53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

Local Properties of Economic NMPC, Dissipativity and Dynamic Programming

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***Abstract*— Classical model predictive control (MPC) implementations rely on a quadratic, positive definite cost function. In economic MPC on the contrary, the stage cost can be any generic function. Recent research has focused on establishing the conditions for stability of economic MPC where strict dissipativity has been shown to play an important role. In this paper, starting from the linear quadratic case and successively extending the analysis to the nonlinear case, we attempt at clarifying the relationship between strict dissipativity and the properties of the MPC optimal control problem.**

I. Introduction

Economic Model Predictive Control (MPC) has recently gained popularity, as it directly optimizes a given performance index, as opposed to tracking MPC, which minimizes the deviation from a given reference. The main advantage of economic MPC over tracking MPC becomes apparent in transients, when the system is steered to steady state while minimizing the given performance index. Stability theory for economic MPC is, however, not as mature as for the tracking case. Important contributions have been given in [1], [2], [3], [6], [8], [9], where the concepts of rotated cost and strict dissipativity play a fundamental role for the stability proofs.

In [9], the linear quadratic case has been analyzed without the use of strict dissipativity. In this paper, the analysis proposed in [9] is related to the strict dissipativity condition [2]. The properties of linear quadratic optimal control are further studied for the case of a linear quadratic stage cost. This establishes the connection between [9] and [3], where a linear storage function was proposed.

Under a stabilizability assumption, the linear quadratic analysis is informative of the local properties of nonlinear economic MPC. However, in order to also capture its global properties, further analyses are needed.

In this paper, the meaning of the storage function and the strict dissipativity condition for nonlinear economic MPC is studied in the framework of dynamic programming. A time-varying storage function satisfying an alternative strict dissipativity condition is proposed and the link is made with the analysis given in [1].

The paper is structured as follows. In Section II, the linear quadratic case is analyzed. The analysis is extended to the general nonlinear case in Section III. An illustrative example is presented in Section IV and conclusions are drawn in Section V.

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II. Linear Quadratic Economic MPC

Let us consider linear systems of the form

$$x_{k+1} = Ax_k + Bu_k, \tag{1}$$

where x and u denote, respectively the state and control input. We define the following quadratic stage cost

$$l(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^T H \begin{bmatrix} x \\ u \end{bmatrix}, \qquad H = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}, \tag{2}$$

with $H \not\succ 0$. Let us define $z_k^T = [x_k^T \ u_k^T]$, then the unconstrained MPC problem of horizon N is given by

$$\min_{z_0, z_1, \dots, z_{N-1}, x_N} \sum_{k=0}^{N-1} z_k^T H z_k + x_N^T P_N x_N \tag{3a}$$

$$\text{s.t. } x_0 = x, \ x_{k+1} = Ax_k + Bu_k, \tag{3b}$$

where matrix P_N defines the terminal cost. The LQR problem can be defined as an infinite horizon MPC problem, i.e. using $N \to \infty$ in (3).

In the case matrix H is indefinite, the LQR (MPC) problem is not of a tracking type and the problem is labeled economic LQR (MPC). The conditions for stability for the economic LQR have been given in [5], [7]. In [9], the link has been established between these stability conditions and the existence of a tracking LQR (i.e. with $H \succ 0$) that delivers the same solution as the economic one. The proof has also been extended to the case of linear quadratic economic MPC, i.e. with a finite horizon and a terminal cost matrix P_N different from the LQR cost-to-go matrix P.

In the following of the section, we will analyze the conditions for stability of economic LQR/MPC first in the case of a purely quadratic stage cost and then in case a linear term is present in the cost function. We will give a new perspective to the results presented in [9], which are based on a *rotation* procedure for a purely quadratic stage cost. Next, our contribution will consist in studying the properties and the structure of the rotation procedure. We will then establish the connection with the analysis given in [3] in the case of a linear system and a linear quadratic stage cost. We will conclude the section by showing how the linear quadratic analysis can be of interest also in the case of nonlinear MPC.

A. Preliminaries: Purely Quadratic Stage Cost

Let us consider first an economic LQR, i.e. using $N \to \infty$ in (3). The cost-to-go is given by

$$V(x) = \min_u \ l(x,u) + V(Ax+Bu) \tag{4}$$

with stage cost $l(x,u)$ given by (2) and cost-to-go $V(x) = x^T P x$, both quadratic and possibly indefinite. Let us define the following linear matrix inequality (LMI) condition

$$\exists\ \tilde{P}\ s.t.\ \begin{bmatrix} Q+\tilde{P}-A^T\tilde{P}A & S^T-A^T\tilde{P}B \\ S-B^T\tilde{P}A & R-B^T\tilde{P}B \end{bmatrix} \succ 0. \quad (5)$$

Proposition 1: If the pair (A,B) is stabilizable and the LMI condition (5) holds, then the LQR exists and is stabilizing for the system $x_{k+1} = Ax_k + Bu_k$.

Proof: The proof is provided in [5], [7], [9]. □

Proposition 2: If the system is stabilizable and condition (5) holds, then there exists an LQR with strictly positive definite stage cost which delivers the same solution as the original problem. The cost-to-go of the modified problem is given by

$$\bar{V}(x) = \min_u L(x,u) + \bar{V}(Ax+Bu), \quad (6)$$

with

$$L(x,u) = l(x,u) + \lambda(x) - \lambda(Ax+Bu), \quad (7a)$$
$$\bar{V}(x) = V(x) + \lambda(x), \quad (7b)$$

and $\lambda(x) = x^T\hat{P}x$. Note that the choice of $\hat{P}$ is not unique [9]. In the following, we will label Problem (6) the *rotated* problem. Equivalently, we will refer to functions $L(x,u)$ and $\bar{V}(x)$, both positive definite, as *rotated stage cost* and *rotated cost-to-go* respectively.

Proof: The proof and the details on how to compute matrix $\hat{P}$ are given in [9]. □

Remark 1: The proposed rotation procedure can be extended to the MPC framework, where the horizon is finite and the terminal cost matrix P_N is in general different from the LQR cost-to-go matrix P. Also for MPC, the rotation can be done using Equation (7). For more details on the topic we refer to [9].

B. Properties of the Rotated Problem

Let us now analyze the properties of the rotated problem and thus fill a gap in the analysis proposed in [9].

Proposition 2 implies that $\bar{V}(x) > 0$, moreover, by applying the LQR feedback $u^* = -Kx$ to the system, it follows that $V(f(x,-Kx)) - V(x) = -L(x,-Kx) < 0$. The rotated cost-to-go $\bar{V}(x)$ is therefore decreasing along the trajectory, which proves that it is a Lyapunov function for the economic LQR. This result also applies to economic MPC, provided that the horizon length or the terminal cost satisfy some technical assumptions [1], [4]. From the point of view of Lyapunov stability theory, the rotation procedure allows us to prove the existence of a Lyapunov function.

Let us now look at the same transformation from the point of view of quadratic programming (QP). The QP to be solved for linear MPC is

$$\min_w\ w^T M w \qquad \text{s.t.}\ \ Gw + g = 0, \quad (8)$$

where all states and controls are lumped in vector $w = [x_0, u_0, x_1, \ldots, x_N]$.

The Hessian of the original problem has a block diagonal structure $M = \mathrm{diag}(H,\ldots,H,P_N)$. The proposed rotation procedure adds terms of the form $\lambda(x_k) - \lambda(Ax_k + Bu_k)$ to the cost. As this modification only depends on variables at stage k, the block diagonal structure of the original problem is preserved in the rotated problem, i.e. the rotated Hessian becomes $\bar{M} = \mathrm{diag}(\bar{H},\ldots,\bar{H},\bar{P}_N)$ with $\bar{P}_N = P_N + \hat{P} \succ 0$,

$$\bar{H} = \begin{bmatrix} Q+\hat{P}-A^T\hat{P}A & S^T-A^T\hat{P}B \\ S-B^T\hat{P}A & R-B^T\hat{P}B \end{bmatrix} \succ 0,$$

and $\hat{P}$ computed as proposed in [9].

Moreover, the sum of the terms introduced by the rotation procedure, is given by

$$\lambda(x_0) + \sum_{k=0}^{N-1} \underbrace{\lambda(x_{k+1}) - \lambda(Ax_k + Bu_k)}_{=0},$$

where all terms in the telescopic sum are forced to 0 by the dynamic constraints $x_{k+1} = Ax_k + Bu_k$. Due to the initial constraint $x_0 = x$, the term $\lambda(x_0)$ is constant and given by $\lambda(x_0) = \lambda(x) = x^T\hat{P}x$. Note that the last term of the sum, i.e. $\lambda(x_N)$, is due to the rotated terminal cost. For all feasible trajectories, the rotation only adds a constant term to the cost and the reduced Hessian is not changed while the full Hessian becomes positive definite and remains block diagonal. Hence, from the optimization problem point of view, the rotated problem does not differ from the original one.

It should be noted that, while there are also other approaches that make the full Hessian positive definite while leaving the reduced Hessian unchanged, the rotation procedure proposed in [9] preserves the block diagonal structure of the full Hessian.

C. Quadratic Affine Term in the Stage Cost

While the case of a quadratic affine stage cost can be reconducted to a purely quadratic form, we will show in the following how this case can be related to the analysis proposed in [3].

Let us consider a problem with stage cost

$$l(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^T H \begin{bmatrix} x \\ u \end{bmatrix} + q^T \begin{bmatrix} x \\ u \end{bmatrix},$$

with an indefinite matrix H. The steady state is defined as

$$(x_s, u_s) = \underset{x,u}{\mathrm{argmin}}\ l(x,u) \qquad \text{s.t.}\ x = Ax + Bu. \quad (9)$$

The concept of rotated cost was used in [3] with $H \succ 0$, where it is proposed to rotate the cost using the Lagrange multiplier μ_s^T associated to the system dynamic constraint in Problem (9) in order to prove stability. The rotated cost is then given by $\bar{l}(x,u) = l(x,u) + \mu_s^T x - \mu_s^T(Ax - Bu)$.

Stability can be proven under the assumption that $\bar{l}(x,u) \geq \alpha(x - x_s)$, with α a function of class $\mathcal{K}_\infty$. This assumption is satisfied only if matrix H is positive definite. In the case of an indefinite matrix H, the rotated cost $\bar{l}(x,u)$ is indefinite,

and it is necessary to also modify the curvature when rotating the cost to prove stability. The cost can therefore be rotated using

$$L(x,u) = l(x,u) - l(x_\mathrm{s},u_\mathrm{s}) + \lambda(x) - \lambda(Ax+Bu), \qquad (10)$$

with $\lambda(x) = x^T\bar{P}x + \bar{p}^T x$.

By construction, the rotation of the stage cost does not change the solution and the rotated steady state problem delivers the same solution as the original one, i.e.

$$\begin{aligned}(x_\mathrm{s},u_\mathrm{s}) = \underset{x,u}{\operatorname{argmin}}\ & l(x,u) + x^T\bar{P}x - (Ax+Bu)^T\bar{P}(Ax+Bu) \\ \text{s.t. }\ & x = Ax+Bu. \end{aligned} \qquad (11)$$

Analogously to the procedure proposed in [3], the linear term in function $\lambda(x)$, is chosen as the Lagrange multiplier $\bar{\mu}_\mathrm{s}$ of the system dynamic constraint of the rotated Problem (11), i.e. $\bar{p} = \bar{\mu}_\mathrm{s}$. It can be verified that this choice of vector $\bar{p}$ implies that $L(x_\mathrm{s},u_\mathrm{s}) = 0$ and $\nabla L(x_\mathrm{s},u_\mathrm{s}) = 0$. Hence the problem becomes of tracking type and the stability theory for tracking problems applies.

D. Local Properties of Nonlinear Economic MPC

In this section, we will briefly discuss the use of the linear quadratic analysis to study the local properties of nonlinear economic MPC.

Let us consider the nonlinear system $x_{k+1} = f(x_k,u_K)$ and an economic stage cost $l(x_k,u_k)$. The optimal steady state is defined as

$$(x_\mathrm{s},u_\mathrm{s}) = \underset{x,u}{\operatorname{argmin}}\ l(x,u) \qquad \text{s.t. } x = f(x,u).$$

The linearized system at steady state is given by $x_{k+1} = Ax_k + Bu_k$ with

$$\begin{aligned} &x_{k+1} = Ax_k + Bu_k, \\ &A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x_\mathrm{s},u_\mathrm{s})}, \qquad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x_\mathrm{s},u_\mathrm{s})}. \end{aligned} \qquad (12)$$

Provided that the pair (A,B) is stabilizable, the linear quadratic approximation of the nonlinear problem is informative of the local properties of nonlinear economic MPC. Two practical implications are detailed in the following.

Strict dissipativity will be introduced in Section III for nonlinear economic MPC. In the case of a linear quadratic problem, strict dissipativity is equivalent to the LMI condition (5), as will be shown in Remark 2. If there is no solution to the LMI, then the stabilizable linear system (12) is not strictly dissipative with respect to the stage cost, which entails that also the nonlinear system is not strictly dissipative.

In [9] it has been proposed to use the linear quadratic rotation procedure in order to compute the stage cost for a tracking nonlinear MPC scheme which approximately solves economic MPC while providing asymptotic stability guarantees. For more details on the topic, we refer to [9].

It should be noted that, if the pair (A,B) is not stabilizable, then the linear quadratic framework cannot be applied. Note that in this case the LQR does not exist. This can happen e.g. due to a loss of local controllability of the nonlinear system. Only a nonlinear analysis can then be used to study the local and global properties of economic MPC.

III. Dynamic Programming and Strict Dissipativity

In this section, we will focus on nonlinear economic MPC. We will briefly recall the main stability result available in the literature, which relies on strict dissipativity and a rotated stage cost. We will analyze the properties of the rotated cost. We will then show the equivalence between dissipativity and infinite horizon optimal control. We will conclude the section by proposing a different proof of stability for economic MPC, based on a time varying storage function with the aim of clarifying how strict dissipativity relates to dynamic programming and MPC stability.

Let us consider the system dynamics $x_{k+1} = f(x_k,u_k)$, with state $x \in \mathbb{X} \subset \mathbb{R}^n$ and control $u \in \mathbb{U} \subset \mathbb{R}^m$. The feasible set $\mathbb{Z}_N$ on the horizon N is defined as

$$\begin{aligned}\mathbb{Z}_N(x) = \{(x_0,u_0,x_1,u_1,\dots,x_N) \mid\ & x_0 = x,\ x_{k+1} = f(x_k,u_k), \\ & (x_k,u_k) \in \mathbb{Z},\ k = 0,\dots,N-1,\ x_N \in \mathbb{X}_\mathrm{f}\},\end{aligned}$$

where the compact sets $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$ and $\mathbb{X}_\mathrm{f}$ define respectively the mixed state-input constraints and the terminal constraint.

The set of feasible controls is given by

$$\begin{aligned}\mathbb{U}_N(x) = \{(u_0,u_1,\dots,u_{N-1}) \mid\ & x_0 = x,\ x_{k+1} = f(x_k,u_k), \\ & (x_k,u_k) \in \mathbb{Z},\ k = 0,\dots,N-1,\ x_N \in \mathbb{X}_\mathrm{f}\}.\end{aligned}$$

Let us consider the following MPC problem over a finite horizon N, with terminal cost $V_N(x) = V_\mathrm{f}(x)$. Without loss of generality, let us assume for simplicity that $l(x_\mathrm{s},u_\mathrm{s}) = 0$.

$$\min_{x_0,u_0,x_1,u_1,\dots,x_N} \ \sum_{k=0}^{N-1} l(x_k,u_k) + V_N(x_N) \qquad (13a)$$

$$\text{s.t.} \quad (x_0,u_0,x_1,u_1,\dots,x_N) \in \mathbb{Z}_N(x) \qquad (13b)$$

Definition 1: A system is strictly dissipative [2] with respect to the supply rate $s : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ if there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$ and a positive definite function $\rho : \mathbb{X} \to \mathbb{R}$ such that for all $(x,u) \in \mathbb{Z}$

$$\lambda(f(x,u)) - \lambda(x) \le -\rho(x-x_\mathrm{s}) + s(x,u). \qquad (14)$$

Assumption 1: The set $\mathbb{Z}$ is compact.

Assumption 2: The stage cost $l(\cdot,\cdot)$ and system dynamics $f(\cdot,\cdot)$ are continuous on $\mathbb{Z}$. The terminal cost function $V_\mathrm{f}(\cdot)$ is continuous on a terminal region $\mathbb{X}_\mathrm{f}$.

Assumption 3: The system $x_{k+1} = f(x_k,u_k)$ is strictly dissipative with respect to the supply rate $s(x,u) = l(x,u) - l(x_\mathrm{s},u_\mathrm{s})$. Moreover, the storage function $\lambda(\cdot)$ is continuous on $\mathbb{Z}$.

Assumption 4: There exists a compact terminal region $\mathbb{X}_\mathrm{f} \subseteq \mathbb{X}$ containing x_s in its interior, and a control law $\kappa : \mathbb{X}_\mathrm{f} \to \mathbb{U}$ such that

$$V_\mathrm{f}(f(x,\kappa(x))) - V_\mathrm{f}(x) \le -l(x,\kappa(x)), \qquad \forall\ x \in \mathbb{X}_\mathrm{f}.$$

Proposition 3: If Assumptions 1, 2, 3, 4 hold, then the steady state solution x_s is an asymptotically stable equilibrium for the nonlinear system.

Proof: The proof is given in [1] and relies on the definition of rotated stage cost and terminal cost, defined respectively as

$$L(x,u) = l(x,u) + \lambda(x) - \lambda(Ax+Bu), \tag{15a}$$
$$\bar{V}(x) = V(x) + \lambda(x). \tag{15b}$$

□

Remark 2: Note that the rotated stage and terminal cost as from [1] has been deliberately written in the same form as the rotated cost (7) proposed for the linear quadratic case. Indeed, the linear quadratic framework has been developed independently of the concept of strict dissipativity, but it is closely related to it. In Equations (7) and (10), the proposed rotation relies on a quadratic function $\lambda(\cdot)$, which is a storage function that satisfies strict dissipativity.

Note also that condition (5) is equivalent to strict dissipativity for linear systems [9]. Indeed, in the linear quadratic case, strict dissipativity reads as

$$(Ax+Bu)^T \bar{P}(Ax+Bu) - x^T \bar{P} x \leq -x^T W x + \begin{bmatrix} x \\ u \end{bmatrix}^T H \begin{bmatrix} x \\ u \end{bmatrix},$$

with $W \succ 0$. By reorganizing the terms one gets

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q+\bar{P}-A^T\bar{P}A & S^T-A^T\bar{P}B \\ S-B^T\bar{P}A & R-B^T\bar{P}B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \geq x^T W x,$$

which is equivalent to (5).

Remark 3: In Section II-B, we analyzed the properties of the QP of the rotated MPC. The optimization problem associated to nonlinear economic MPC is a nonlinear programming problem (NLP). Looking at the rotated nonlinear MPC problem from the NLP point of view, one can draw similar conclusions to the linear quadratic case. Analogously to the linear quadratic case, the cost of the rotated problem differs from the original one only by the following sum

$$\sum_{k=0}^{N-1} \lambda(x_k) - \lambda(f(x_k,u_k)) + \lambda(x_N) = \lambda(x_0) + \sum_{k=0}^{N-1} \underbrace{\lambda(x_{k+1}) - \lambda(f(x_k,u_k))}_{=0},$$

where all terms in the telescopic sum are forced to 0 by the dynamic constraints $x_{k+1} = f(x_k,u_k)$ and the term $\lambda(x_0) = \lambda(x)$ is constant due to the initial constraint $x_0 = x$. The rotated nonlinear programming problem (NLP) is hence equivalent to the original one for all feasible trajectories and the rotation is only needed to prove the existence of a Lyapunov function for the nonlinear system.

Analogously to the linear quadratic case, the rotation of stage k only involves variables at stage k and the block diagonal structure of the Hessian is preserved.

In the following, we will show how dissipativity is related to optimal control and why strict dissipativity is needed to prove stability.

A. On the Dissipativity of Optimal Control Problems

In the following, we will show the equivalence of the notion of dissipativity with optimality of infinite horizon problems. Let us consider the MPC problem (13) in an infinite horizon setting, i.e. for $N \to \infty$. Its value function is given by

$$V(x) = \min_u \; l(x,u) + V(f(x,u)), \tag{16}$$

where, without loss of generality, it is assumed that $l(x_s,u_s) = 0$.

Let us define $\bar{l}(x,u) = l(x,u) - V(x) + V(f(x,u))$. As the initial state x is not an optimization variable in Problem (16), it holds that

$$0 = \bar{V}(x) = \min_u \; \bar{l}(x,u), \tag{17}$$

has the same minimizer u^* as (16). The definition of stage cost $\bar{l}(x,u)$ corresponds to rotating the original stage cost using function $\lambda(x) = -V(x)$, the system is always dissipative with respect to the supply rate $s(x,u) = l(x,u)$, i.e.

$$\lambda(f(x,u)) - \lambda(x) \leq l(x,u) \Leftrightarrow V(x) \leq V(f(x,u)) + l(x,u).$$

This implies that dissipativity holds for any infinite horizon problem with a bounded solution and it coincides with the definition of optimality. Dissipativity is therefore not enough to guarantee asymptotic stability of the nonlinear system, hence the need of strict dissipativity in the stability proof.

B. A Time-Varying Storage Function for MPC

In the following, we will attempt at clarifying the role played by strict dissipativity in the dynamic programming framework. As opposed to the standard approach proposed in [1], we will propose a time-varying storage function whose definition is based on optimal trajectories rather than feasible but possibly suboptimal ones. At the end of the section, we will establish the connection between the two approaches.

Let us consider the MPC problem (13) and let us define the cost-to-go at stage k as

$$V_j(x) = \min_{x_j,u_j,x_{j+1},u_{j+1},\ldots,x_j} \; \sum_{k=j}^{N-1} l(x_k,u_k) + V_N(x_N) \quad \text{s.t.} \quad x_j = x, \qquad x_{k+1} = f(x_k,u_k).$$

The optimal control policy at stage k is defined as $u_k^* = u_k^*(x) = \text{argmin}_u \; l(x,u) + V_{k+1}(x)$.

Assumption 5: There exists a compact terminal region $\mathbb{X}_f \in \mathbb{X}$, such that $x_s \in \mathbb{X}_f$ and

$$V_f(f(x,u_{N-1}^*(x))) - V_f(x) \leq -l(x,u_{N-1}^*(x)), \quad \forall\, x \in \mathbb{X}_f.$$

Note that Assumption 5 follows from Assumption 4 as, by definition, $V_f(f(x,\kappa(x))) + l(x,\kappa(x)) \geq V_f(f(x,u_{N-1}^*(x))) + l(x,u_{N-1}^*(x))$.

Definition 2: The system $x_{k+1} = f(x_k, u_k)$ is optimally strictly dissipative with respect to the supply rate $s(x,u) = l(x,u)$, if there exists a storage function $\lambda_N : \mathbb{X}_\mathrm{f} \to \mathbb{R}$ and a positive definite function $\rho : \mathbb{X}_\mathrm{f} \to \mathbb{R}$ such that for all $x \in \mathbb{X}_\mathrm{f}$

$$\begin{aligned} &\exists\ \lambda_N(x) \text{ s.t.} \\ &\quad \lambda_N(x) \geq \lambda_N(f(x, u^*_{N-1}(x))) - l(x, u^*_{N-1}(x)) + \rho(x - x_\mathrm{s}). \end{aligned} \tag{18}$$

Assumption 6: There exists a continuous terminal storage function $\lambda_N(\cdot)$ that satisfies the optimal strict dissipativity condition (18). Moreover, without loss of generality, $\lambda_N(x_\mathrm{s}) = 0$.

Assumption 7: The system is suboptimally operated off steady state, i.e. $u^*_k(x_\mathrm{s}) = u_\mathrm{s}$.

Theorem 1: If Assumptions 2, 5, 6, 7 hold, then the problem can be rotated with a time-varying storage function, the rotated cost-to-go $\bar{V}_0(x)$ is a Lyapunov function and the steady state x_s is an asymptotically stable equilibrium for the nonlinear system.

Proof: We construct a time varying storage function recursively as

$$\lambda_k(x) = \lambda_{k+1}(f(x, u^*_k)) - l(x, u^*_k) + \sigma_k(x - x_\mathrm{s}), \qquad k = 0, \ldots, N-1, \tag{19}$$

with $\sigma_k : \mathbb{X} \to \mathbb{R}$ a series of positive definite functions. By continuity of the stage cost, system dynamics and terminal storage function λ_N, all storage functions λ_k are continuous. The rotated cost becomes $L_k(x,u) = l(x,u) + \lambda_k(x) - \lambda_{k+1}(f(x,u))$ and, by construction $L_k(x, u^*_k) = \sigma_k(x - x_\mathrm{s})$, i.e. at the optimum all stage costs are bounded from below by a positive definite function.

The cost-to-go at each stage k is defined as $V_k(x) = \min_u\ l(x,u) + V_{k+1}(f(x,u))$. Defining the rotated cost-to-go as $\bar{V}_k(x) = V_k(x) + \lambda_k(x)$, one obtains

$$\bar{V}_k(x) = \min_u\ L_k(x,u) + \bar{V}_{k+1}(x).$$

The rotated cost-to-go is then given by

$$\begin{aligned} \bar{V}_k(x) &= L_k(x, u^*_k) + \bar{V}_{k+1}(f(x, u^*_k)) \\ &= \sigma_k(x - x_\mathrm{s}) + \bar{V}_{k+1}(f(x, u^*_k)), \end{aligned}$$

and is continuous by construction.

The rotated terminal cost $\bar{V}_f(\cdot)$ and cost-to-go $\bar{V}_k(\cdot)$ are bounded from below by a positive definite function on $\mathbb{X}_\mathrm{f}$ and $\mathbb{X}$ respectively. The proof follows from Assumptions 2, 5, 6 and can be found in [1].

The existence of the upper bound $\bar{V}_k(x) \leq \alpha(x - x_\mathrm{s})$, with $\alpha(\cdot)$ a class $\mathcal{K}$ function, can also be proven applying the arguments given in [1]. Hence it holds that

$$\hat{\alpha}(x - x_\mathrm{s}) \leq \bar{V}_k(x) \leq \alpha(x - x_\mathrm{s}), \tag{20}$$

with $\hat{\alpha}(\cdot)$ a class $\mathcal{K}$ function.

In order to prove the descent of the rotated cost-to-go along the closed loop trajectory, let us define the optimal trajectory of the MPC problem (13) as $X^*(x) = [x^*_0(x), x^*_1(x), \ldots, x^*_N(x)] = [x^*_0, x^*_1, \ldots, x^*_N]$. Let us moreover define a candidate feasible trajectory for the next time step as

$$\begin{aligned} \tilde{U}(x^*_1) &= [u^*_1(x^*_1), u^*_2(x^*_2), \ldots, u^*_{N-1}(x^*_{N-1}), \bar{u}_N], \\ \tilde{X}(x^*_1) &= [x^*_1, x^*_2, \ldots, x^*_N, \bar{x}_{N+1}], \end{aligned}$$

with $\bar{x}_{N+1} = f(x^*_N, \bar{u}_N)$ and $\bar{u}_N = u^*_{N-1}(x^*_N)$. The rotated cost $\tilde{V}_0(x^*_1)$ associated with this trajectory is given by

$$\tilde{V}_0(x^*_1) = \lambda_0(x^*_1) + \sum_{k=1}^{N-1} l(x^*_k, u^*_k(x^*_k)) + l(x^*_N, \bar{u}_N) + V_\mathrm{f}(\bar{x}_{N+1}).$$

This implies that

$$\begin{aligned} \tilde{V}_0(x^*_1) - \bar{V}_0(x) = {} & \lambda_0(x^*_1) - \lambda_0(x) - l(x, u^*_k(x)) \\ & \underbrace{+ l(x^*_N, u^*_{N-1}(x^*_N)) + V_\mathrm{f}(\bar{x}_{N+1}) - V_\mathrm{f}(x^*_N)}_{\leq 0} \\ \leq {} & \lambda_0(x^*_1) - \lambda_1(x^*_1) - \rho(x - x_\mathrm{s}). \end{aligned}$$

where $\lambda_0(x) + l(x, u^*_k(x)) = \lambda_1(x^*_1) + \rho(x - x_\mathrm{s})$ has been used.

The recursion for the time varying storage function implies

$$\begin{aligned} \lambda_0(x^*_1) - \lambda_1(x^*_1) = {} & \lambda_1(x^*_2) - \lambda_2(x^*_2) \\ = {} & \lambda_{N-1}(x^*_N) - \lambda_N(x^*_N) \\ = {} & \lambda_N(\bar{x}_{N+1}) - l(x^*_N, u^*_{N-1}(x^*_N)) \\ & + \rho(x^*_N - x_\mathrm{s}) - \lambda_N(x^*_N). \end{aligned}$$

Therefore, conditions (18) and (19) guarantee that

$$\bar{V}_0(x^*_1) - \bar{V}_0(x) \leq \tilde{V}_0(x^*_1) - \bar{V}_0(x) \leq -\rho(x - x_\mathrm{s}).$$

Function $\bar{V}_0(x)$ is thus a Lyapunov function and the system is asymptotically stable. □

Remark 4: If there exists a function $\lambda(\cdot)$ which satisfies the strict dissipativity condition (14), one can choose $\lambda_N(\cdot) = \lambda_k(\cdot) = \lambda(\cdot)$.

IV. Numerical Example

Let us introduce a simple example in order to illustrate the theoretical developments presented in this paper. We consider the following system

$$x_{k+1} = a\, x_k + 0.1\, u_k x_k^2,$$

with parameter a, indefinite stage cost

$$l(x,u) = x^2 + u^2 + 2.2\, x u,$$

and steady state $(x_\mathrm{s}, u_\mathrm{s}) = (0,0)$. Let us consider the following feasible sets

$$\mathbb{X}_\mathrm{f} = \mathbb{X} = \{x \mid -1 \leq x \leq 1\}, \qquad \mathbb{U} = \mathbb{R}.$$

The linearization of the system at steady state yields $x_{k+1} = a\, x_k$. If $a = 1$, the linear system is not stabilizable, thus the linear quadratic local approximation cannot be used to determine the local properties of the nonlinear problem. The cost-to-go $V(\cdot)$ and the optimal feedback law are displayed in Figure 1. It can also be seen that the cost-to-go at steady state is nonsmooth, therefore, it is not possible to compute a quadratic approximation of the cost-to-go at steady state.

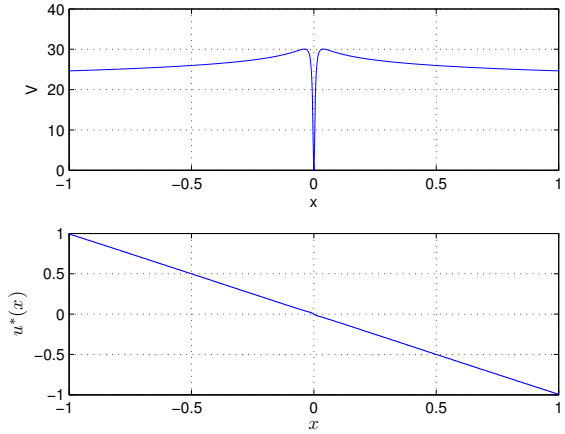


Fig. 1. Value function V and feedback law for the nonlinear infinite horizon optimal control problem with $a = 1$.

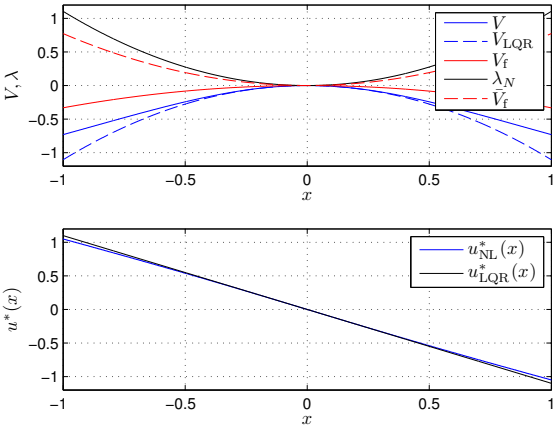


Fig. 2. Top graph: Cost-to-go of the nonlinear economic MPC (blue line) and cost-to-go of the LQR (black line), both with $a = 0.9$. Possible terminal cost $V_{\mathrm{f}}(\cdot)$ and storage function $\lambda_N(\cdot)$ are also displayed, together with the resulting rotated terminal cost $\bar{V}_{\mathrm{f}}(\cdot)$. Bottom graph: Feedback law for the nonlinear infinite horizon optimal control problem $u^*_{\mathrm{NL}}(x)$ and LQR $u^*_{\mathrm{LQR}}(x)$.

Taking $a = 0.9$, the linear system is stabilizable, thus the linear quadratic analysis does provide local information on the nonlinear problem. For instance, it can be noted that the LMI condition (5) is satisfied by $\tilde{P} = -2$, hence the linear system is strictly dissipative with respect to the supply rate $s(x,u) = x^2 + u^2 + 2.2xu$.

The cost-to-go of the nonlinear system $V(x)$ and the cost-to-go obtained by the second order expansion at steady state $V_{\mathrm{LQR}}(x)$ are displayed in Figure 2. For both cases, the cost-to-go is negative definite, but the economic MPC is stabilizing the system.

It can be noted that, for this example, the cost-to-go of the linear quadratic approximation is smaller than the one of the nonlinear problem, i.e. $V_{\mathrm{LQR}}(x) = x^T P x \leq V(x)$. For the terminal cost and storage function one can choose e.g. $V_{\mathrm{f}}(x) = 0.3\,V_{\mathrm{LQR}}(x)$ and $\lambda_N(x) = -V_{\mathrm{LQR}}(x)$. It can be verified that these choices satisfy the assumption on the terminal cost Assumption 5 and both optimal strict dissipativity Assumption 6 and strict dissipativity Assumption 3. The resulting rotated terminal cost $\bar{V}_{\mathrm{f}}(x)$, given by (15), is also displayed in Figure 2.

The control law obtained with the nonlinear optimal control problem $u^*_{\mathrm{NL}}(x)$ and the one obtained with the linear quadratic approximation $u^*_{\mathrm{LQR}}(x) = -Kx$ are displayed in Figure 2. In Figure 2 it can also be seen that the linear quadratic framework is a local approximation of the nonlinear problem.

V. Conclusions

In this paper, we have established the connection between several recent results on stability of economic MPC. Starting from the linear quadratic case, and generalizing to the nonlinear case, we have shown how strict dissipativity is related to dynamic programming. The MPC optimization problem is not changed by rotating the cost, but rotation allows to prove the existence of a Lyapunov function.

In order to clarify the role played by strict dissipativity in proving asymptotic stability, we have proposed an alternative stability proof, which relies on a time-varying storage function. Finally, we have illustrated the theoretical results by means of a simple example.

Acknowledgments

The authors would like to thank Professor Lars Grüne for the fruitful exchange of ideas and collaboration on economic MPC. This research was supported by Research Council KUL: PFV/10/002 Optimization in Engineering Center OPTEC, GOA/10/09 MaNet and GOA/10/11 Global real- time optimal control of autonomous robots and mechatronic systems. Flemish Government: FWO: PhD/postdoc grants; IWT: PhD Grants, projects: Eurostars SMART; Belgian Federal Science Policy Office: IUAP P7 (DYSCO, Dynamical systems, control and optimization, 2012-2017); EU: FP7-SADCO (MC ITN-264735), FP7-TEMPO(MC ITN-607957), ERC HIGHWIND (259 166).

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