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Indefinite linear MPC and approximated economic MPC for nonlinear systems☆

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ABSTRACT

The stability proof for economic model predictive control (MPC) relies on strict dissipativity, which is in general hard to check. In this contribution, we will first analyze the linear quadratic case with possibly indefinite cost. The conditions for stability will be recalled and it will be shown that every stabilizing LQR/MPC has an equivalent positive definite LQR/MPC which yields the same open loop and closed loop behavior. This analysis will then be used to formulate an approximated nonlinear economic MPC scheme, which has stability guarantees. An example will be used to illustrate the proposed technique and show its potential in terms of performance.

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1. Introduction

Economic model predictive control (MPC) has recently gained popularity, as it directly optimizes a given performance index, as opposed to tracking MPC, which minimizes the deviation from a given reference. The main advantage of economic MPC over tracking MPC becomes obvious in transients, when the system is steered to steady state so as to minimize the given performance index.

The stability theory of economic MPC initially considered linear systems and convex objectives [1,2]. For nonlinear systems, an analysis of average performance bounds was proposed in [3] and average constraints were considered in [4]. Lyapunov stability of economic MPC was first proven in [5] under a strong duality assumption and generalized in [6,7] under a strict dissipativity assumption. The necessity of strict dissipativity for optimal steady-state operation has been analyzed in [11,12]. A stability proof in the absence of terminal constraints is provided in [8]. The extension of the stability results to periodic systems has been considered in [9,10]. Economic MPC schemes based on Lyapunov techniques have been proposed in [13–15].

In the nonlinear case, the strict dissipativity condition can be extremely hard to check, thus making it difficult to ensure stability. This contribution focuses on the linear quadratic case with possibly indefinite stage costs. The analysis becomes simpler in this case and the conditions that ensure stability for the LQR controller have been studied in [16,17].

The main contributions of this paper are (a) the analysis of linear quadratic economic MPC and (b) the construction of a tracking nonlinear MPC (NMPC) scheme which closely mimics the economic NMPC behavior while providing stability guarantees. The main results for the linear quadratic case are given in Theorems 1 and 2, which state that any possibly indefinite stabilizing LQR or linear quadratic MPC can be reformulated as a positive definite one that delivers the same trajectory. This result is then used for economic NMPC, to construct an approximated economic NMPC scheme of tracking type which provides stability guarantees.

The paper is structured as follows. Section 2 proposes a motivational example which shows how tracking NMPC can yield nearly the same performance as economic NMPC. In Section 3, the conditions for having a stabilizing LQR are recalled. Section 4 shows that, given any indefinite stabilizing LQR, there exists a positive definite LQR which yields the same closed loop trajectory. Section 5 is the analogue of Section 4 for the case of a finite horizon (MPC). In Section 6 a linear example is proposed to illustrate the theory. A nonlinear example well known in the economic NMPC literature is proposed in Section 7 to illustrate how the proposed

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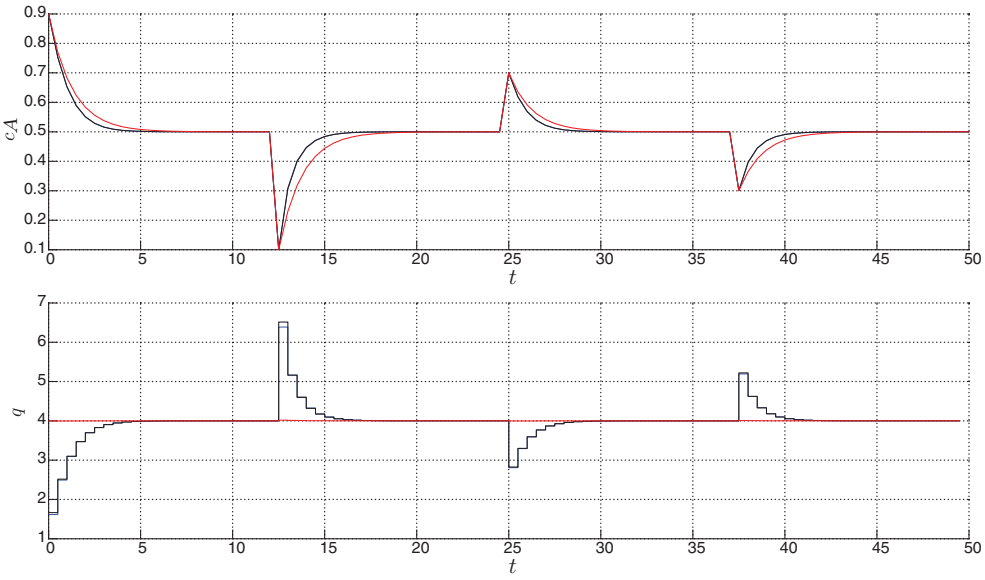

Fig. 1. State and input profiles for the economic NMPC scheme (black line), tracking scheme 0 (blue line) and tracking scheme 1 (red line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

framework can be used for approximated economic NMPC with stability guarantees in the nonlinear case.

2. Motivational example

As a motivation to this article, we would like to start with an example that shows how a tracking NMPC formulation can yield a good performance compared to an economic NMPC formulation. We propose the following CSTR example from [5,7]. It considers a single first-order, irreversible chemical reaction $A \rightarrow B$ with reaction rate $k_r c_A$, where $k_r = 0.4\,\mathrm{l/(mol\,min)}$ is the rate constant. Quantities c_A and c_B denote the molar concentrations of A and B respectively. The process dynamics are given by

$$\dot{c}_A = \frac{q}{V_R}(c_{Af} - c_A) - k_r c_A \tag{1}$$

$$\dot{c}_B = \frac{q}{V_R}(c_{Bf} - c_B) + k_r c_A \tag{2}$$

where $c_{Af} = 1\,\mathrm{mol/l}$, $c_{Bf} = 0\,\mathrm{mol/l}$ denote the feed concentrations of A and B respectively and $V_R = 10\,\mathrm{l}$ is the volume of the reactor. Here, the flow through the reactor q is a control variable. In order to ensure local controllability at steady state, let us assume that $c_A + c_B = 1$. The states are thus defined as $x = c_A$ and the controls as $u = q$. Moreover, the flow rate is upper bounded by $q \leq 20\,\mathrm{l/min}$.

The economic stage cost is given by

$$\ell(x, u) = -(2q(1 - c_A) - 0.5q) + 0.1(q - 4)^2, \tag{3}$$

which gives the steady state $x^s = c_A^s = 0.5\,\mathrm{mol/l}$ and $q^s = 4\,\mathrm{l/min}$.

Consider an alternative tracking formulation where the stage cost is positive definite and defined as

$$\ell_{\mathrm{tr}}^{\star}(x, u) = \begin{bmatrix} c_A - c_A^s \\ q - q^s \end{bmatrix}^T H^{\star} \begin{bmatrix} c_A - c_A^s \\ q - q^s \end{bmatrix}, \quad \star \in \{0, 1\} \tag{4}$$

$$H^0 = \begin{bmatrix} 14.227 & 0.825 \\ 0.825 & 0.066 \end{bmatrix}, \quad H^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5}$$

A simulation has been run in order to compare the economic performance of the tracking NMPC scheme with $\star$=0 and the economic NMPC scheme. For completeness, also the performance a more standard tracking scheme with $\star$=1 has been compared. All simulations are run in Python using Casadi [18] to discretize the optimal control problem (OCP) and Ipopt [19] to solve the nonlinear programming problem (NLP). The continuous time dynamics have been discretized with a sampling time $T_s = 0.5\,\mathrm{min}$ using a fixed step size Runge-Kutta integrator of order 4. The chosen prediction horizon is $N = 100$ and the endpoint constraint $x_N = x^s$ has been introduced.

The proposed scenario lasts 50 s and considers a system at steady state that is perturbed every 12.5 s with the following perturbation $d = [0.4, -0.4, 0.2, -0.2]$ which alters the state as follows: $c_A(k12.5) = c_A(k12.5) + d(k)$, $k = 1, 2, 3, 4$. The metric G has been proposed in [20] to measure the gain obtained by using economic MPC over tracking:

$$G^{\star} = 100\frac{\overline{P}_{\mathrm{eco}} - \overline{P}_{\mathrm{tr}}^{\star}}{\overline{P}_s}, \quad \star = 0, 1, \tag{6}$$

where $\overline{P}_{\mathrm{eco}}$, $\overline{P}_{\mathrm{tr}}^{\star}$, $\overline{P}_s$ denote the average profit obtained by economic NMPC, tracking NMPC $\star$ and steady state operation respectively.

The simulation results yield $G^0 = 0.0009$ and $G^1 = 1.139$, which suggest that scheme 0 has a negligible performance loss compared to the economic NMPC scheme, while scheme 1 has a small performance loss which is however sensibly higher than the one of scheme 0. The state and input trajectories are displayed in Fig. 1, where it can be seen that scheme 0 delivers an input trajectory which is very close to the one delivered by economic NMPC, while the state trajectory is even closer. Scheme 1, on the other hand, delivers significantly different trajectories.

Proving stability for economic NMPC schemes can be very involved. Moreover, tracking NMPC problems typically have a least squares structure which can be exploited in efficient algorithms. For economic NMPC, on the contrary, such structure is not present and the complexity of the problem tends to be significantly higher. This motivates our interest in tracking NMPC formulations that mimic economic NMPC behavior.

3. Existence of a stabilizing LQR

In this section we will focus on linear quadratic discrete time infinite horizon OCPs. The linear quadratic regulator (LQR) is defined by the following OCP

$$V(\bar{x}_0) = \inf_{x,u} \sum_{k=0}^{\infty} l(x_k, u_k) \tag{7a}$$

$$\text{s.t.} \quad x_0 = \bar{x}_0, \tag{7b}$$

$$x_{k+1} = Ax_k + Bu_k, \quad k \geq 0, \tag{7c}$$

$$\lim_{k\to\infty} x_k = 0, \tag{7d}$$

with state $x = [x_0, x_1, \ldots]$ and control $u = [u_0, u_1, \ldots]$. The stage cost is defined as $l(x_k, u_k) = \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T H \begin{bmatrix} x_k \\ u_k \end{bmatrix}$, with matrix $H = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}$ symmetric and possibly indefinite. This optimization problem has been widely studied in the literature and excellent analyses can be found in [17,21] for discrete time systems and in [16] for continuous time systems.

The contributions above define a set of necessary conditions for the boundedness of problem (3) and establish the link to the existence of solutions to the algebraic Riccati equation. In [16], moreover, the connection is made between the algebraic Riccati equation, a linear matrix inequality, a frequency condition and a dissipativity condition and the properties of a set of optimal control problems. Most of the results for continuous time can be directly translated to discrete time. In this section, we will briefly recall the conditions for the existence of a stabilizing LQR with a possibly indefinite stage cost.

It is well known that the solution to the LQR problem (3) can be obtained by computing the stabilizing solution of the discrete algebraic Riccati equation (DARE), defined as

$$\begin{aligned} \mathcal{D}(A, B, Q, R, S) := \{(P, K) | \quad & Q + A^T PA - P - (S^T + A^T PB)K = 0, \\ & K = (R + B^T PB)^{-1}(S + B^T PA), \\ & \rho(A - BK) < 1\}, \end{aligned} \tag{8}$$

where ρ denotes the spectral radius, i.e. $\rho(A) = \max\{|\lambda| | \lambda \text{ is eigenvalue of } A\}$. In general, the DARE has many solutions but only one, if it exists, is stabilizing the system (A, B), yields the solution to the LQR problem and the cost-to-go is then given by $V(\bar{x}_0) = x_0^T P x_0$. Note that, in the case of an indefinite cost, i.e. $H \not\succeq 0$, also the cost-to-go P can be indefinite.

One necessary but not sufficient condition for the existence of a solution to (8) is the invertibility of matrix $(R + B^T PB)$. In the following, we will recall the conditions for which the LQR problem (3) has a stabilizing solution, which also guarantee that $(R + B^T PB) \succeq 0$ is invertible and the DARE (8) does have a necessarily unique stabilizing solution.

If stabilizability is assumed and any of the following equivalent conditions is satisfied, then the LQR has a unique stabilizing solution.

- The linear matrix inequality (LMI): $\exists\, P$ s.t.

$$\begin{bmatrix} Q + A^T PA - P & S^T + A^T PB \\ S + B^T PA & R + B^T PB \end{bmatrix} \succeq 0; \tag{9}$$

- The quadratic matrix inequality (QMI): $\exists\, P$ s.t.

$$Q + A^T PA - P - (S^T + A^T PB)(R + B^T PB)^{-1}(S + B^T PA) \succeq 0, \tag{10a}$$

$$R + B^T PB \succeq 0; \tag{10b}$$

- The frequency domain inequality (FDI): $\forall\, z$ s.t. $|z| = 1$

$$\begin{aligned} \Psi(z) := \; & R + S(Iz - A)^{-1}B + B^T(Iz^{-1} - A^T)^{-1}S^T \\ & + B^T(Iz^{-1} - A^T)^{-1}Q(Iz - A)^{-1}B \succeq 0; \end{aligned} \tag{11}$$

- The dissipation inequality (DIE): $\exists \quad \mathcal{V} : \mathbb{R}^{n_x} \to \mathbb{R}$ s.t. $\forall \quad x_k, u_k$

$$l(x_k, u_k) + \mathcal{V}(Ax_k + Bu_k) \geq \mathcal{V}(x_k), \tag{12}$$

with n_x the state dimension.

The connection between the FDI and detectability of $(A, Q^{1/2})$ has been established in [17].

In order to show that these conditions are indeed necessary for the existence of a stabilizing LQR, let us adapt two theorems from [16] to the case of discrete-time systems. Those theorems analyze the boundedness of similar OCPs and relate it to the concept of dissipativity. This allows one to establish boundedness of the LQR OCP, thus proving the existence of a stabilizing solution to the DARE (8). Let us define the feasible set $\mathbb{Z}_n$ on the horizon n as

$$\begin{aligned} \mathbb{Z}_n(\bar{x}_0) = \{&(x_0, \ldots, x_n, u_0, \ldots, u_{n-1}) | x_0 = \bar{x}_0, \\ & x_{k+1} = Ax_k + Bu_k, \quad k = 0, \ldots, n\}. \end{aligned} \tag{13}$$

Let us introduce the following optimal control problems (OCP)

$$V_f^+(\bar{x}_0) = \inf_{(x,u)\in\mathbb{Z}_\infty(\bar{x}_0)} \sum_{k=0}^{\infty} l(x_k, u_k), \tag{14}$$

$$V^+(\bar{x}_0) = \inf_{(x,u)\in\mathbb{Z}_\infty(\bar{x}_0)} \sum_{k=0}^{\infty} l(x_k, u_k), \quad \text{s.t.} \quad \lim_{k\to\infty} x_k = 0, \tag{15}$$

$$V^-(\bar{x}_0) = -\inf_{(x,u)\in\mathbb{Z}_\infty(\bar{x}_0)} \sum_{k=-\infty}^{0} l(x_k, u_k), \quad \text{s.t.} \quad \lim_{k\to-\infty} x_k = 0, \tag{16}$$

$$V_N^+(\bar{x}_0) = \inf_{(x,u)\in\mathbb{Z}_N(\bar{x}_0), N\geq 0} \sum_{k=0}^{N} l(x_k, u_k). \tag{17}$$

Optimality implies that $V_N^+ \leq 0$ and $V_N^+ \leq V_f^+ \leq V^+$. If stabilizability is assumed, then $V_N^+, \quad V_f^+, \quad V^+ < \infty$ and if controllability is assumed $V^- > -\infty$.

Theorems 1 and 2 from [16] establish the existence of a stabilizing LQR, by proving the boundedness of V^+. It is interesting to note how boundedness of V^+ can be established by looking at the boundedness of problems that can be quite different from it. Both theorems can be directly translated to discrete time as follows.

Proposition 1. *Assume that the system $x_{k+1} = Ax_k + Bu_k$ is controllable, then the following conditions are equivalent:*

1 $\sum_{k=0}^{N} l(x_k, u_k) \geq 0$ *for every* $N \geq 0$ *and every* $(x, u) \in \mathbb{Z}_N(0)$;
2 $V^-(x) \leq 0$;
3 $V_f^+(x) \leq 0$;
4 $V_N^+(x) \leq 0$;
5 *There exists a function* $\mathcal{V}(x) \leq 0$ *which satisfies the DIE (12).*

Moreover, whenever any of these conditions is satisfied, then $-\infty \leq V^-(x) \leq V_N^+(x) \leq V_f^+(x) \leq V^+(x) \leq \infty$. *Finally, any function* $\mathcal{V}(x)$ *satisfying the DIE (12) satisfies* $V^-(x) \leq \mathcal{V}(x) \leq V^+(x)$ *and* $V^-(x) \leq \mathcal{V}(x) \leq V_N^+(x)$ *if* $\mathcal{V}(x) \leq 0$.

Proof. The proof follows the same arguments as in [16]. □

Proposition 2. *Assume that the system $x_{k+1} = Ax_k + Bu_k$ is controllable, then the following conditions are equivalent:*

1 $\sum_{k=0}^{N} l(x_k, u_k) \geq 0$ *for every* $N \geq 0$ *and every* $(x, u) \in \mathbb{Z}_N(0)$ *with* $x_N = 0$;
2 $V^+(x) \geq -\infty$;
3 $V^-(x) \leq \infty$;
4 *There exists a function* $\mathcal{V}(x)$ *which satisfies the DIE (12).*

Moreover, whenever any of these conditions is satisfied, then $-\infty \leq V^-(x) \leq \mathcal{V}(x) \leq V^+(x) \leq \infty$.

Proof. The proof follows the same arguments as in [16]. □

Remark 1. The first condition of Propositions 1 and 2 is trivially satisfied when $H \geq 0$. Boundedness of the infima is hence always guaranteed for positive semidefinite cost functions, provided controllability is assumed.

Remark 2. Note that Condition 1 in both Propositions 1 and 2 implies that the system is optimally operated at steady state, though our definition of optimal operation at steady state differs slightly from the one given in [22,11].

The importance of the DIE (12) for the existence of a stabilizing LQR is established in Proposition 2. For quadratic functions $\mathcal{V}(x) = x^T P x$, the DIE (12) is equivalent to the LMI (9). This is easily seen by replacing the dynamic equations $x_{k+1} = Ax_k + Bu_k$ in the DIE, which directly gives the LMI. The condition $V^+ > -\infty$ is then satisfied if and only if there exists a real symmetric solution $P = P^T$ to the LMI (see [16], Theorem 3). The QMI is the Schur complement of the LMI, thus it is equivalent to the LMI if $R + B^T P B \geq 0$ holds. Note also that the FDI implies the LMI and it allows to establish the boundedness of the infima a priori, given the system matrices and the stage cost only.

Provided that the pair (A, B) is stabilizable, the existence of a necessarily unique symmetric solution $P = P^T$ to $\mathcal{D}(A, B, Q, R, S)$ such that $|\lambda(A - BK)| < 0$ is guaranteed if, for all $|z| = 1$, the FDI is satisfied, i.e. $\Psi(z) > 0$, see [17,21].

Remark 3. Note that, in order to check the LMI condition, the matrix P that satisfies it does not need to be a solution to the DARE. This condition can hence be checked by solving the LMI for P: if a solution exists, then the condition can be satisfied, the system is dissipative and a stabilizing LQR exists. On the contrary, if the solution does not exist, the condition cannot be satisfied and the system is not dissipative, thus the system cannot be stabilized using the proposed cost function. Note that this check can be formulated as a convex problem that is efficiently solvable.

Remark 4. If the LMI condition is not satisfied, one might be interested in defining a new cost for which the condition holds. In order to modify the initial cost as little as possible, one can solve the following convex problem

$$\min_{P,T} \|T\|^2 \tag{18}$$

$$\text{s.t.} \quad \begin{bmatrix} Q + A^T P A - P & S^T + A^T P B \\ S + B^T P A & R + B^T P B \end{bmatrix} + T \succeq 0, \tag{19}$$

where $T = \begin{bmatrix} T_Q & T_S^T \\ T_S & T_R \end{bmatrix}$ and $\|\cdot\|$ can be any matrix norm. The new cost is then defined by $\tilde{Q} = Q + T_Q$, $\tilde{R} = R + T_R$, $\tilde{S} = S + T_S$.

In the following section, we will assume that the aforementioned conditions are satisfied, thus that a stabilizing LQR exists.

4. Equivalent LQR formulations

In Section 3, the conditions for the existence of a stabilizing LQR have been given. In general, the stage cost does not need to be positive definite, i.e. $H \not\succ 0$ and the solution P to the DARE can also be indefinite. In this section we will show that any stabilizing LQR can be reformulated as a positive definite one, i.e. $P \succ 0$ and $H \succ 0$, which yields the same trajectory.

Let us introduce the following notation. The DARE $\mathcal{D}(A, B, Q, R, S)$ delivers a solution (P, K), where $P = P^T$ is a real symmetric matrix and K is a feedback gain. Let us define the LQR trajectory as $\phi(\bar{x}_0) := \mathcal{P}(A, B, Q, R, S)$, i.e. the LQR problem $\mathcal{P}(A, B, Q, R, S)$ has associated DARE $(P, K) = \mathcal{D}(A, B, Q, R, S)$ and delivers the closed loop trajectory $\phi(x_0) = (x_k)_{k \geq 0}$ with $x_{k+1} = (A - BK)x_k$. In the following, we will consider two problems equivalent if they deliver the same trajectory, i.e. $\mathcal{P}(A_1, B_1, Q_1, R_1, S_1) = \mathcal{P}(A_2, B_2, Q_2, R_2, S_2)$.

Definition 1. Problem $\mathcal{P}(A_2, B_2, Q_2, R_2, S_2)$ is a positive definite reformulation of the possibly indefinite LQR problem $\mathcal{P}(A_1, B_1, Q_1, R_1, S_1)$ if $\begin{bmatrix} Q_2 & S_2^T \\ S_2 & R_2 \end{bmatrix} \succ 0$ and $\mathcal{P}(A_2, B_2, Q_2, R_2, S_2) = \mathcal{P}(A_1, B_1, Q_1, R_1, S_1)$.

Theorem 1. *Any stabilizing LQR problem can be reformulated as a positive definite LQR problem.*

Before proving the theorem, two lemmas are introduced that will help in establishing the desired result. The following lemma considers the case of a system (A, B) which is pre-stabilized using a linear feedback $\overline{K}$. It proves that, by suitably modifying the cost, an LQR can be defined for the pre-stabilized system $(A_{\overline{K}}, B)$, which is equivalent to the original LQR defined for (A, B).

Lemma 1. *Let us assume that a stabilizing LQR exists for system (A, B), with weighting matrices Q, R, S. The equivalence $\mathcal{P}(A, B, Q, R, S) = \mathcal{P}(A_{\overline{K}}, B, Q_{\overline{K}}, R, S_{\overline{K}})$ holds for any arbitrary feedback matrix $\overline{K}$, when choosing $A_{\overline{K}} = A - B\overline{K}$, $Q_{\overline{K}} = Q - S^T\overline{K} - \overline{K}^T S + \overline{K}^T R\overline{K}$, $S_{\overline{K}} = S - R\overline{K}$. Moreover, the real symmetric matrix $P_{\overline{K}}$, computed as $(P_{\overline{K}}, \overline{K}) = \mathcal{D}(A_{\overline{K}}, B, Q_{\overline{K}}, R, S_{\overline{K}})$, satisfies $P_{\overline{K}} = P$ with $(P, K) = \mathcal{D}(A, B, Q, R, S)$. The feedback gain $K_{\overline{K}}$ is given by $K_{\overline{K}} = K - \overline{K}$.*

Proof. The proof follows by substitution of $A_{\overline{K}}$, $Q_{\overline{K}}$ and $S_{\overline{K}}$ into (8) and simplification [17]. The details are provided in A. Note that the reformulation proposed in Lemma 1 is equivalent to choosing for the control input $u_k = -\overline{K}x_k + v_k$, i.e.

$$x_{k+1} = (A - B\overline{K})x_k + Bv_k = A_{\overline{K}}x_k + Bv_k. \tag{20}$$

□

Remark 5. Note that, from $u_k = -\overline{K}x_k + v_k$ and $v_k = -K_{\overline{K}}x_k$, one obtains

$$-Kx_k = u_k = -\overline{K}x_k + v_k = -\overline{K}x_k - K_{\overline{K}}x_k,$$

which also entails $K_{\overline{K}} = K - \overline{K}$.

Corollary 1. *By choosing $\overline{K} = K$, the DARE reduces to $Q_K + A_K^T P A_K - P = 0$. This entails that, for any arbitrarily chosen matrix R_K such that $R_K + B^T P B \succ 0$, the solution remains unchanged, i.e. $\mathcal{P}(A_K, B, Q_K, R, S_K) = \mathcal{P}(A_K, B, Q_K, R_K, S_K)$. One obtains thus a problem whose solution is $K_K = 0$ and $v_k = 0$.*

Proof: The DARE corresponding to $\mathcal{P}(A_K, B, Q_K, R_K, S_K)$ is given by

$$Q_K + A_K^T P A_K - P - (S_K^T + A_K P B)(R_K + B^T P B)^{-1}(S_K + B^T P A_K) = 0.$$

The last term is always zero, as can be seen by replacing the expression for K:

$$\begin{aligned} S_K + B^T P A_K &= S - RK + B^T PA - B^T PBK \\ &= S + B^T PA - (R + B^T PB)K \\ &= S + B^T PA - (R + B^T PB)(R + B^T PB)^{-1}(S + B^T PA) \\ &= 0. \end{aligned}$$

Regardless of the choice of matrix R_K, the DARE simplifies and P is defined by

$$Q_K + A_K^T P A_K - P = 0.$$

Remark 6. The condition $R_K + B^T PB \succ 0$ is required for the LQR to exist, see e.g. (3). This condition can alternatively be derived from dynamic programming arguments.

The following lemma states that the LQR cost can be modified such that the trajectory $\phi(\bar{x}_0)$ remains unchanged while the cost-to-go matrix P can be any arbitrary symmetric matrix.

Lemma 2. *Let us assume that a stabilizing LQR exists for system (A, B), with weighting matrices Q, R, S. Given any real symmetric matrix $\overline{P}$, the equivalence $\mathcal{P}(A, B, Q, R, S) = \mathcal{P}(A, B, Q_{\overline{P}}, R_{\overline{P}}, S_{\overline{P}})$ holds when choosing $Q_{\overline{P}} = Q + A^T \overline{P} A - \overline{P}$, $S_{\overline{P}} = S + A^T \overline{P} B$ and $R_{\overline{P}} = R + B^T \overline{P} B$. The solution $P_{\overline{P}}$ to $\mathcal{D}(A, B, Q_{\overline{P}}, R_{\overline{P}}, S_{\overline{P}})$ is given by $P_{\overline{P}} = P - \overline{P}$ and the resulting feedback gain is $K_{\overline{P}} = K$.*

Proof. By writing the DARE $\mathcal{D}(A, B, Q_{\overline{P}}, R_{\overline{P}}, S_{\overline{P}})$, and replacing the expressions for $Q_{\overline{P}}, R_{\overline{P}}, S_{\overline{P}}$, one immediately obtains $\mathcal{D}(A, B, Q, R, S)$, which is solved by matrix P and feedback gain K.

By definition $P = \overline{P} + P_{\overline{P}}$ and one can write

$$\begin{aligned} 0 &= Q + A^T PA - P - (S^T + A^T PB)(R + B^T PB)^{-1}(S + B^T PA) \\ &= Q + A^T \overline{P} A - \overline{P} + A^T P_{\overline{P}} A - P_{\overline{P}} - (S^T + A^T \overline{P} B + A^T P_{\overline{P}} B) \\ &\quad \times (R + B^T \overline{P} B + B^T P_{\overline{P}} B)^{-1}(S + B^T \overline{P} A + B^T P_{\overline{P}} A) \\ &= Q_{\overline{P}} + A^T P_{\overline{P}} A - P_{\overline{P}} - (S_{\overline{P}}^T + A^T P_{\overline{P}} B)(R_{\overline{P}} + B^T P_{\overline{P}} B)^{-1}(S_{\overline{P}} + B^T P_{\overline{P}} A) \end{aligned}$$

□

Note that choosing $\overline{P} = P$, one obtains $P_{\overline{P}} = 0$.

Corollary 2. *By successively applying Lemma 1 using $\overline{K} = K$ and Lemma 2, one obtains that $\mathcal{P}(A_K, B, Q_K, R_K, S_K) = \mathcal{P}(A_K, B, Q_{K,\overline{P}}, R_{K,\overline{P}}, S_{K,\overline{P}})$, where $Q_{K,\overline{P}} = Q_K + A_K^T \overline{P} A_K - \overline{P}$, $S_{K,\overline{P}} = S_K + A_K^T \overline{P} B$ and $R_{K,\overline{P}} = R_K + B^T \overline{P} B$. Note that, as R_K can be chosen arbitrarily large, also $R_{K,\overline{P}}$ can be chosen arbitrarily large.*

Using these lemmas and corollaries, it is easier to prove Theorem 1.

Proof. (Theorem 1) Using the equivalence given in Corollary 2, it follows that $\mathcal{P}(A, B, Q, R, S) = \mathcal{P}(A_K, B, Q_{K,\overline{P}}, R_{K,\overline{P}}, S_{K,\overline{P}})$ and $K_{K,\overline{P}} = 0$. As we are interested in possibly indefinite but stabilizing LQR formulations, by hypothesis, the feedback gain K is stabilizing, i.e. $\rho(A_K) = \rho(A - BK) < 1$, where ρ denotes the spectral radius, i.e. $\rho(A) = \max\{|\lambda| \mid \lambda \text{ iseigenvalueof } A\}$. By Lyapunov stability theory, for any symmetric matrix $Q_L \succ 0$, there exists a matrix P_L such that $A_K^T P_L A_K - P_L + Q_L = 0$. Choosing $\overline{P} = P - P_L$, and recalling the result of Corollary 1, one obtains $A_K^T \overline{P} A_K - \overline{P} = Q_L - Q_K$. Then $Q_{K,\overline{P}} = Q_K + A_K^T \overline{P} A_K - \overline{P} = Q_L \succ 0$. One last step is needed to make matrix $H_{K,\overline{P}} = \begin{bmatrix} Q_{K,\overline{P}} & S_{K,\overline{P}}^T \\ S_{K,\overline{P}} & R_{K,\overline{P}} \end{bmatrix}$ positive definite. By Corollary 2, one can arbitrarily choose matrix $R_{K,\overline{P}}$ arbitrarily large without changing the solution $\phi(\bar{x}_0)$, nor the cost-to-go $\overline{P}$. The result $H_{K,\overline{P}} \succ 0$ is then obtained by selecting $R_{K,\overline{P}} \succ S_{K,\overline{P}} Q_{K,\overline{P}}^{-1} S_{K,\overline{P}}^T$. While this reformulated problem already yields an LQR with positive definite cost, it applies to the system (A_K, B). By applying Lemma 1 using $\overline{K} = -K$ one obtains an LQR with positive definite cost, which applies to the system (A, B), i.e. $\mathcal{P}(A, B, \tilde{Q}, \tilde{R}, \tilde{S}) = \mathcal{P}(A, B, Q, R, S)$, where $\tilde{Q} = Q_{K,\overline{P}} + S_{K,\overline{P}}^T K + K^T S_{K,\overline{P}} + K^T R_{K,\overline{P}} K$, $\tilde{S} = S_{K,\overline{P}} + R_{K,\overline{P}} K$ and $\tilde{R} = R_{K,\overline{P}}$. This can be shown by applying the Schur complement to check positive definiteness. Matrix $\tilde{R}$ is positive definite, so it remains to prove that $\tilde{Q} \succ \tilde{S}^T \tilde{R}^{-1} \tilde{S}$. This follows immediately from the definition:

$$\tilde{Q} = Q_{K,\overline{P}} + S_{K,\overline{P}}^T K + K^T S_{K,\overline{P}} + K^T R_{K,\overline{P}} K,$$

$$\tilde{S}^T \tilde{R}^{-1} \tilde{S} = S_{K,\overline{P}}^T R_{K,\overline{P}}^{-1} S_{K,\overline{P}} + S_{K,\overline{P}}^T K + K^T S_{K,\overline{P}} + K^T R_{K,\overline{P}} K.$$

The result $\tilde{Q} \succ \tilde{S}^T \tilde{R}^{-1} \tilde{S}$ directly follows from $Q_{K,\overline{P}} \succ S_{K,\overline{P}}^T R_{K,\overline{P}}^{-1} S_{K,\overline{P}}$. □

Remark 7. In order to compute the equivalent positive definite LQR matrices $\tilde{Q}$, $\tilde{R}$, $\tilde{S}$ one can solve the following SDP

$$\min_{\tilde{P}, \tilde{Q}, \tilde{R}, \tilde{S}, \tilde{H}} \quad \|\tilde{P} - I\|^2 + \|\tilde{H} - I\|^2 \tag{21a}$$

$$\text{s.t.} \quad \tilde{H} = \begin{bmatrix} \tilde{Q} & \tilde{S}^T \\ \tilde{S} & \tilde{R} \end{bmatrix} \tag{21b}$$

$$\tilde{H} \succeq 0 \tag{21c}$$

$$\tilde{P} \succeq 0 \tag{21d}$$

$$\tilde{Q} + A^T \tilde{P} A - \tilde{P} - (\tilde{S}^T + A^T \tilde{P} B)K = 0, \tag{21e}$$

$$(\tilde{R} + B^T \tilde{P} B)K - (\tilde{S} + B^T \tilde{P} A) = 0. \tag{21f}$$

Due to our choice of objective, the problem is convex and, by Theorem 1 it is also feasible, thus the solution to this SDP always exists and is unique. One advantage of solving the proposed SDP to compute matrices $\tilde{H}$ and $\tilde{P}$ is that the numerical properties of those matrices are optimal in the sense of the chosen matrix norm and reference.

5. Equivalent MPC formulations

In this section, we will analyze the MPC problem. We will restrict our attention to the case of no active inequality constraint at steady state. Note that this does not exclude the possibility of having state and input constraints in the MPC formulation, as long as they are not active at steady state. When considering an MPC problem, the time horizon becomes finite and the problem to be solved is

$$V(\bar{x}_0) = \inf_{x,u} \sum_{k=0}^{N} l(x_k, u_k) + x_N^T P_N x_N \tag{22a}$$

$$\text{s.t.} \quad x_0 = \bar{x}_0, \tag{22b}$$

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, \ldots, N-1, \tag{22c}$$

with $l(x_k, u_k) = \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T H \begin{bmatrix} x_k \\ u_k \end{bmatrix}$, matrices $H = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}$ and P_N symmetric and possibly indefinite. When no inequality constraints are imposed, the solution to this problem can be obtained by

propagating backwards the discrete Riccati equation (DRE)

$$\begin{aligned}\mathcal{R}_N(A,B,Q,R,S,P_N) := \{&(P_0,P_1,\ldots,P_N,K_0,K_1,\ldots,K_{N-1})\\ |P_{k-1} &= Q + A^TP_kA - (S^T + A^TP_kB)K_{k-1},\\ K_{k-1} &= (R+B^TP_kB)^{-1}(S+B^TP_kA),\\ k &= N,\ldots,1.\}.\end{aligned} \tag{23a}$$

The closed loop MPC trajectory can then be computed by simulating forward in time the system

$$x_{k+1} = A_{K_0}x_k = (A - BK_0)x_k,$$

while the MPC predicted trajectory can be computed by simulating forward in time

$$x_{k+1} = A_{K_k}x_k = (A - BK_k)x_k.$$

Remark 8. Note that, if the terminal cost matrix is chosen as $P_N = P$, where P is obtained by solving the LQR, then the theory from Section 4 applies. This section focuses on the case $P_N \neq P$.

Let us define the MPC predicted trajectory as $\psi(\bar{x}_0) := \mathcal{P}_N(A,B,Q,R,S,P_N)$, i.e. the associated DRE is $(P_0,P_1,\ldots,P_N,K_0,K_1,\ldots,K_{N-1}) = \mathcal{R}_N(A,B,Q,R,S,P_N)$ and the closed loop trajectory is given by $\psi(x_0) = (x_k)_{0\le k\le N}$ with $x_{k+1} = (A - BK_k)x_k$. In the following, we will define two problems equivalent if they deliver the same predicted trajectory, i.e. $\mathcal{P}_N(\tilde{A},\tilde{B},\tilde{Q},\tilde{R},\tilde{S},\tilde{P}_N) = \mathcal{P}_N(A,B,Q,R,S,P_N)$.

Definition 2. Problem $\mathcal{P}_N(\tilde{A},\tilde{B},\tilde{Q},\tilde{R},\tilde{S},\tilde{P}_N)$ is a positive definite reformulation of the possibly indefinite MPC problem $\mathcal{P}_N(A,B,Q,R,S,P_N)$ if $\begin{bmatrix}\tilde{Q} & \tilde{S}^T\\ \tilde{S} & \tilde{R}\end{bmatrix} \succ 0$, $\tilde{P}_N \succ 0$ and $\mathcal{P}_N(\tilde{A},\tilde{B},\tilde{Q},\tilde{R},\tilde{S},\tilde{P}_N) = \mathcal{P}_N(A,B,Q,R,S,P_N)$.

Theorem 1 considered the case of an infinite horizon linear quadratic OCP (LQR problem). The following theorem considers the case of a finite horizon OCP with a terminal cost which is different from the LQR cost to go (MPC problem).

Theorem 2. *Every stabilizing possibly indefinite MPC scheme of the form (5) with time invariant matrices A, B, H can be formulated as an equivalent positive definite MPC scheme.*

Before proving the theorem, let us first introduce the following lemma.

Lemma 3. *Two DREs $\mathcal{R}_N(A,B,Q,R,S,P_N)$ and $\mathcal{R}_N(A,B,\tilde{Q},\tilde{R},\tilde{S},\tilde{P}_N)$ yield the same feedback sequence $K_0,\ldots,K_{N-1}$ if*

$$\tilde{P}_N - \tilde{P} = P_N - P,\ \tilde{R} + B^T\tilde{P}B = R + B^TPB,\ \tilde{S} + B^T\tilde{P}A = S + B^TPA, \tag{24}$$

with P and $\tilde{P}$ computed by solving the associated DAREs $\mathcal{D}(A,B,Q,R,S)$ and $\mathcal{D}(A,B,\tilde{Q},\tilde{R},\tilde{S})$.

Proof. Let us first consider the two equivalent LQR problems $\mathcal{P}(A,B,Q,R,S) = \mathcal{P}(A,B,\tilde{Q},\tilde{R},\tilde{S})$ with associated DAREs $(P,K) = \mathcal{D}(A,B,Q,R,S)$ and $(\tilde{P},K) = \mathcal{D}(A,B,\tilde{Q},\tilde{R},\tilde{S})$. Defining $\Delta_k = P_k - P$ and $\tilde{\Delta}_k = \tilde{P}_k - \tilde{P}$, the two DREs can be rewritten as $\mathcal{R}_N(A,B,Q,R,S,P+\Delta_N)$ and $\mathcal{R}_N(A,B,\tilde{Q},\tilde{R},\tilde{S},\tilde{P}+\tilde{\Delta}_N)$, i.e.

$$P + \Delta_{k-1} = Q + A^TPA + A^T\Delta_kA - (S^T + A^TPB + A^T\Delta_kB)K_{k-1}, \tag{25a}$$

$$K_{k-1} = (R + B^TPB + B^T\Delta_kB)^{-1}(S + B^TPA + B^T\Delta_kA), \tag{25b}$$

and

$$\tilde{P} + \tilde{\Delta}_{k-1} = \tilde{Q} + A^T\tilde{P}A + A^T\tilde{\Delta}_kA - (\tilde{S}^T + A^T\tilde{P}B + A^T\tilde{\Delta}_kB)K_{k-1}, \tag{26a}$$

$$K_{k-1} = (\tilde{R} + B^T\tilde{P}B + B^T\tilde{\Delta}_kB)^{-1}(\tilde{S} + B^T\tilde{P}A + B^T\tilde{\Delta}_kA). \tag{26b}$$

The identity $\Delta_k = \tilde{\Delta}_k$ is obtained if condition (24) is satisfied. This can be seen by replacing the values of P and $\tilde{P}$ from the DARE into Eqs. (Proof) and (Proof), which gives

$$\Delta_{k-1} = A^T\Delta_kA - (S^T + A^TPB)K - (S^T + A^TPB + A^T\Delta_kB)K_{k-1}, \tag{27a}$$

$$K_{k-1} = (R + B^TPB + B^T\Delta_kB)^{-1}(S + B^TPA + B^T\Delta_kA), \tag{27b}$$

and

$$\tilde{\Delta}_{k-1} = A^T\tilde{\Delta}_kA - (\tilde{S}^T + A^T\tilde{P}B)K - (\tilde{S}^T + A^T\tilde{P}B + A^T\tilde{\Delta}_kB)K_{k-1}, \tag{28a}$$

$$K_{k-1} = (\tilde{R} + B^T\tilde{P}B + B^T\tilde{\Delta}_kB)^{-1}(\tilde{S} + B^T\tilde{P}A + B^T\tilde{\Delta}_kA). \tag{28b}$$

By replacing condition (24) into (Proof) one immediately obtains (Proof). □

Remark 9. Lemma 3 provides sufficient conditions for the equivalence of two DREs but does not provide any guarantee on the existence of two DREs which satisfy those conditions. It will be clear in the proof of Theorem 2 that this is indeed possible.

Proof. (Theorem 2) The proof follows from combining the results of Theorem 1 and Lemma 3. In particular, conditions (24) can be easily enforced in Theorem 1. From Corollary 1 we have that $S_K^T + B^TPA_K = S_{K,\overline{P}}^T + B^TP_{\overline{P}}A_K = 0$, for any choice of matrix $\overline{P}$. As matrix $R_{K,\overline{P}}$ can be arbitrarily chosen, one can choose $R_{K,\overline{P}} = R + B^TPB - B^TP_{\overline{P}}B$.

As in Theorem 1, we can choose $\overline{P} = P - P_L$, to obtain $P_{\overline{P}} = P_L$. In order to guarantee positive definiteness, let us choose Q_L arbitrarily but positive definite and scale it by a factor $\gamma > 0$. The Lyapunov equation becomes $\gamma A_K^TP_LA_K - \gamma P_L + \gamma Q_L = 0$ and, subsequently,

$$\begin{aligned}S_{K,\overline{P}}^T &= -\gamma B^TP_LA_K,\\ R_{K,\overline{P}} &= R + B^TPB - \gamma B^TP_LB.\end{aligned}$$

The condition for ensuring positive definiteness $R_{K,\overline{P}} \succ S_{K,\overline{P}}^TQ_{K,\overline{P}}^{-1}S_{K,\overline{P}}$ thus becomes

$$R + B^TPB \succ \gamma B^TP_LB + \gamma(B^TA_K^TP_LB)Q_L^{-1}(B^TP_LA_KB), \tag{29}$$

which holds for γ sufficiently small. The transformation back from system (A_K,B) to system (A,B) can then be done as in Theorem 1. □

Remark 10. In the same way as for Theorem 1, also for Theorem 2, the MPC reformulation can be computed by solving the following SDP

$$\min_{\tilde{P},\tilde{Q},\tilde{R},\tilde{S},\tilde{H}} \|\tilde{P} - I\|^2 + \|\tilde{H} - I\|^2 \tag{30a}$$

$$\text{s.t. } \tilde{H} = \begin{bmatrix}\tilde{Q} & \tilde{S}^T\\ \tilde{S} & \tilde{R}\end{bmatrix} \tag{30b}$$

$$\tilde{H} \succeq 0 \tag{30c}$$

$$\tilde{P} \succeq 0 \tag{30d}$$

$$\tilde{Q} + A^T\tilde{P}A - \tilde{P} - (\tilde{S}^T + A^T\tilde{P}B)K = 0, \tag{30e}$$

$$(\tilde{R} + B^T\tilde{P}B)K - (\tilde{S} + B^T\tilde{P}A) = 0, \tag{30f}$$

$$\tilde{R} + B^T\tilde{P}B = R + B^TPB, \tag{30g}$$

$$\tilde{S} + B^T\tilde{P}A = S + B^TPA. \tag{30h}$$

The new terminal cost is then given by $\tilde{P}_N = P_N - P + \tilde{P}$.

Remark 11. The reformulation obtained in Theorem 2 yields a convexified stage cost which is independent of the chosen terminal cost. It can be verified that the result is also valid for the case of an end point constraint in the MPC scheme.

6. Numerical example

In order to illustrate the results presented in the previous section, let us consider the linear system

$$x_{k+1} = Ax_k + Bu_k,$$
$$A = \begin{bmatrix} -0.3319 & 0.7595 & 1.5399 \\ -0.3393 & 0.1250 & 0.4245 \\ -0.5090 & 0.9388 & 0.8864 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1060 \\ -1.3835 \\ -0.1496 \end{bmatrix}.$$

The stage cost is chosen indefinite and defined as

$$H = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} = \begin{bmatrix} -1.0029 & -0.0896 & 1.1050 & -0.0420 \\ -0.0896 & 1.6790 & -0.5762 & 0.2112 \\ 1.1050 & -0.5762 & -0.4381 & -0.2832 \\ -0.0420 & 0.2112 & -0.2832 & 0.6192 \end{bmatrix}. \tag{31}$$

To obtain a positive definite equivalent LQR one can either apply the procedure developed in the proof of Theorem 1 or solve SDP (7), which results in matrices $\tilde{H}$ and $\tilde{P}$ with the following properties

$$\lambda(\tilde{H}_{\text{Th1}}) = \begin{bmatrix} 615.9651 \\ 1.0000 \\ 0.9516 \\ 0.0017 \end{bmatrix}, \quad \lambda(\tilde{H}_{\text{SDP}}) = \begin{bmatrix} 1.3444 \\ 0.6252 \\ 0.0010 \\ 0.0010 \end{bmatrix},$$
$$\lambda(\tilde{P}_{\text{Th1}}) = \begin{bmatrix} 33.0529 \\ 10.4083 \\ 1.0004 \end{bmatrix}, \quad \lambda(\tilde{P}_{\text{SDP}}) = \begin{bmatrix} 1.0140 \\ 0.2345 \\ 0.0383 \end{bmatrix}.$$

In particular, it can be seen that solving SDP (7) yields better conditioned matrices, as an effect of the chosen cost function, which aims at finding matrices $\tilde{H}$ and $\tilde{P}$ as close to the identity as possible. In this example, the Frobenius norm was used in the objective function.

To visualize the results, a simulation has been run starting from $x_0 = [1, 0, 0]$. The resulting trajectory and the value of the three LQR formulations are displayed in Fig. 2. It can be seen that all formulations stabilize the system and the obtained trajectories coincide. However, only the two reformulated LQR yield a positive definite value function, which is also a Lyapunov function for the system.

7. Approximated economic NMPC

Economic NMPC consists in optimizing a cost which is not necessarily positive definite, thus standard stability proofs do not hold. A stability proof for economic NMPC has been proposed in [7]. It relies on the concept of strict dissipativity, which is unfortunately extremely hard to check in the general case. This motivates a search for NMPC schemes that approximate economic behavior, but for which it is rather easy to prove stability.

Local information about the system can be obtained by looking at the second order expansion at steady-state. The optimal steady state is computed by solving the following NLP

$$\min_{x,u} \ \ell(x,u) \quad \text{s.t.} \quad x = f(x,u), \tag{32}$$

where x and u denote the state and control vectors and function $f(x, u)$ is the considered discrete time system. Using $w^T = [x^T\ u^T]$, the Lagrangian of Problem (32) is given by $\mathcal{L}(w, \lambda) = \ell(x, u) - \lambda^T(x - f(x, u))$, where λ is the vector of Lagrange multipliers associated with the equality constraints. The second order expansion at steady state is then obtained by using the linear system (A, B) with

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{w_s}, \quad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{w_s},$$

where x_s and u_s denote the state and control vectors at steady state. The cost function is given by $\ell_2(x, u) = 1/2(w - w_s)^T H(w - w_s)$, where matrix H is given by

$$H = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \left.\frac{\partial^2 \mathcal{L}(w,\lambda)}{\partial w^2}\right|_{w_s}.$$

Note that matrix H is in general indefinite, hence an NMPC scheme using $\ell_2(\cdot, \cdot)$ as stage cost is in general not of tracking type and stability is guaranteed if the system is dissipative. Checking if the linear system is dissipative is straightforward: if the LMI, QMI or FDI condition is satisfied, then the system is dissipative. Alternatively one can check if the cost, rotated as in [5] is convex.

If the nonlinear system is locally controllable at steady state but not dissipative, steady-state operation is not optimal [12]. If nevertheless one is interested in stabilizing the system at steady state, the objective can be regularized so as to make the system dissipative, see e.g. [5,7]. Adding regularization can make the system dissipative, but the stage cost can nevertheless still be indefinite. As the system is dissipative, the indefinite LQR or linear MPC scheme will be stabilizing for the linear system. According to Theorem 2, one can convexify the indefinite stage cost to compute a positive definite stage cost and the convexified LQR and MPC problems are then equivalent to the original ones. This new positive definite stage cost can then be used in an NMPC context to stabilize the nonlinear plant. As the cost is quadratic positive definite, the NMPC scheme is of tracking type. The classical NMPC stability proofs are then applicable [23]. Moreover, the stage cost is a second order approximation of the nonlinear economic stage cost.

Note that the convexification procedure proposed in this paper relies on a second order approximation of the economic NMPC, which fails to exist if the system is not locally stabilizable at steady state, i.e. if (A, B) is not stabilizable. Techniques to overcome this limitation are the subject of ongoing research.

7.1. Numerical example

In order to illustrate these concepts, let us look at the CSTR example from [5,7]. Analogously as in Section 2, we will assume that $c_A + c_B = 1$ in order to have local stabilizability at steady state. The real economic stage cost is

$$\ell(x,u) = -(2q(1 - c_A) - 0.5q), \tag{33}$$

and the quadratic term $\gamma(q - 4)^2$ with $\gamma = 0.1$ is added in order to make the system dissipative. While in Section 2 we considered economic NMPC with the regularized stage cost, let us now consider the original economic stage cost (33).

The optimal steady state computed by solving Problem (32) is $c_A = 0.5$ mol/L and $q = 4$ L/min. The second order expansion at steady state gives

$$A = 0.6757, \quad B = 0.0203,$$
$$H = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} 0 & 1.1832 \\ 1.1832 & -0.0229 \end{bmatrix}.$$

As anticipated, this system is not dissipative, hence, the LMI condition (9) cannot be satisfied. In [5], the regularization $0.505(c_A + c_B + q)$ is added to the cost. In [7], the regularization $\gamma(q - 4)^2$ is added to the cost with $\gamma = 0.1$. In the framework proposed in this paper, the regularization can be chosen as the minimal one which is needed in order to satisfy the LMI (9) using the approach proposed in Remark 4. Choosing to restrict the regularization to the form $\gamma(q - 4)^2$ as proposed in [7] and solving SDP (7), one obtains that the needed regularization for the linear system is $\gamma = 0.0275$. Note that this value is only valid for the linear system and does not guarantee stability of the economic NMPC scheme a

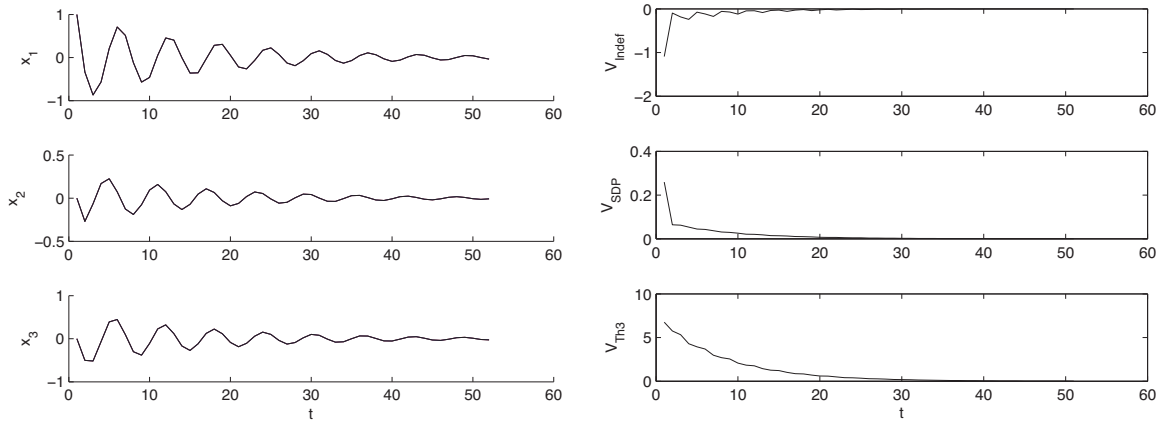

Fig. 2. Trajectory and value function for the three LQR formulations.

priori. It is instead guaranteed that the tracking NMPC using the regularized and convexified cost is stabilizing for the nonlinear plant. By convexifying the cost in the framework of Theorem 2 using the minimal regularization $\gamma = 0.0275$, one obtains

$$H^2 = \begin{bmatrix} 25.0142 & 0.5528 \\ 0.5528 & 0.0132 \end{bmatrix}.$$

In the following, we will call *scheme* 2 the NMPC scheme which uses a tracking stage cost defined by matrix H^2 and *regularized economic NMPC* the economic NMPC scheme which uses the stage cost proposed in [7], i.e. cost (33) regularized with $0.1(q-4)^2$. Using $\gamma = 0.1$, one would obtain matrix H^0 used in Section 2. This explains why scheme 0 yields state and input profiles that closely mimic the ones obtained with the regularized economic NMPC (cf. Section 2 and Fig. 1).

The tracking NMPC scheme 2 has been compared to the regularized economic NMPC formulation in the same scenario as in Section 2. In this second example, though, gain $G^\star$ is computed using the average profit of each scheme computed using the actual stage cost (33). For the tracking scheme that uses H^2, one obtains $G^2 = -8.0599$, which indicates that the economic NMPC formulation, regularized with the term $0.1(q-4)^2$ performs worse than the tracking NMPC formulation which uses H^2. For the tracking implementation which uses H^1 instead, one obtains $G^1 = 2.1907$, which indicates that the regularized economic NMPC formulation performs better than this second tracking NMPC formulation.

The state and input profiles are displayed in Fig. 3, where it can be seen that the tracking scheme 2 outputs trajectories that are significantly different from the regularized economic NMPC scheme. In particular, tracking scheme 2 has a more aggressive behavior and it activates the input bounds in several occasions.

Provided that either the terminal cost and constraint are chosen adequately [23] or the horizon is chosen long enough [24], the tracking NMPC scheme has the advantage of immediately providing stability guarantees. Moreover, the underlying OCP can be efficiently solved by numerical algorithms for fast NMPC, e.g. the real time iteration (RTI) scheme [25] with the Gauss-Newton Hessian approximation. For economic NMPC instead, the underlying OCP is in general more difficult to solve.

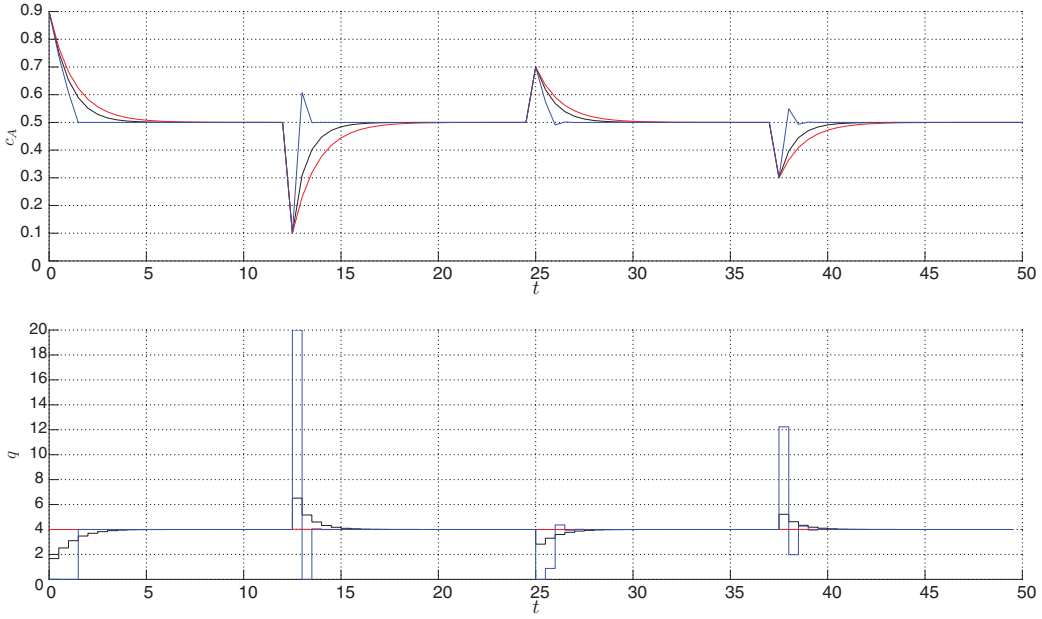

Fig. 3. State and input trajectories obtained with the economic NMPC scheme (black line), with the tracking scheme 2 (blue line) and with the tracking scheme 1 (red line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

8. Conclusions

In this paper we have analyzed economic MPC for the linear quadratic case. Both the cases of finite and infinite prediction horizons have been considered and it has been shown that every stabilizing LQR or MPC scheme can be reformulated such that the cost is positive definite, while preserving the same closed loop behavior.

Approximated economic NMPC with stability guarantees has been proposed, based on the previous results. The application to a simple well known example has shown the potential of this approach also in terms of economic performance.

Future work will focus on extending the proposed framework to more general settings, including periodic systems and active constraints at steady state.

Appendix A. Proof of Lemma 1

By replacing the definitions of $A_{\overline{K}}$, $Q_{\overline{K}}$ and $S_{\overline{K}}$ in $\mathcal{D}(A_{\overline{K}}, B, Q_{\overline{K}}, R, S_{\overline{K}})$, one obtains

$$\begin{aligned}
0 &= Q_{\overline{K}} + A_{\overline{K}}^T P_{\overline{K}} A_{\overline{K}} - P_{\overline{K}} - (S_{\overline{K}}^T + A_{\overline{K}}^T P_{\overline{K}} B)(R + B^T P_{\overline{K}} B)^{-1}(S_{\overline{K}} + B^T P_{\overline{K}} A_{\overline{K}}) \\
&= Q - S^T\overline{K} - \overline{K}^T S + \overline{K}^T R\overline{K} + (A - B\overline{K})^T P_{\overline{K}}(A - B\overline{K}) - P_{\overline{K}} \\
&\quad -(S^T - \overline{K}^T R + (A - B\overline{K})^T P_{\overline{K}} B)(R + B^T P_{\overline{K}} B)^{-1} \\
&\quad \times(S - R\overline{K} + B^T P_{\overline{K}}(A - B\overline{K}))
\end{aligned}$$

The last term of the sum can be rewritten as

$$\begin{aligned}
&(S^T - \overline{K}^T R + (A - B\overline{K})^T P_{\overline{K}} B)(R + B^T P_{\overline{K}} B)^{-1}(S - R\overline{K} + B^T P_{\overline{K}}(A - B\overline{K})) \\
&= -\overline{K}^T \underbrace{(R + B^T P_{\overline{K}} B)(R + B^T P_{\overline{K}} B)^{-1}}_{I}(S - R\overline{K} + B^T P_{\overline{K}}(A - B\overline{K})) \\
&\quad + (S^T + A^T P_{\overline{K}} B)(R + B^T P_{\overline{K}} B)^{-1}(S - R\overline{K} + B^T P_{\overline{K}}(A - B\overline{K})) \\
&= -\overline{K}^T S + \overline{K}^T R\overline{K} - \overline{K}^T B^T P_{\overline{K}} A + \overline{K}^T B^T P_{\overline{K}} B\overline{K} \\
&\quad + (S^T + A^T P_{\overline{K}} B)(R + B^T P_{\overline{K}} B)^{-1}(S + B^T P_{\overline{K}} A) \\
&\quad - (S^T + A^T P_{\overline{K}} B)\underbrace{(R + B^T P_{\overline{K}} B)^{-1}(R + B^T P_{\overline{K}} B)}_{I}\overline{K} \\
&= -\overline{K}^T S + \overline{K}^T R\overline{K} - \overline{K}^T B^T P_{\overline{K}} A + \overline{K}^T B^T P_{\overline{K}} B\overline{K} \\
&\quad - S^T\overline{K} - A^T P_{\overline{K}} B\overline{K} + (S^T + A^T P_{\overline{K}} B)(R + B^T P_{\overline{K}} B)^{-1}(S + B^T P_{\overline{K}} A).
\end{aligned}$$

Replacing this expression in $\mathcal{D}(A_{\overline{K}}, B, Q_{\overline{K}}, R, S_{\overline{K}})$ and simplifying one obtains

$$Q + A^T P_{\overline{K}} A - P_{\overline{K}} - (S^T + A^T P_{\overline{K}} B)(R + B^T P_{\overline{K}} B)^{-1}(S + B^T P_{\overline{K}} A) = 0,$$

which directly entails $P_{\overline{K}} = P$.

The result $K_{\overline{K}} = K - \overline{K}$ is obtained as follows

$$\begin{aligned}
K_{\overline{K}} &= (R + B^T PB)^{-1}(S_{\overline{K}} + B^T PA_{\overline{K}}) \\
&= (R + B^T PB)^{-1}(S - R\overline{K} + B^T PA - B^T PB\overline{K}) \\
&= (R + B^T PB)^{-1}(S + B^T PA) - (R + B^T PB)^{-1}(R + B^T PB)\overline{K} \\
&= K - \overline{K}.
\end{aligned}$$

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