On invariant subspaces of matrices: A new proof of a theorem of Halmos[☆]

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Abstract

P. Halmos proved that if A is a matrix and if E is an A-invariant subspace, then there exist matrices B and C such that $BA = AB$, $CA = AC$, E is the kernel of B and E is the range of C. We present an elementary proof of this result and show that there exist B and C that additionally satisfy $BC = CB = \mathbb{O}$.

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Let $A, B, C \in M^{n\times n}$ ($n \times n$ matrices over $\mathbb{C}$). If $AB = BA$ and $AC = CA$ then both the kernel of B and the range of C are A-invariant. P. Halmos [6] proved the converse: any A-invariant subspace E is the kernel (range) of some matrix B (C) that commutes with A.

A generalization to C_0 contractions on Hilbert spaces is due to H. Bercovici [2, Proposition 5.33], [3, Corollary 2.11] and P. Wu [9, Theorem 1.2], [10, Theorem 5]). This generalization also yields a description of invariant subspaces of direct sums of Riemann-Liouville operators [4, Theorem 3.3].

Halmos's original proof as well as other known proofs [1, 5] are not trivial.

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In this note we present an elementary proof of this result. Moreover, the following theorem slightly generalizes it.

Theorem 1. *Let $A \in M^{n\times n}$ and let E be an invariant subspace for A. Then there exist $B, C \in M^{n\times n}$ such that*

(i) $AB = BA$ *and* $E = \ker B$;

(ii) $AC = CA$ *and* $E = \text{Range } C$;

(iii) $BC = CB = \mathbb{O}$.

Proof. Consider a partition of A with respect to the orthogonal decomposition $\mathbb{C}^n = E \oplus E^{\perp}$:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ \mathbb{O} & A_{22} \end{pmatrix}. \tag{1}$$

Since every complex matrix is similar to its transpose [7, p. 134], there exist nonsingular matrices X, K, and R such that

$$AX = XA^T, \qquad A_{22}^T K = KA_{22}, \qquad A_{11}R = RA_{11}^T. \tag{2}$$

Hence,

$$\begin{aligned} A^T \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & K \end{pmatrix} &= \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & A_{22}^T K \end{pmatrix} = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & KA_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & K \end{pmatrix} A, \\ A \begin{pmatrix} R & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} &= \begin{pmatrix} A_{11}R & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} = \begin{pmatrix} RA_{11}^T & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} = \begin{pmatrix} R & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} A^T. \end{aligned} \tag{3}$$

Let us set

$$B := X \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & K \end{pmatrix}, \qquad C := \begin{pmatrix} R & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} X^{-1}. \tag{4}$$

Since X, K, R are nonsingular, it follows that $E = \ker B = \text{Range } C$.

The equations $AB = BA$, $AC = CA$, and $BC = CB = \mathbb{O}$ follow from (1)-(4). □

Remark 1. *Theorem 1 holds also for a matrix A with coefficients in any field K. The proof is based on the Taussky-Zassenhaus result: every matrix over K is similar to its transpose [8].*

References

[1] M. Barraa, B. Charles, Sous-espaces invariants d'un opérateur nilpotent sur un espace de banach, Linear Algebra Appl. 153 (1991) 177 – 182.

[2] H. Bercovici, On the Jordan model of C_0 operators II, Acta Sci. Math. (Szeged) 42 (1980) 43 – 56.

[3] H. Bercovici, Operator theory and arithmetic in H^∞, American Mathematical Society, Providence, RI, 1988.

[4] I.Y. Domanov, M.M. Malamud, On the Spectral Analysis of Direct Sums of Riemann-Liouville Operators in Sobolev Spaces of Vector Functions, Integral Equations Operator Theory 63 (2009) 181 – 215.

[5] A. Faouzi, On the orbit of invariant subspaces of linear operators in finite-dimensional spaces (new proof of a Halmos's result), Linear Algebra Appl. 329 (2001) 171 – 174.

[6] P.R. Halmos, Eigenvectors and adjoints, Linear Algebra Appl. 4 (1971) 11 – 15.

[7] R. Horn, C. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1990.

[8] O. Taussky, H. Zassenhaus, On the similarity transformation between a matrix and its transpose, Pacific J. Math. 9 (1959) 893 – 896.

[9] P. Wu, On a conjecture of Sz.-Nagy and Foias, Acta Sci. Math. (Szeged) 42 (1980) 331 – 338.

[10] P. Wu, Which $C_{.0}$ -contraction is quasisimilar to its Jordan model?, Acta Sci. Math. (Szeged) 47 (1984) 449 – 455.