-Modem Total Power Constraints (PMTPC). We observe that the rates are independent of the weights given to the different users. This is due to the lower

R1/R2 (Mbps)	w1=0.0	w1=0.1	w1=0.2
User 1 first	0/69.75	94.62/68.80	101.0/67.76
User 2 first	0/69.75	94.62/68.80	101.0/67.76
w1=0.3	w1=0.4	w1=0.5	w1=0.6
105.3/66.37	107.4/65.29	110.0/63.28	117.0/61.11
105.3/66.37	107.4/65.29	110.0/63.28	117.0/61.11
w1=0.7	w1=0.8	w1=0.9	w1=1.0
112.9/58.68	114.2/54.83	115.3/48.12	116.2/0
112.9/58.68	114.2/54.83	115.3/48.12	116.2/0

TABLE I

RATES UNDER TOTAL POWER CONSTRAINT IN THE BC

R1/R2(Mbps)	w1=0.0	w1=0.1	w1=0.2
User 1 first	0/63.88	109.8/63.47	109.8/63.47
User 2 first	0/63.88	109.8/63.47	109.8/63.47
w1=0.3	w1=0.4	w1=0.5	w1=0.6
109.8/63.47	109.8/63.47	109.8/63.47	109.8/63.47
109.8/63.47	109.8/63.47	109.8/63.47	109.8/63.47
w1=0.7	w1=0.8	w1=0.9	w1=1.0
109.8/63.47	109.8/63.47	109.8/63.47	109.9/0
109.8/63.47	109.8/63.47	109.8/63.47	109.9/0

TABLE II

RATES UNDER PER-MODEM TOTAL POWER CONSTRAINT IN THE BC

degree of freedom used in the optimization process compared to a total power constraint case. **Figure 2** summarizes the two first tables by plotting the rates. The Diagonalizing Precoder (DP) of [10] achieves 109.8 Mbps for the first user and 63.47 Mbps for the second user. Therefore the DP achieves most of the capacity under AWGN again owing to the diagonal dominance of the channel matrix.

2) In the second simulation, we consider independent complex Gaussian channels with mean 0 and power 1 and maximum transmit power P_j^{tot}=14.5 dBm per transmitter. We set the number of transmitters $T = 2$, the number of users $K = 2$ and the number of receivers for user 1 and 2 are $M_1 = M_2 = 1$. The weighted rate sums are calculated in **Table 3** and **Table 4** under a total power constraint and under per-modem total power constraints respectively. The channel matrix is no longer diagonally dominant compared to the VDSL2 channels of the first simulations. In this case, the detection order clearly plays a role and leads to different weighted rate sums. The main observation is that the weights define the detection order, as stated in [7], [8]. The user with the largest weight has to be encoded first and the user with the smallest weight has to be encoded last in order to achieve the largest weighted rate sum. In **Figure 3** the different rates are plotted, leading to four different rate regions with total power constraint and per-modem total power constraints. The optimal rates obtained on this channel with the DP [10] are R1=151.80 Mbps and R2=151.50 Mbps. Therefore the MIMO BC OSB algorithm achieves better performance than the DP when the channel matrix is not diagonally dominant.

3) The third set of simulation results involves a VDSL2 scenario with a 0 to 30 MHz bandwidth and with Differential Mode (DM) lines of 400 meters also exploiting Phantom Mode (PM) transmission [11] and a maximum transmit power P_j^{tot}=14.5 dBm per line. We set the number of transmitters $T = 3$, the number of users $K = 2$ and the number of

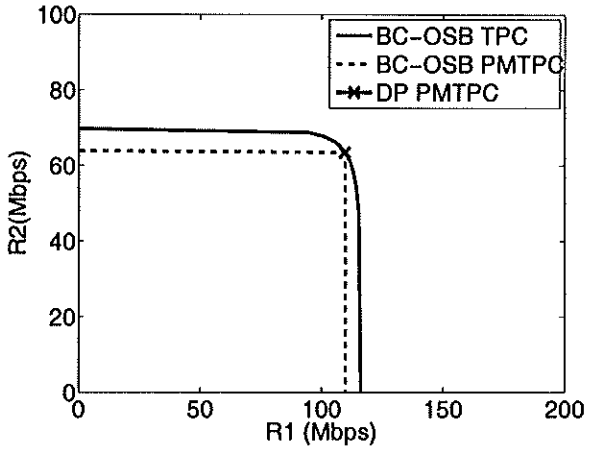


Fig. 2. Rate region of BC OSB in a VDSL2 system

w1R1+w2R2	w1=0.0	w1=0.1	w1=0.2	w1=0.3	w1=0.4
User 1 first	164.0	157.2	155.8	155.6	155.9
User 2 first	168.4	163.8	161.6	159.9	157.0
w1=0.5	w1=0.6	w1=0.7	w1=0.8	w1=0.9	w1=1.0
157.4	157.7	159.9	161.6	163.6	168.3
156.6	155.9	155.5	155.6	157.0	163.8

TABLE III

WEIGHTED RATE SUM UNDER TOTAL POWER CONSTRAINT IN THE BC

w1R1+w2R2	w1=0.0	w1=0.1	w1=0.2	w1=0.3	w1=0.4
User 1 first	167.1	153.0	153.9	154.8	155.7
User 2 first	167.8	160.5	159.5	158.6	157.6
w1=0.5	w1=0.6	w1=0.7	w1=0.8	w1=0.9	w1=1.0
156.6	157.5	158.4	159.3	160.2	167.5
156.6	155.7	154.8	153.8	152.9	166.9

TABLE IV

WEIGHTED RATE SUM UNDER PER-MODEM TOTAL POWER CONSTRAINT IN THE BC

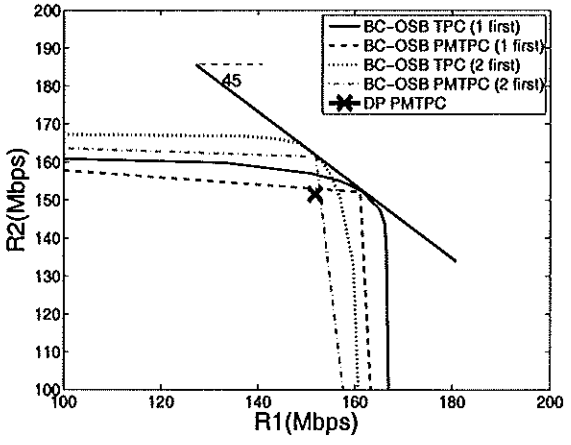


Fig. 3. Rate region of BC OSB in complex Gaussian channels

Rates (Mbps)	SVD TPC	SVD PMTPC	DP	BC OSB
Total 2x3 Total	324.57	306.54	285.89	295.42

TABLE V

COMPARISON BETWEEN DIFFERENT SCHEMES

receivers $M_1 = M_2 = 1$. In this case, 2 DM lines and 1 PM line are used for downlink with external noise coming from 2 VDSL2 DM lines of similar length whose PSD's are set at -60 dBm/Hz. We keep the FDD band plan of VDSL2 with similar per-modem total power constraints and we further transmit in the full 12-30 MHz bandwidth.

Table 5 shows a comparison between existing algorithms [10], [12] with equal weight $w_1 = w_2 = 0.5$ (2-sided coordination algorithms versus 1-sided coordination algorithms). The algorithms using Singular Value Decomposition (SVD) under a TPC and SVD under PMTPC provide the optimal rate sum [12]. Owing to the external noise, there is a rate loss between SVD-based schemes (coordination at both sides) and the DP PMTPC or the MIMO BC OSB PMTPC (coordination at the transmit side). The MIMO BC OSB shows a increased rate compared to the DP. This is due to the linear structure of the DP which makes it difficult to meet the per-modem total power constraints for non-square matrices.

V. CONCLUSION

In this paper, we have derived a unified framework for MIMO BC that can cope with a total power constraint as well as with per-modem total power constraints and non-zero SNR gap, which is relevant in xDSL and wireless contexts. More specifically we have devised an algorithm called MIMO BC Optimal Spectral Balancing (OSB) which finds the data rate region for per-modem total power constraints and non-zero SNR gap scenarios. We have shown that all other configurations can easily be derived from this algorithm and for all SNR operating points. Simulation results were given for VDSL2 channels and complex Gaussian channels, as well as for a bonded VDSL2 scenario exploiting Phantom Mode (PM) transmission with external noise coming from VDSL2 disturbing lines. The MIMO BC OSB algorithm did not show any significant improvement compared to existing algorithms like DP in the VDSL2 scenario with AWGN owing to the diagonal dominance of the channel matrix. However, when the channel matrix is non-diagonally dominant, the MIMO BC OSB algorithm shows better results than DP. This is also the case in a VDSL2 channel with non-square channel matrices when the number of transmitters is larger than the number of users (extra differential mode line or phantom mode transmission).

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ANNEX

Algorithm 1 MIMO BC OSB Algorithm under PMTPC and non-zero SNR gap

1 for $t = 1 \dots T$
2 init $\lambda_t = 1$
3 init $step_t = 2$
4 init $b_t = 0$
5 init $(\mathbf{Q'}_{ik})_{i=1\dots N_c}^{k=1\dots K} = 0$
6 end for
7 while $\exists t$ s.t. $|\sum_{i=1}^{N_c}[\mathbf{\Lambda}^{-1/2}(\sum_{k=1}^{K}\mathbf{Q'}_{ik})\mathbf{\Lambda}^{-1/2}]_{tt} - P_t^{tot}| > \epsilon_t$
8 Exhaustive search $\max_{(\mathbf{\Phi'}_i)_{i=1\dots N_c}} \mathcal{L}^{MAC}(\mathbf{\Lambda}, (\mathbf{\Phi'}_i)_{i=1\dots N_c})$
9 MAC-BC Duality
10 for k=1 to K $\forall i = 1 \dots N_c$
11 $\mathbf{A'}_{ik} = 1 + \mathbf{H'}_{ik}(\sum_{j=1}^{k-1}\mathbf{Q'}_{ij})\mathbf{H'}_{ik}^{H}$
12 $\mathbf{B'}_{ik} = \mathbf{I} + \sum_{j=k+1}^{K}\mathbf{H'}_{ij}^{H}\mathbf{\Phi'}_{ij}\mathbf{H'}_{ij}$
13 $\mathbf{Q'}_{ik} = \mathbf{B'}_{ik}^{-1/2}\frac{1}{\sqrt{\Gamma_M}}\mathbf{F'}_{ik}\mathbf{G'}_{ik}^{H}\sqrt{\Gamma_B}\mathbf{A'}_{ik}$ $\mathbf{\Phi'}_{ij}\mathbf{A'}_{ik}\sqrt{\Gamma_B}\mathbf{G'}_{ik}\mathbf{F'}_{ik}^{H}\frac{1}{\sqrt{\Gamma_M}}\mathbf{B'}_{ik}^{-1/2}$
14 end for
15 if $\sum_{i=1}^{N_c}[\mathbf{\Lambda}^{-1/2}(\sum_{k=1}^{K}\mathbf{Q'}_{ik})\mathbf{\Lambda}^{-1/2}]_{tt} - P_t^{tot} < 0$
16 $b_t = b_t + 1$
17 $\lambda_t = \lambda_t / step_t$
18 $step_t = step_t - 1/2^{b_t}$
19 end if
20 $\lambda_t = \lambda_t * step_t$
21 end while