Multiuser Spectrum Balancing under Inter-Carrier Interference

Martin Wolkerstorfer, Marc Moonen, and Driton Statovci

***Abstract*—Inter-carrier interference (ICI) in multicarrier systems results in a coupling among all subcarriers. This greatly complicates the problem of multiuser bit and power allocation, also referred to as spectrum balancing. We propose a Lagrange relaxation of the ICI coupling terms, and an optimal algorithm for solving the resulting relaxed spectrum balancing problem. While this gives us a rate-performance upper bound, based on it we also propose a novel heuristic for computing a lower bound. The relaxation is exemplarily applied to digital subscriber lines (DSL) with asynchronous far-end crosstalk in near-far scenarios. The results confirm the existence of scenarios where the performance upper and lower bounds obtained by the relaxation are significantly tighter than those previously proposed.**

***Index Terms*—Interference channels, optimization methods, OFDM modulation, DSL.**

I. Introduction

MULTICARRIER modulation techniques have found application in numerous communication systems due to their spectral efficiency. They allow to exploit the frequency-selectivity of the channel by multiuser bit and transmit-power allocation, commonly referred to as "spectrum balancing" in the context of multicarrier digital subscriber lines (DSL). The allocation problem is greatly simplified by the assumption of orthogonality among transmissions over distinct subcarriers. However, this hinges on several assumptions such as a cyclic prefix length that is longer than the channel impulse response, or perfect time and frequency synchronization among transceivers. Otherwise inter-carrier interference (ICI) may occur, both, to a single user as well as among several users. Analysis of the effect of ICI on various modulation techniques and previously proposed bit and power allocation techniques under ICI can be found in [1]–[7].

We study the problem of spectrum balancing for frequency-selective multiuser ICI channels. Our main contribution is the formulation and optimal solution of a Lagrange dual relaxation based performance bound to the optimal multiuser spectrum balancing problem with ICI. Furthermore, we propose two novel heuristics for computing lower bounds on the achievable rate-region. An exemplary proof-of-concept for this relaxation approach is provided by applying it to downstream near-far DSL scenarios suffering from symbol-level asynchronism of the far-end crosstalk [5], [6], [8], [9]. However, we emphasize that our analysis and proposed algorithms similarly apply to other multicarrier systems and ICI scenarios, including for instance self-interference [1] or near-end crosstalk [7]. Our results show that the inter-carrier coupling terms *provably* have a non-negligible impact on the achievable rate-region.

Various heuristics for the studied resource allocation problem have appeared in [5], [6], [8]–[10]. Additionally, in [5] a performance bound has been computed by neglecting the ICI coupling terms and solving the Lagrange dual problem optimally [11]. Lagrange relaxation has previously been considered inapplicable to the allocation problem with ICI [5], [6], [8], [9] as the ICI couplings prevent a direct application of the decomposition algorithm in [11]. Differently, in order to make the Lagrange relaxation approach tractable we propose to create an artificial variable [12] for each of the U users and C subcarriers, modeling the received interference from all *other* users and subcarriers. Correspondingly UC additional constraints are formulated, enforcing that these artificial variables are no less than the true ICI. We will solve the Lagrange relaxation of the resulting spectrum balancing problem through "column generation" [13] and a nested per-subcarrier power optimization based on the branch-and-bound search in [14]. This search is modified by a novel linear program formulation on each subcarrier involving the adjustment of both, all users' transmit power and artificial interference power.

II. System and Optimization Model

We consider a DSL network with users denoted by $\mathcal{U} = \{1, \ldots, U\}$, each employing discrete multi-tone (DMT) modulation with subcarriers indexed by $\mathcal{C} = \{1, \ldots, C\}$. The sets of subcarriers and users excluding indices u and c will be denoted by $\mathcal{U} \setminus u$ and $\mathcal{C} \setminus c$, respectively. The power allocation and received background noise of user $u \in \mathcal{U}$ on subcarrier $c \in \mathcal{C}$ are written as p_c^u and N_c^u, respectively, and the allocation variables on subcarrier c are compactly written in vector form as $\mathbf{p}_c \in \mathcal{R}^U$. The transfer coefficient from user t and subcarrier k to user u and subcarrier c is denoted by H_{kc}^{tu}. For example, H_{cc}^{uu} represents the direct channel coefficient of user u on subcarrier c, and coefficients with indices $c \neq k$ denote ICI couplings. We model the achievable rate per DMT symbol for $u \in \mathcal{U}, c \in \mathcal{C}$, through two-dimensional constellations by the gap approximation of capacity [15], denoted by

$$r_c^u\left(\mathbf{p}_c, X_c^u\right) = \log_2\Big(1 + \frac{H_{cc}^{uu} p_c^u}{\Gamma\left(\sum_{t \in \mathcal{U}\setminus u} H_{cc}^{tu} p_c^t + X_c^u + N_c^u\right)}\Big), \tag{1}$$

where $X_c^u = \sum_{t \in \mathcal{U}, k \in \mathcal{C}\setminus c} H_{kc}^{tu}\, p_k^t$ denotes the ICI part of the interference noise, and Γ is the signal-to-noise ratio (SNR)

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gap according to a specific bit-error rate requirement.[1]

Our objective is the maximization of the weighted sum-rate of all users, constrained by the maximum allowed transmit power P_u per user $u \in \mathcal{U}$, as well as maximum power and bit-allocation limits $\hat{p}_c^u$ and $\hat{B}$ for user u on subcarrier c, respectively. For brevity we omit any minimum targeted sum-rate constraints. However, such constraints can be incorporated into the problem relaxation and the proposed algorithm in Section III, similarly to more general objective functions [13], [16]. Our problem is compactly written as

$$\underset{\{p_c^u,\, x_c^u,\, \forall u \in \mathcal{U}, c \in \mathcal{C}\}}{\text{maximize}} \quad \sum_{u\in\mathcal{U}} w_u \sum_{c\in\mathcal{C}} r_c^u\left(\mathbf{p}_c,\, x_c^u\right) \tag{2a}$$

subject to

$$r_c^u\left(\mathbf{p}_c, x_c^u\right) \in \mathbb{Z} \cap [0, \hat{B}], \qquad \forall u \in \mathcal{U}, c \in \mathcal{C}, \tag{2b}$$

$$0 \le p_c^u \le \hat{p}_c^u, \qquad \forall u \in \mathcal{U}, c \in \mathcal{C}, \tag{2c}$$

$$0 \le x_c^u \le \sum_{t\in\mathcal{U}, k\in\mathcal{C}\setminus c} H_{kc}^{tu}\hat{p}_k^t, \quad \forall u \in \mathcal{U}, c \in \mathcal{C}, \tag{2d}$$

$$\sum_{c\in\mathcal{C}} p_c^u \;\le\; P_u, \qquad \forall u \in \mathcal{U}, \tag{2e}$$

$$\sum_{t\in\mathcal{U}, k\in\mathcal{C}\setminus c} H_{kc}^{tu} p_k^t \;\le\; x_c^u, \qquad \forall u \in \mathcal{U}, c \in \mathcal{C}, \tag{2f}$$

where the weights w_u, $u \in \mathcal{U}$, in (2a) will allow us to trace out the rate-region, and the bit-allocation $r_c^u\left(\mathbf{p}_c, x_c^u\right)$ is restricted to discrete values in (2b) due to the finite constellation sizes used in practice. Furthermore, we introduced artificial interference variables x_c^u modeling the ICI part of the crosstalk noise and constrained by the actual ICI noise in (2f) and by the maximum ICI noise in (2d) as defined by the power mask. The motivation for these variables will become clearer in the following section where all coupling constraints in (2f) and (2e) will be relaxed. Furthermore, while the constraints in (2d) are redundant at this point, they will become effective after Lagrange relaxation (see the problem in (5) of Section III).

III. Proposed Lagrange Relaxation Upper-Bound

The spectrum balancing problem with ICI differs from its counterpart without ICI [11] by the additional coupling among subcarriers in (2f). One way of computing a performance upper bound to the problem in (2) is by neglecting the additional constraints in (2f) (implying that the ICI noise terms x_c^u can be set to zero), relaxing the remaining coupling constraints in (2e), and optimally solving the resulting Lagrange dual problem [5]. We will compute an upper bound by optimally solving a different Lagrange dual problem, obtained after relaxing all coupling constraints in (2e) and (2f). Since we only relax the constraints in (2f) but do not completely neglect them, we obtain a tighter performance bound to that in [5].

The optimal dual objective value can be equivalently computed by solving the time-sharing relaxation [13], where each possible per-subcarrier allocation is assigned to a continuous time-share between 0 and 1. The sum-rate and sum-power are obtained by optimal convex combination (or "time-averaging") of all possible discrete bit and power allocations. The mechanism for iteratively solving this relaxation without explicitly knowing the prohibitively large number of possible allocations is column generation [17, Ch. 23]. This scheme results in a decomposition of the problem into a linear master problem which provides iteratively updated dual multipliers for the constraints in (2e) and (2f), and a subproblem on each subcarrier targeting the optimization of the Lagrangian. More precisely, at any iteration the master problem assumes a certain subset of known power and artificial-interference allocations on each subcarrier $c \in \mathcal{C}$ indexed by a set $\mathcal{I}_c$. Each of these allocations $p_c^{u,(i)}$ and $x_c^{u,(i)}$ with time-share $\xi_c^{(i)}$ and $i \in \mathcal{I}_c$ are feasible with respect to the per-subcarrier constraints in (2b)–(2d). The master problem optimizing the time sharing over these known allocations has the form[2]

$$\underset{\xi_c^{(i)}\ge 0, i\in\mathcal{I}_c, c\in\mathcal{C}}{\text{maximize}} \quad \sum_{u\in\mathcal{U}} w_u \sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I}_c} r_c^u\left(\mathbf{p}_c^{(i)},\, x_c^{u,(i)}\right)\xi_c^{(i)} \tag{3a}$$

subject to

$$\sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I}_c} p_c^{u,(i)}\xi_c^{(i)} \;\le\; P_u, \qquad \forall u \in \mathcal{U}, \tag{3b}$$

$$\sum_{i\in\mathcal{I}_c}\sum_{t\in\mathcal{U}, k\in\mathcal{C}\setminus c} H_{kc}^{tu} p_k^{t,(i)} \;\le\; \sum_{i\in\mathcal{I}_c} x_c^{u,(i)}, \forall u \in \mathcal{U}, c \in \mathcal{C}, \tag{3c}$$

$$\sum_{i\in\mathcal{I}_c} \xi_c^{(i)} \;=\; 1, \qquad \forall c \in \mathcal{C}. \tag{3d}$$

The only difference to the corresponding master problem without ICI [13] is the constraint in (3c) where we enforce that the ICI noise that has been accounted for in the known allocations is no less than the actual one *on average*. This modification towards constraints that only hold on average over the set of possible solutions is equivalent to a Lagrange relaxation of the constraints [13]. In the ICI-free case a solution using time-sharing on few subcarriers was shown to exist [13]. However, the same argumentation does not lead to such a restriction in the ICI case due to the larger number of constraints. Denoting the Lagrange multipliers for the constraints in (3b) and (3c) by $\boldsymbol{\nu} \in \mathcal{R}^U$ and $\boldsymbol{\kappa} \in \mathcal{R}^{U\cdot C}$, respectively, the nonlinear discrete dual subproblem on subcarrier $c \in \mathcal{C}$, optimizing the Lagrangian of the problem in (2) for relaxed constraints in (2e) and (2f), is given as

$$\underset{p_c^u, x_c^u, \forall u\in\mathcal{U}}{\text{maximize}} \quad \sum_{u\in\mathcal{U}} \left(w_u r_c^u(\mathbf{p}_c, x_c^u) - \hat{w}_c^u p_c^u + \kappa_c^u x_c^u\right) \tag{4a}$$

$$\text{subject to} \quad r_c^u\left(\mathbf{p}_c, x_c^u\right) \in \mathbb{Z} \cap [0, \hat{B}], \qquad \forall u \in \mathcal{U}, \tag{4b}$$

$$0 \le p_c^u \le \hat{p}_c^u, \qquad \forall u \in \mathcal{U}, \tag{4c}$$

$$0 \le x_c^u \le \sum_{t\in\mathcal{U}, k\in\mathcal{C}\setminus c} H_{kc}^{tu}\hat{p}_k^t, \quad \forall u \in \mathcal{U}, \tag{4d}$$

where $\hat{w}_c^u = \nu_u + \sum_{t\in\mathcal{U}, k\in\mathcal{C}\setminus c} H_{ck}^{ut}\kappa_k^t$. Note that the second term in this expression accounts for the ICI-impact of a user u's power allocation on subcarrier c onto all other subcarriers for all users. The complete algorithm for solving the Lagrange relaxation of the original problem in (2) is summarized in Algorithm 1, where we suggest to use the branch-and-bound (BnB) search in [14] to solve the subproblem in (4) on all subcarriers. More precisely, this scheme searches in the space

[1] In order to stress the applicability of our model we consider ICI due to other users as well as the same user. However, for the exemplary DSL problem studied in Section V we have $H_{kc}^{uu} = 0, \forall u \in \mathcal{U}, c \in \mathcal{C}, k \in \mathcal{C} \setminus c$.

[2] We refer to [13] for a means to ensure feasibility of the master problem and to eventually detect infeasibility of the time-sharing relaxation.

Algorithm 1 Dual-Optimal Multiuser Spectrum Balancing under Inter-Carrier Interference (OSB-ICI)

1: **repeat**
2: Solve the problem in (3) by a primal-dual LP solver, thereby obtaining dual multipliers $\boldsymbol{\nu}$ and $\boldsymbol{\kappa}$
3: Solve the problem in (4) by branch-and-bound [14], with the nested linear programs in (5), thereby obtaining novel power and ICI allocations
4: Add only new allocations to the master problem in (3), thereby updating the index-sets $\mathcal{I}_c, \forall c \in \mathcal{C}$
5: **until** Convergence

of discrete bit allocations, and for any bit-allocation b_c^u the constraint in (4b) becomes $r_c^u\left(\mathbf{p}_c, x_c^u\right) \geq b_c^u$. Based on (4) this leads to various linear programs in the form of

$$\underset{p_c^u, x_c^u, \forall u \in \mathcal{U}}{\text{maximize}} \quad \sum_{u \in \mathcal{U}} \Big(- \hat{w}_c^u p_c^u + \kappa_c^u x_c^u \Big) \tag{5a}$$

$$\text{subject to} \quad r_c^u\left(\mathbf{p}_c, x_c^u\right) \geq b_c^u, \quad \forall u \in \mathcal{U}, \tag{5b}$$

$$\text{Constraints in (4c) and (4d)}, \tag{5c}$$

that have to be solved during the BnB search. Bounds on the objective function in (4a) can be computed similarly as in the ICI-free case [14], and additionally using the upper bound in (4d) on the artificial interference variables. Algorithm 1 terminates when the gap between the relaxed primal objective in (3) and the best found dual function value found through optimization in (4) vanishes. Convergence in a finite number of iterations is guaranteed [13], e.g., our simulations in Section V took in the order of ten column generation iterations.

IV. Proposed Heuristic Lower-Bounds

In order to compute lower bounds to the optimal objective of the original problem in (2) we compare two discrete bit-allocation heuristics that are summarized in Algorithm 2.

The first heuristic is based on the average solution obtained in the final iteration of the column generation scheme in Algorithm 1, cf. Line 2. Based on it we compute a *continuous* bit-allocation table in Line 4, where $[\cdot]^{\hat{B}}$ denotes the projection onto the interval between zero and $\hat{B}$. In Lines 6–10 we round this table to a feasible *discrete* bit-allocation. Feasibility of the rounding is checked according to [5] in Line 10. In Line 12 the obtained bit-allocation table is used to initialize the multi-user bit-adding heuristic for ICI channels in [5].

The second proposed heuristic differs from the first one in that the modified iterative waterfilling (IWF) scheme [6] in Line 3 is used instead of Algorithm 1 in order to initialize the mentioned bit-rounding and bit-allocation heuristics, respectively. While it has been empirically observed [6] to perform superior to the discrete bit-allocation schemes in [5], it does not obey the power mask constraints in (2c) or the discreteness of the bit-allocation table enforced in (2b). Hence, we initially neglect these constraints and run the IWF scheme [6] initialized by a uniform allocation of the maximum sum-power over subcarriers. Both constraints are reinforced in Line 3 and the following bit-rounding procedure.

Note that the results of both heuristics in Algorithm 2 are guaranteed to produce feasible solutions to the original problem in (2). This is due to the true rate computation under ICI in Line 4, the infeasibility check in Line 10, and the fact that the bit-adding process in Line 12 stops when no feasible improvement can be made.

Algorithm 2 Bit-Allocation Heuristics under ICI

1: *Initialization Phase*
2: *Option A:* Run Algorithm 1 to obtain $\xi_c^{(i)}, i \in \mathcal{I}_c, c \in \mathcal{C}$
$\tilde{p}_c^u = \sum_{i \in \mathcal{I}_c} p_c^{u,(i)} \xi_c^{(i)}, \forall c \in \mathcal{C}, u \in \mathcal{U}$
3: *Option B:* Run modified IWF [6] to obtain $\mathbf{p}_c, c \in \mathcal{C}$,
$\tilde{p}_c^u = \min\{p_c^u, \hat{p}_c^u\}, \forall u \in \mathcal{U}, c \in \mathcal{C}$,
4: Compute bit-table $\tilde{b}_c^u = \left[r_c^u \left(\tilde{\mathbf{p}}_c, \tilde{X}_c^u \right) \right]^{\hat{B}}$,
where $\tilde{X}_c^u = \sum_{t \in \mathcal{U}, k \in \mathcal{C} \setminus c} H_{kc}^{tu}\, \tilde{p}_k^t, \forall c \in \mathcal{C}, u \in \mathcal{U}$
5: *Rounding Phase*
6: $\delta_c^u = \lceil \tilde{b}_c^u \rceil - \tilde{b}_c^u$
7: if $\delta_c^u < 0.5$ then $\tilde{b}_c^u = \lfloor \tilde{b}_c^u \rfloor$, $\delta_c^u = 0$, $\forall c \in \mathcal{C}, u \in \mathcal{U}$
8: **while** any $\delta_c^u > 0$ **do**
9: $\{c, u\} = \text{argmin}\{\delta_k^t\}$, $\tilde{b}_c^u = \lceil \tilde{b}_c^u \rceil$, $\delta_c^u = 0$
10: if $r_c^u\left(\mathbf{p}_c, X_c^u\right) \geq \tilde{b}_c^u$, $\forall u \in \mathcal{U}, c \in \mathcal{C}$, is infeasible under (2c) and (2e), then $\tilde{b}_c^u = \tilde{b}_c^u - 1$
11: *Bit-Adding Phase*
12: Run the bit-adding algorithm in [5] starting from the initial bit-allocation $\tilde{b}_c^u, c \in \mathcal{C}, u \in \mathcal{U}$

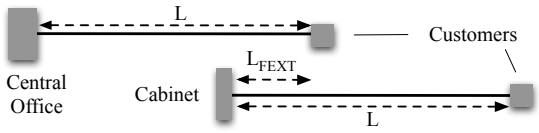


Fig. 1. Near-far DSL scenario.

V. Simulations

In this section we show simulation results of the upper and lower bounds proposed in Sections III and IV on the achievable two-user rate-region by multiuser spectrum balancing with ICI according to the discrete formulation in (2).

We assume asymmetric DSL 2 (ADSL2) systems conforming to [18, Appendix A.1.3], with a maximum aggregate sum-power of 19.9 dBm, a passband transmit power mask of -40 dBm/Hz, $\Gamma = 12.8$ dB, $\hat{B} = 15$, and a flat background noise at -140 dBm/Hz. The channel model is based on a 24 AWG twisted-pair cable model in [19, Sec. B.3.3], and the empirical 99 % worst-case crosstalk coupling model in [19, A.2.2.2], cf. [20] for a publicly available DSL simulator. The network topology is similar to those in [5] and shown in Figure 1, with two loops of length $L \in \{2, 3, 4\}$ km, interfering over a FEXT-length of $L_{\text{FEXT}} = 1$ km. As in various related studies [5], [6], [8], [10] we compute the ICI couplings according to the simplified worst-case model in [5], which is independent of the exact offset between the received direct and FEXT DMT symbols.[3] Another feature of this model is that the FEXT coupling coefficients $H_{cc}^{tu}, t \neq u, t \in \mathcal{U}, u \in \mathcal{U}$, on a single subcarrier $c \in \mathcal{C}$ correspond to those in the

[3]However, we note that the equivalent results obtained under the delay-dependent models in [5], [9] and a maximum asynchronism of half the DMT symbol length are qualitatively similar.

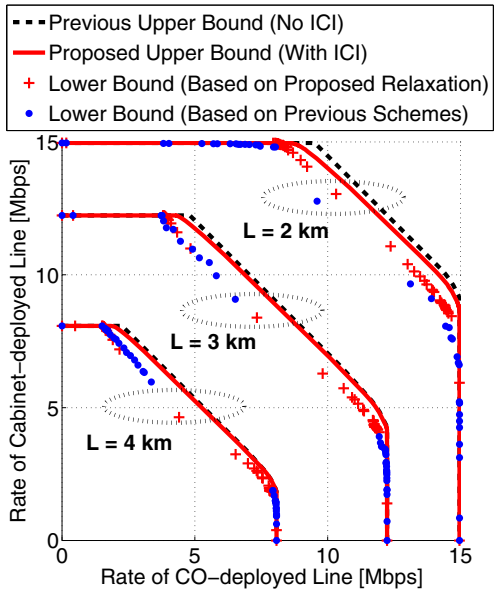


Fig. 2. Rate regions under different line-lengths L.

case of synchronous FEXT. Therefore, the upper bound [5] obtained by neglecting ICI is equivalent to the performance of a synchronized network.

In Figure 2 we show the rate-regions for different line-lengths L, obtained by using weights $w_1 \in \{0, 1/60, \ldots, 1\}, w_2 = 1 - w_1$. Especially in the scenario with $L = 2\,\text{km}$ we find that the proposed Lagrange relaxation based upper bound is significantly tighter than the bound obtained previously under Lagrange relaxation and neglected ICI couplings. Furthermore, the heuristic based on the proposed Lagrange relaxation partly improves the achieved rate-region compared to that based on the previously proposed heuristics (IWF and discrete bit-allocation).[4] Comparing this previously proposed heuristic to the previously proposed upper bound (neglecting ICI) under sum-rate maximization, its resulting upper bound on the suboptimality is between $6.7\,\%$ and $7.5\,\%$ in the three network scenarios. However, comparing the heuristic and upper bound based on the proposed Lagrange relaxation to each other, we obtain a resulting upper bound on the sum-rate suboptimality of only between $3.0\,\%$ and $4.1\,\%$.

VI. Conclusions

We have demonstrated the applicability of Lagrange relaxation to multiuser spectrum balancing in multicarrier systems with inter-carrier interference (ICI). A proof-of-concept was given by solving the relaxation in two-user digital subscriber line scenarios with asynchronous far-end crosstalk. Simulations show that the relaxation gives a significantly tighter performance bound compared to that obtained by completely neglecting ICI. Furthermore, it is shown to lend itself as the basis for discrete bit-allocation heuristics, leading to tighter performance lower bounds in certain scenarios. Hence, the proposed optimal Lagrange relaxation provides a useful performance reference for less complex spectrum balancing heuristics in scenarios with few users.

[4]The heuristics' results in Figure 2 exclude dominated sum-rate points.

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