NEW SIMULTANEOUS GENERALIZED SCHUR DECOMPOSITION METHODS FOR THE COMPUTATION OF THE CANONICAL POLYADIC DECOMPOSITION

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ABSTRACT

In signal processing several problems have been formulated as Simultaneous Generalized Schur Decomposition (SGSD) problems. Applications are found in blind source separation and multidimensional harmonic retrieval. Furthermore, SGSD methods for computing a third-order Canonical Polyadic (CP) decomposition have been proposed. The original SGSD method requires that all three matrix factors of the CP decomposition have full column rank. We first propose a new version of the SGSD method for computing a third-order CP decomposition. The proposed method mainly differs from the existing method in the way the triangular matrices are computed. Second, we propose an alternative SGSD method which only requires that two of the matrix factors of the CP decomposition have full column rank.

Index Terms— tensor, canonical decomposition, polyadic decomposition, parafac, simultaneous generalized Schur decomposition.

1. INTRODUCTION

Simultaneous Generalized Schur Decomposition (SGSD) problems appear in blind source separation [13], [5], blind underdetermined mixture identification [6] and in multidimensional harmonic retrieval [7], [1]. In the mentioned problems the SGSD is used as a mean to compute a third order Canonical Polyadic (CP) decomposition with a partial Hermitian symmetry. It was later adapted to the general unsymmetric third order CP decomposition case in [3]. The SGSD method mainly requires that two of its matrix factors have full column rank. This may sound as a limitation of the method, but as shown in [4] a CP decomposition problem with one full column rank matrix factor can often be converted to a CP decomposition problem with all full column rank matrix factors.

The original SGSD method [13], here called SGSD1-CP, requires that all three matrix factors of the given third order CP decomposition have full column rank. Moreover, it does not fully take the structure among the involved triangular matrices into account, as will be explained in section 2. We will first propose a new SGSD method, called SGSD2-CP, for the computation of a third order CP decomposition with three full column rank matrix factors. This method attempts to take the structure among the involved triangular matrices into account. The method proposed in this paper is based on a general procedure for computing the parameters of a structured tensor [2]. The second contribution of this paper is a more relaxed SGSD method, called SGSD3-CP, which only requires that two of the matrix factors of the given CP decomposition have full column rank.

The paper is organized as follows. The rest of the introduction will present our notation followed by a quick review of the CP and SGSD decompositions. Next, in section 2 we review the original SGSD1-CP method for computing a CP decomposition. In section 3 and 4 we propose the new SGSD2-CP and SGSD3-CP methods for computing a CP decomposition. Section 5 presents numerical experiments while section 6 concludes the paper.

1.1. Notations

Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively. The symbol $\otimes$ denotes the Kronecker product

$$\mathbf{A} \otimes \mathbf{B} \triangleq \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

and $\odot$ denotes the Khatri-Rao product

$$\mathbf{A} \odot \mathbf{B} \triangleq \begin{bmatrix} \mathbf{a}_1 \otimes \mathbf{b}_1 & \mathbf{a}_2 \otimes \mathbf{b}_2 & \dots \end{bmatrix},$$

where $\mathbf{a}_r$ and $\mathbf{b}_r$ denote the rth column vector of $\mathbf{A}$ and $\mathbf{B}$, respectively. Furthermore, the outer product of three

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vectors $\mathbf{a}^{(n)} \in \mathbb{R}^{I_n}$ is denoted by $\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \mathbf{a}^{(3)} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ and it satisfies the relation

$$\left(\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \mathbf{a}^{(3)}\right)_{i_1 i_2 i_3} = a^{(1)}_{i_1} a^{(2)}_{i_2} \cdots a^{(N)}_{i_N}.$$

Further, $(\cdot)^T$, $(\cdot)^\dagger$, $\|\cdot\|_F$ and $\mathrm{Col}(\cdot)$ denote the transpose, Moore-Penrose pseudo-inverse, Frobenius norm and column space of a matrix, respectively.

The identity matrix is denoted by $\mathbf{I}_R \in \mathbb{R}^{R \times R}$. A matrix $\mathbf{A} \in \mathbb{R}^{I \times J}$ with $I \geq J$ is said to be semi-orthogonal if $\mathbf{A}^T\mathbf{A} = \mathbf{I}_J$. Matlab index notation will be used to denote submatrices of a given matrix. For example, $\mathbf{A}(1:k,:)$ denotes the submatrix of $\mathbf{A}$ consisting of the rows from 1 to k.

The notation $\mathrm{triu}(\cdot)$ and $\mathrm{diag}(\cdot)$ is used to denote the operator that sets the strictly lower triangular elements and off-diagonal elements of a matrix equal to zero, respectively. Let $\mathbf{A} \in \mathbb{R}^{J \times K}$ with $J, K \geq R$, then $\mathrm{triu}_R(\mathbf{A})$ is equal to

$$\mathrm{triu}_R(\mathbf{A}) = \begin{bmatrix} \mathrm{triu}(\mathbf{A}(1:R,1:R)) & \mathbf{0}_{R,K-R} \\ \mathbf{0}_{J-R,R} & \mathbf{0}_{J-R,K-R} \end{bmatrix} \in \mathbb{R}^{J \times K},$$

where $\mathbf{0}_{M,N} \in \mathbb{R}^{M \times N}$ denotes an all-zero matrix.

Moreover, let $\mathbf{A} \in \mathbb{R}^{I \times J}$, then $\mathrm{Vec}(\mathbf{A}) \in \mathbb{R}^{IJ}$ denotes the column vector defined by $(\mathrm{Vec}(\mathbf{A}))_{i+(j-1)I} = (\mathbf{A})_{ij}$. Let $\mathbf{a} \in \mathbb{R}^{IJ}$, then the reverse operation is $\mathrm{Unvec}(\mathbf{a}) = \mathbf{A} \in \mathbb{R}^{I \times J}$ such that $(\mathbf{a})_{i+(j-1)I} = (\mathbf{A})_{ij}$. Let $\mathbf{A} \in \mathbb{R}^{I \times I}$, then $\mathrm{Vecd}(\mathbf{A}) \in \mathbb{R}^{I}$ denotes the column vector defined by $(\mathrm{Vecd}(\mathbf{A}))_i = (\mathbf{A})_{ii}$. Let $\mathbf{A} \in \mathbb{R}^{I \times J}$, then $D_k(\mathbf{A}) \in \mathbb{R}^{J \times J}$ denotes the diagonal matrix holding row k of $\mathbf{A}$ on its diagonal.

The k-rank of a matrix $\mathbf{A}$ is denoted by $\mathrm{k}(\mathbf{A})$. It is equal to the largest integer $\mathrm{k}(\mathbf{A})$ such that every subset of $\mathrm{k}(\mathbf{A})$ columns of $\mathbf{A}$ is linearly independent.

1.2. CP Decomposition

A third order rank-1 tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ is defined as the outer product of non-zero vectors $\mathbf{a}^{(n)} \in \mathbb{R}^{I_n}$, $n \in [1,3]$, such that $\mathcal{X}_{i_1 i_2 i_3} = \prod_{n=1}^{3} a^{(n)}_{i_n}$. The rank of a tensor $\mathcal{X}$ is equal to the minimal number of rank-1 tensors that yield $\mathcal{X}$ in a linear combination. Assume that the rank of $\mathcal{X}$ is R, then it can be written as

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{a}^{(1)}_r \circ \mathbf{a}^{(2)}_r \circ \mathbf{a}^{(3)}_r, \tag{1}$$

where $\mathbf{a}^{(n)}_r \in \mathbb{R}^{I_n}$. This decomposition will be referred to as the Canonical Polyadic (CP) decomposition of $\mathcal{X}$ [8]. Let us stack the vectors $\{\mathbf{a}^{(n)}_r\}$ into the matrices

$$\mathbf{A}^{(n)} = \left[\, \mathbf{a}^{(n)}_1, \ \cdots \ , \mathbf{a}^{(n)}_R \,\right] \in \mathbb{R}^{I_n \times R}, \quad n \in [1,3]. \tag{2}$$

The matrices $\mathbf{A}^{(n)}$ in (2) will be referred to as the matrix factors of the tensor $\mathcal{X}$ in (1). The following three matrix representations of a CP decomposition of the third order tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ will be used throughout the paper. Let $\mathbf{X}^{(i_1 \cdot\cdot)} \in \mathbb{R}^{I_2 \times I_3}$ denote the matrix such that $\mathbf{X}^{(i_1 \cdot\cdot)}_{i_2 i_3} = \mathcal{X}_{i_1 i_2 i_3}$, then $\mathbf{X}^{(i_1 \cdot\cdot)} = \mathbf{A}^{(2)} D_{i_1}\left(\mathbf{A}^{(1)}\right) \mathbf{A}^{(3)T}$ and

$$\mathbb{R}^{I_1 I_2 \times I_3} \ni \mathbf{X}_{(1)} \triangleq \begin{bmatrix} \mathbf{X}^{(1\cdot\cdot)} \\ \vdots \\ \mathbf{X}^{(I_1 \cdot\cdot)} \end{bmatrix} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right) \mathbf{A}^{(3)T}.$$

Similarly, let the matrices $\mathbf{X}^{(\cdot i_2 \cdot)} \in \mathbb{R}^{I_3 \times I_1}$ be constructed such that $\mathbf{X}^{(\cdot i_2 \cdot)}_{i_3 i_1} = \mathcal{X}_{i_1 i_2 i_3}$, then $\mathbf{X}^{(\cdot i_2 \cdot)} = \mathbf{A}^{(3)} D_{i_2}\left(\mathbf{A}^{(2)}\right) \mathbf{A}^{(1)T}$ and

$$\mathbb{R}^{I_2 I_3 \times I_1} \ni \mathbf{X}_{(2)} \triangleq \begin{bmatrix} \mathbf{X}^{(\cdot 1 \cdot)} \\ \vdots \\ \mathbf{X}^{(\cdot I_2 \cdot)} \end{bmatrix} = \left(\mathbf{A}^{(2)} \odot \mathbf{A}^{(3)}\right) \mathbf{A}^{(1)T}.$$

Finally, let $\mathbf{X}^{(\cdot\cdot i_3)} \in \mathbb{R}^{I_1 \times I_2}$ satisfy $\mathbf{X}^{(\cdot\cdot i_3)}_{i_1 i_2} = \mathcal{X}_{i_1 i_2 i_3}$, then $\mathbf{X}^{(\cdot\cdot i_3)} = \mathbf{A}^{(1)} D_{i_3}\left(\mathbf{A}^{(3)}\right) \mathbf{A}^{(2)T}$ and

$$\mathbb{R}^{I_1 I_3 \times I_2} \ni \mathbf{X}_{(3)} \triangleq \begin{bmatrix} \mathbf{X}^{(\cdot\cdot 1)} \\ \vdots \\ \mathbf{X}^{(\cdot\cdot I_3)} \end{bmatrix} = \left(\mathbf{A}^{(3)} \odot \mathbf{A}^{(1)}\right) \mathbf{A}^{(2)T}.$$

1.3. CP Decomposition as a SGSD Problem

The idea of considering a CP decomposition as a SGSD problem was proposed in [13] and it will be briefly reviewed in this subsection. Let us find $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$ by minimizing the cost function

$$f\left(\{\mathbf{A}^{(n)}\}\right) = \sum_{i_3=1}^{I_3} \left\| \mathbf{X}^{(\cdot\cdot i_3)} - \mathbf{A}^{(1)} D_{i_3}\left(\mathbf{A}^{(3)}\right) \mathbf{A}^{(2)T} \right\|_F^2. \tag{3}$$

Moreover, let $\mathbf{A}^{(1)} = \mathbf{QR}$ be the QR factorization of $\mathbf{A}^{(1)}$ where $\mathbf{Q} \in \mathbb{R}^{I_1 \times I_1}$ is a orthogonal matrix and

$$\mathbf{R} = \begin{bmatrix} \overline{\mathbf{R}} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{I_1 \times R}, \tag{4}$$

with $\overline{\mathbf{R}} \in \mathbb{R}^{R \times R}$ being an upper triangular matrix. Similarly, let $\mathbf{A}^{(2)} = \mathbf{ZL}$ be the QL factorization of $\mathbf{A}^{(2)}$ where $\mathbf{Z} \in \mathbb{R}^{I_2 \times I_2}$ is a orthogonal matrix and

$$\mathbf{L} = \begin{bmatrix} \overline{\mathbf{L}} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{I_2 \times R}, \tag{5}$$

with $\overline{\mathbf{L}} \in \mathbb{R}^{R \times R}$ being a lower triangular matrix. Then (3) can be written as

$$f\left(\mathbf{Q}, \mathbf{Z}, \mathbf{R}, \mathbf{L}, \mathbf{A}^{(3)}\right) = \sum_{i_3=1}^{I_3} \left\| \mathbf{X}^{(\cdot\cdot i_3)} - \mathbf{QR}\, D_{i_3}\left(\mathbf{A}^{(3)}\right) \mathbf{L}^T \mathbf{Z}^T \right\|_F^2.$$

By replacing the matrices $\{\overline{\mathbf{R}} D_{i_3}\left(\mathbf{A}^{(3)}\right) \overline{\mathbf{L}}^T\}$ by the upper triangular matrices $\{\overline{\mathbf{R}}^{(\cdot\cdot i_3)}\}$ we obtain the SGSD problem

$$g\left(\mathbf{Q},\mathbf{Z},\{\mathbf{R}^{(\cdot\cdot i_3)}\}\right)=\sum_{i_3=1}^{I_3}\left\|\mathbf{X}^{(\cdot\cdot i_3)}-\mathbf{Q}\mathbf{R}^{(\cdot\cdot i_3)}\mathbf{Z}^T\right\|_F^2, \tag{6}$$

where

$$\mathbf{R}^{(\cdot\cdot i_3)}=\begin{bmatrix}\overline{\mathbf{R}}^{(\cdot\cdot i_3)} & \mathbf{0}_{R,I_2-R}\\ \mathbf{0}_{I_1-R,R} & \mathbf{0}_{I_1-R,I_2-R}\end{bmatrix}\in\mathbb{R}^{I_1\times I_2}.$$

Minimizing the cost function (6) is equivalent to maximizing the cost function

$$h\left(\mathbf{Q},\mathbf{Z}\right) \;=\; \sum_{i_3=1}^{I_3}\left\|\mathrm{triu}_R\left(\mathbf{Q}^T\mathbf{X}^{(\cdot\cdot i_3)}\mathbf{Z}\right)\right\|_F^2. \tag{7}$$

Thus, in the SGSD method we first compute the orthogonal matrices $\mathbf{Q}$ and $\mathbf{Z}$ which will make the matrices $\{\mathbf{X}^{(\cdot\cdot i_3)}\}$ as upper triangular as possible by maximizing (7). This means that we have gone from the original non-orthogonal optimization problem (3) to the orthogonal optimization problem (7).

Various methods have been proposed to numerically maximize the cost function (7). For instance, in [13] the exended QZ method was proposed for the case when $I_1 = I_2 = R$. In the numerical experiments presented in section 5 we will apply the generalized version of the extended QZ method to case $I_1, I_2 \geq R$ as proposed in [11].

The following sections 2, 3 and 4 will discuss methods to find $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$ once $\mathbf{Q}$ and $\mathbf{Z}$ have been found.

2. SGSD1-CP

In this section the original SGSD1-CP method proposed in [13], [3] will be reviewed. Let $\mathcal{X}\in\mathbb{R}^{I_1\times I_2\times I_3}$ be a tensor of rank R constructed from the full column rank matrix factors $\mathbf{A}^{(n)}\in\mathbb{R}^{I_n\times R}$, $n\in\{1,2,3\}$.

Assume that $\mathbf{Q}$ and $\mathbf{Z}$ have been found and recall that $\mathrm{triu}_R\left(\mathbf{Q}^T\widetilde{\mathbf{X}}^{(\cdot\cdot i_3)}\mathbf{Z}\right)=\mathbf{R}D_{i_3}\left(\mathbf{A}^{(3)}\right)\mathbf{L}^T$. Furthermore, let the tensor $\mathcal{R}\in\mathbb{R}^{R\times R\times I_3}$ be constructed such that $\mathcal{R}_{i_1i_2i_3}=\mathbf{R}^{(\cdot\cdot i_3)}_{i_1i_2}=\mathrm{triu}\left(\mathbf{Q}^T\widetilde{\mathbf{X}}^{(\cdot\cdot i_3)}\mathbf{Z}\right)_{i_1i_2}$, then $\mathbf{R}^{(\cdot\cdot i_3)}=\overline{\mathbf{R}}D_{i_3}\left(\mathbf{A}^{(3)}\right)\overline{\mathbf{L}}^T$. We have that $\mathrm{diag}\left(\mathbf{R}^{(\cdot\cdot i_3)}\right)=D_{i_3}\left(\mathbf{A}^{(3)}\right)\Lambda,\quad\forall i_3\in[1,I_3]$, where $\Lambda=\mathrm{diag}\left(\overline{\mathbf{R}}\right)\mathrm{diag}\left(\overline{\mathbf{L}}\right)$. Due to the inherent scaling ambiguity of the CP decomposition and the non-zero diagonal entries of $\overline{\mathbf{R}}$ and $\overline{\mathbf{L}}$ we can set $\Lambda=\mathbf{I}_R$. Thus

$$\mathbf{A}^{(3)}=\begin{bmatrix}\mathrm{Vecd}\left(\mathbf{R}^{(\cdot\cdot 1)}\right)^T\\ \mathrm{Vecd}\left(\mathbf{R}^{(\cdot\cdot 2)}\right)^T\\ \vdots\\ \mathrm{Vecd}\left(\mathbf{R}^{(\cdot\cdot I_3)}\right)^T\end{bmatrix}. \tag{8}$$

Assume that $\mathbf{A}^{(3)}$ has been found and recall that $\mathbf{X}_{(1)}=\left(\mathbf{A}^{(1)}\odot\mathbf{A}^{(2)}\right)\mathbf{A}^{(3)T}$. Let $\mathbf{F}=\mathbf{X}_{(1)}\left(\mathbf{A}^{(3)T}\right)^{\dagger}$, then this implies that the matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ follow from the R decoupled best rank-1 matrix problems

$$\min_{\mathbf{a}_r^{(1)},\mathbf{a}_r^{(2)}}\left\|\mathrm{Unvec}\left(\mathbf{f}_r\right)-\mathbf{a}_r^{(2)}\mathbf{a}_r^{(1)T}\right\|_F^2,\quad r\in[1,R], \tag{9}$$

where $\mathbf{a}_r^{(1)}$, $\mathbf{a}_r^{(2)}$ and $\mathbf{f}_r$ denotes the rth column vector of $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{F}$, respectively. The problem (9) can be solved by standard numerical methods. In the presence of noise a refinement of $\mathbf{A}^{(3)}$ is often necessary and therefore $\mathbf{A}^{(3)}$ follows from $\mathbf{A}^{(3)T}=\left(\mathbf{A}^{(1)}\odot\mathbf{A}^{(2)}\right)^{\dagger}\mathbf{X}_{(3)}$.

3. SGSD2-CP

In this section the new SGSD2-CP method for computing a third order CP decomposition is presented. When computing $\mathbf{A}^{(3)}$, then the SGSD1-CP method only exploits the relation among the diagonals of the triangular matrices $\{\mathbf{R}^{(\cdot\cdot i_3)}\}$. The SGSD2-CP method, on the other hand attempts to fully take the relations among the triangular matrices $\{\mathbf{R}^{(\cdot\cdot i_3)}\}$ into account when computing $\mathbf{A}^{(3)}$.

Let the third order tensor $\mathcal{X}\in\mathbb{R}^{I_1\times I_2\times I_3}$ of rank R be constructed from the full column rank matrix factors $\mathbf{A}^{(n)}\in\mathbb{R}^{I_n\times R}$, $n\in\{1,2,3\}$. Assume that $\mathbf{Q}$ and $\mathbf{Z}$ have been found and let the tensor $\mathcal{R}\in\mathbb{R}^{R\times R\times I_3}$ be constructed such that $\mathbf{R}^{(\cdot\cdot i_3)}_{i_1i_2}=\mathcal{R}_{i_1i_2i_3}=\mathrm{triu}\left(\mathbf{Q}^T\widetilde{\mathbf{X}}^{(\cdot\cdot i_3)}\mathbf{Z}\right)_{i_1i_2}$, then $\mathbf{R}^{(\cdot\cdot i_3)}=\overline{\mathbf{R}}D_{i_3}\left(\mathbf{A}^{(3)}\right)\overline{\mathbf{L}}^T$. Moreover, let $\mathbf{R}^{(i_1\cdot\cdot)}\in\mathbb{R}^{R\times I_3}$ satisfy the relation $\mathbf{R}^{(i_1\cdot\cdot)}_{i_2i_3}=\mathcal{R}_{i_1i_2i_3}$, then $\mathbf{R}^{(i_1\cdot\cdot)}=\overline{\mathbf{L}}D_{i_1}\left(\overline{\mathbf{R}}\right)\mathbf{A}^{(3)T}$. Stack the matrices $\{\mathbf{R}^{(i_1\cdot\cdot)}\}$ into the matrix

$$\mathbb{R}^{R^2\times I_3}\ni\mathbf{R}_{(1)}=\begin{bmatrix}\mathbf{R}^{(1\cdot\cdot)}\\ \vdots\\ \mathbf{R}^{(R\cdot\cdot)}\end{bmatrix}=\left(\overline{\mathbf{R}}\odot\overline{\mathbf{L}}\right)\mathbf{A}^{(3)T}.$$

Let $\mathbf{R}_{(1)}=\mathbf{U}\Sigma\mathbf{W}^H$ denote the compact SVD of $\mathbf{R}_{(1)}$, where $\mathbf{U}\in\mathbb{R}^{R^2\times R}$ and $\mathbf{W}\in\mathbb{R}^{I_3\times R}$ are semi-orthogonal matrices and $\Sigma\in\mathbb{R}^{R\times R}$ is a positive diagonal matrix. If the matrices $\overline{\mathbf{R}}\odot\overline{\mathbf{L}}$ and $\mathbf{A}^{(3)}$ have full column rank, then there exists a nonsingular matrix $\mathbf{M}\in\mathbb{R}^{R\times R}$ such that

$$\overline{\mathbf{R}}\odot\overline{\mathbf{L}}=\mathbf{U}\mathbf{M}. \tag{10}$$

We can compute $\overline{\mathbf{R}}$ and $\overline{\mathbf{L}}$ from (10) by solving R decoupled linear equations, see [12] for details. This means that we also get $\mathbf{A}^{(3)T}=\left(\overline{\mathbf{R}}\odot\overline{\mathbf{L}}\right)^{\dagger}\mathbf{R}_{(1)}$.

Assume that $\mathbf{A}^{(3)}$ has been found, then as in the SGSD1-CP method, the matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ follow from the R decoupled best rank-1 matrix problems (9).

4. SGSD3-CP

In this section we present the new SGSD3-CP method for computing a third order CP decomposition where only two of the matrix factors are required to have full column rank. This is in contrast to the SGSD1-CP and SGSD2-CP methods which both require that all three of the involved matrix factors have full column rank.

Let the third order tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ of rank R be constructed from the two full column rank matrix factors $\mathbf{A}^{(1)} \in \mathbb{R}^{I_1 \times R}$, $\mathbf{A}^{(2)} \in \mathbb{R}^{I_2 \times R}$ and the matrix factor $\mathbf{A}^{(3)} \in \mathbb{R}^{I_3 \times R}$ with $\mathrm{k}\left(\mathbf{A}^{(3)}\right) \geq 2$. Assume that $\mathbf{Q}$ and $\mathbf{Z}$ have been found and let $\mathbf{R}^{(\cdot\cdot i_3)} = \mathbf{Q}^T \mathbf{X}^{(\cdot\cdot i_3)} \mathbf{Z}$. As in the SGSD1-CP method, we find $\mathbf{A}^{(3)}$ from the relation (8). Let the tensor $\mathcal{R} \in \mathbb{R}^{R \times R \times I_3}$ be constructed such that $\mathbf{R}^{(\cdot\cdot i_3)}_{i_1 i_2} = \mathcal{R}_{i_1 i_2 i_3} = \mathrm{triu}\left(\mathbf{Q}^T \widetilde{\mathbf{X}}^{(\cdot\cdot i_3)} \mathbf{Z}\right)_{i_1 i_2}$, then $\mathbf{R}^{(\cdot\cdot i_3)} = \overline{\mathbf{R}} D_{i_3}\left(\mathbf{A}^{(3)}\right) \overline{\mathbf{L}}^T$. Moreover, let $\mathbf{R}^{(\cdot i_2 \cdot)} \in \mathbb{R}^{R \times I_3}$ satisfy the relation $\mathbf{R}^{(\cdot i_2 \cdot)}_{i_3 i_1} = \mathcal{R}_{i_1 i_2 i_3}$, then $\mathbf{R}^{(\cdot i_2 \cdot)} = \mathbf{A}^{(3)} D_{i_2}\left(\overline{\mathbf{L}}\right) \overline{\mathbf{R}}^T$. Stack the matrices $\{\mathbf{R}^{(\cdot i_2 \cdot)}\}$ into the matrix

$$\mathbb{R}^{RI_3 \times R} \ni \mathbf{R}_{(2)} = \begin{bmatrix} \mathbf{R}^{(\cdot 1 \cdot)} \\ \vdots \\ \mathbf{R}^{(\cdot R \cdot)} \end{bmatrix} = \left(\overline{\mathbf{L}} \odot \mathbf{A}^{(3)}\right) \overline{\mathbf{R}}^T.$$

Let $\mathbf{R}_{(2)} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{R}_{(2)}$. Since $\mathrm{k}\left(\overline{\mathbf{L}}\right) = R$ and $\mathrm{k}\left(\mathbf{A}^{(3)}\right) \geq 2$ we get $\mathrm{k}\left(\overline{\mathbf{L}} \odot \mathbf{A}^{(3)}\right) = R$ [10]. This implies that there exists a nonsingular matrix $\mathbf{M} \in \mathbb{R}^{R \times R}$ such that

$$\mathbf{U}\mathbf{M} = \overline{\mathbf{L}} \odot \mathbf{A}^{(3)}. \tag{11}$$

We can compute $\overline{\mathbf{L}}$ from (11) by solving R decoupled linear equations, see [12] for details.

Once $\overline{\mathbf{L}} \in \mathbb{R}^{R \times R}$ has been obtained, and thereby also $\mathbf{L} = [\overline{\mathbf{L}}^T, \mathbf{0}]^T \in \mathbb{R}^{I_2 \times R}$, we also get $\mathbf{A}^{(2)} = \mathbf{Z}\mathbf{L}$. Finally, from the relation $\mathbf{X}_{(2)} = \left(\mathbf{A}^{(2)} \odot \mathbf{A}^{(3)}\right) \mathbf{A}^{(1)T}$ we obtain $\mathbf{A}^{(1)}$ as follows $\mathbf{A}^{(1)T} = \left(\mathbf{A}^{(2)} \odot \mathbf{A}^{(3)}\right)^{\dagger} \mathbf{X}_{(2)}$.

In the presence of noise a refinement of the matrix factors is often necessary. Let $\mathbf{F} = \mathbf{X}_{(2)}\left(\mathbf{A}^{(1)T}\right)^{\dagger}$, then the matrices $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$ follow from the R decoupled best rank-1 matrix problems

$$\min_{\mathbf{a}_r^{(2)}, \mathbf{a}_r^{(3)}} \left\| \mathrm{Unvec}\left(\mathbf{f}_r\right) - \mathbf{a}_r^{(3)} \mathbf{a}_r^{(2)T} \right\|_F^2, \quad r \in [1, R],$$

where $\mathbf{a}_r^{(2)}, \mathbf{a}_r^{(3)}$ and $\mathbf{f}_r$ denotes the rth column vector of $\mathbf{A}^{(2)}$, $\mathbf{A}^{(3)}$ and $\mathbf{F}$, respectively.

5. NUMERICAL EXPERIMENTS

To simplify notations, let SGSDN-CP, where $N \in [1, 3]$, denote a SGSD method. For example SGSD1-CP corresponds to $N = 1$. In this section comparisons between the popular Alternating Least Squares (ALS) method and the SGSDN-CP methods will be carried out. The entries of all the involved tensors are randomly drawn from a uniform distribution with support $[-\frac{1}{2}, \frac{1}{2}]$. Let $\mathcal{T} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ with rank R denote the tensor we attempt to estimate from the observed tensor $\mathcal{X} = \mathcal{T} + \beta\mathcal{N}$, where $\mathcal{N}$ is an unstructured perturbation tensor and $\beta \in \mathbb{R}$ is a gain factor. The following measures will be used

$$\mathrm{SNR\ [dB]} = 10 \log\left(\frac{\|\mathcal{X}\|_F^2}{\|\beta\mathcal{N}\|_F^2}\right), \quad \mathrm{P}\left(\mathbf{A}^{(1)}\right) = \min_{\Pi\Lambda} \frac{\left\|\mathbf{A}^{(1)} - \widehat{\mathbf{A}}^{(1)}\Pi\Lambda\right\|_F}{\left\|\mathbf{A}^{(1)}\right\|_F},$$

where $\widehat{\mathbf{A}}^{(1)}$ denotes the estimated matrix factor, Π denotes a permutation matrix and Λ denotes a diagonal matrix. In order to find Π and Λ the greedy least squares column matching algorithm between $\mathbf{A}^{(n)}$ and $\widehat{\mathbf{A}}^{(n)}$ proposed in [9] is used.

To measure the elapsed time in seconds used to execute the algorithms in MATLAB, the built-in functions *tic*$(\cdot)$ and *toc*$(\cdot)$ are used.

Let $f\left(\widehat{\mathcal{T}}^{(k)}\right) = \left\|\mathcal{T} - \widehat{\mathcal{T}}^{(k)}\right\|_F$, where $\widehat{\mathcal{T}}^{(k)}$ denotes the estimated tensor at iteration k, then we decide that the ALS method has converged when $\left|f\left(\widehat{\mathcal{T}}^{(k)}\right) - f\left(\widehat{\mathcal{T}}^{(k+1)}\right)\right| < 1e-8$ or when the number of iterations exceed 2000.

Similarly, let $g\left(\mathbf{Q}^{(k)}, \mathbf{Z}^{(k)}\right) = \sum_{i_3=1}^{I_3} \left\|\mathrm{triu}\left(\mathbf{Q}^{(k)T} \mathbf{X}^{(\cdot\cdot i_3)} \mathbf{Z}^{(k)}\right)\right\|_F^2$, where $\mathbf{Q}^{(k)}$ and $\mathbf{Z}^{(k)}$ denote the estimates of the orthogonal matrices in the SGSDN-CP method, then we decide that the given SGSDN-CP method has converged when $\left|g\left(\mathbf{Q}^{(k)}, \mathbf{Z}^{(k)}\right) - g\left(\mathbf{Q}^{(k+1)}, \mathbf{Z}^{(k+1)}\right)\right| < 1e-5$ or when the number of iterations exceed 200.

Moreover, if we let the SGSDN-CP method be followed by at most 100 ALS refinement steps, then the SGSDN-CP method will be referred to as SGSDN-CP-ALS.

Let $I_1 = 6$, $I_2 = 7$, $I_3 = 8$ and $R = 5$. The mean $\mathrm{P}\left(\mathbf{A}^{(1)}\right)$ and time values over 101 trials as a function of SNR can be seen on figure 1. We first notice that the SGSD2-CP method performs better than the SGSD1-CP method. We also notice that above 25 dB, the SGSD1-CP-ALS and SGSD2-CP-ALS methods perform better than the ALS method while below 20 dB the methods seems to yield similar performance. However, the SGSDN-CP-ALS methods are less costly than the ALS method.

Let $I_1 = 6$, $I_2 = 7$, $I_3 = 8$ and $R = 3$. The mean $\mathrm{P}\left(\mathbf{A}^{(1)}\right)$ and time values over 101 trials as a function of SNR can be seen on figure 2. We first notice that the SGSD3-CP method performs slightly worse than the ALS method while the SGSD3-CP-ALS yield similar performance as the ALS method, but at a significantly lower computational cost.

6. CONCLUSION

Based on the link between the SGSD and CP decomposition it is possible to address some CP decomposition problems as SGSD problems. We reviewed the original SGSD1-CP method for computing a third order CP decomposition. The SGSD1-CP method does not fully take the relations among the involved triangular matrices into account. This motivated us to propose the SGSD2-CP which attempts to take the relations among the involved triangular matrices into account. The SGSD1-CP also make the unnecessary assumption that all three matrix factors of the given CP decomposition have full column rank. This motivated us to propose the SGSD3-CP

which only requires that two of the matrix factors of the given CP decomposition to have full column rank.

Numerical experiments showed the proposed SGSD2-CP method can enhance the performance of the SGSD1-CP method. They also showed that the SGSD-CP methods are computationally more efficient than the ALS method. Hence, the SGSD-CP methods could be used to speed up the ALS method.

The discussed SGSD-CP methods can also be used to compute a CP decomposition with a partial symmetry. Moreover, the SGSD-CP methods can also be used to compute the CP decomposition of an Nth-order tensor. For a detailed discussion of these issues we refer to [12].

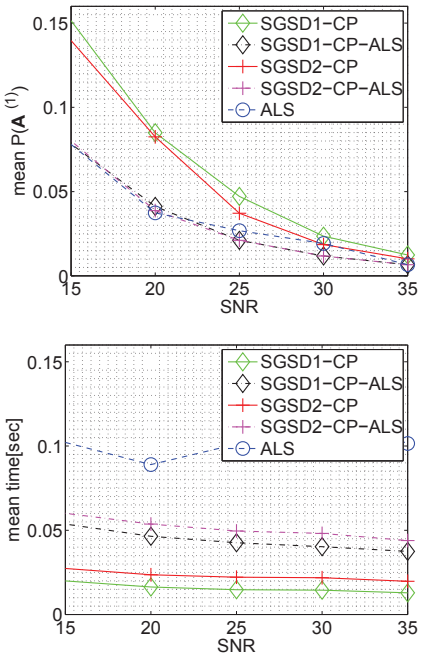

Fig. 1. Mean $\mathrm{P}\left(\mathbf{A}^{(1)}\right)$ and time values for CP decompsition with $I_1 = 6$, $I_2 = 7$, $I_3 = 8$ and $R = 5$.

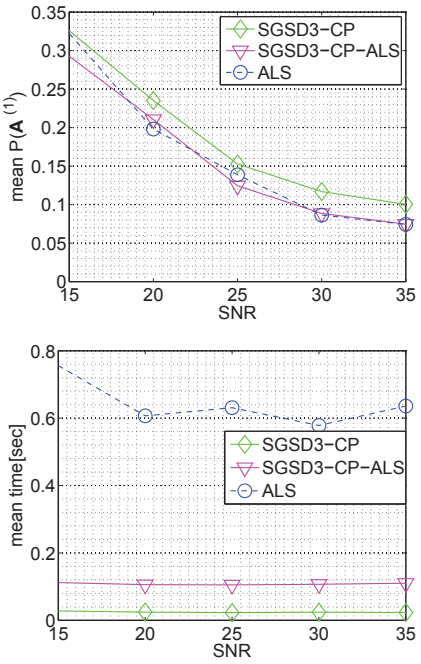

Fig. 2. Mean $\mathrm{P}\left(\mathbf{A}^{(1)}\right)$ and time values for CP decompsition with $I_1 = 6$, $I_2 = 7$, $I_3 = 3$ and $R = 5$.

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