IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received November 18, 2019, accepted January 5, 2020, date of current version January 30, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2967666

Lattice Reduction Aided Precoding Design in Downstream G.fast DSL Networks

WOUTER LANNEER[1], CARL NUZMAN[2], YANNICK LEFEVRE[3], PASCHALIS TSIAFLAKIS[3], WERNER COOMANS[3], AND MARC MOONEN[1], (Fellow, IEEE)

[1]ESAT/STADIUS Center for Dynamical Systems, Signal Processing and Data Analytics, KU Leuven, 3000 Leuven, Belgium
[2]Mathematics of Communication Department, Nokia Bell Labs, Murray Hill, NJ 07974, USA
[3]Fixed Networks Department, Nokia Bell Labs, 2018 Antwerp, Belgium

Corresponding author: Wouter Lanneer (wouter.lanneer@nokia-bell-labs.com)

This research work was carried out at the ESAT Laboratory of KU Leuven in the frame of VLAIO O&O Project nr. HBC.2016.0055 '5GBB' and HBC.2017.1007 'Multi-gigabit Innovations in Access', FWO nr. G.0B1818N 'Real-time adaptive cross-layer dynamic spectrum management for 5th generation broadband copper access networks', Fonds de la Recherche Scientifique - FNRS and the Fonds' Wetenschappelijk Onderzoek - Vlaanderen under EOS Project no 30452698 'MUlti-SErvice WIreless NETwork'. This research work is presented in the Ph.D. dissertation of the first author [1]. The scientific responsibility is assumed by its authors.

ABSTRACT As a non-linear precoding alternative to Tomlinson-Harashima precoding (THP), in this paper, so-called lattice reduction aided precoding (LRP) is considered as a crosstalk precompensation technique for downstream transmission in G.fast DSL networks. First, a practically achievable bit-rate expression for LRP is proposed in function of the precoder and integer matrix. The problem then consists of a joint precoder and integer matrix design in order to maximize the weighted sum-rate (WSR) under per-line power constraints. For a fixed integer matrix, zero-forcing (ZF) precoder matrix design simplifies to gain scaling optimization with complex gain scalars, for which a successive lower bound maximization method is presented. Additionally, it is established that the achievable ZF-LRP sum-rate is upper bounded by the achievable ZF-THP sum-rate at high SNR. For computing the optimal precoder matrix, on the other hand, an efficient method is developed by leveraging on the equivalence between the WSR maximization and the weighted sum of mean squared error (MSE) minimization, leading to a locally-optimal MMSE-LRP solution. Simulations with a measured G.fast cable binder are provided to compare the proposed LRP schemes with THP schemes.

INDEX TERMS DSL, G.fast, dynamic spectrum management, lattice reduction, non-linear precoding (NLP).

I. INTRODUCTION

Ultra-broadband digital subscriber line (DSL) networks like G.fast [2] aim at providing gigabit (i.e. fiber-like) data speeds over very short copper lines (below 100 m) by signaling in high frequencies (up to 212 MHz). The use of such high frequencies leads to increasingly stronger levels of crosstalk interference among the lines within a cable binder [3]. As a result, non-linear precoding (NLP) schemes are considered as an alternative to linear precoding (LP) schemes for far-end crosstalk precompensation in downstream transmission.

NLP schemes include (non-linear) modulo operations at the transmitters and the receivers to transmit equivalent lower power signals. A well-known NLP scheme proposed in DSL is multi-user Tomlinson-Harashima precoding (THP) [4]. However, a disadvantage of THP is the required sequential feedback operation proportional to the number of users (or lines), which in combination with the modulo operations creates a challenging circuit timing problem. In addition, the use of THP schemes may increase the dynamic range at the receiver and is more sensitive to practical non-idealities than LP schemes [5].

In this paper, an alternative NLP scheme is therefore considered, commonly referred to as lattice reduction aided precoding (LRP) [6]–[10]. Although, like THP, LRP uses (non-linear) modulo operations to transmit equivalent lower power signals, it features a complete forward data path (like in LP schemes). This reduces the implementation complexity increase of the NLP scheme and results in a relaxed circuit timing problem compared to THP, especially for scenarios with a large number of users.

A. MAIN CONTRIBUTIONS

In this paper, LRP design is considered for downstream transmission in G.fast DSL networks. Specifically, the LRP scheme precedes the precoder matrix by a (complex) integer

The associate editor coordinating the review of this manuscript and approving it for publication was Mamoun Alazab.

This work is licensed under a Creative Commons Attribution 4.0 License. For more information, see http://creativecommons.org/licenses/by/4.0/

matrix, leading to a "reduced" precoder matrix with typically lower transmit power penalties. Using non-linear scalar modulo operations, the integer matrix can be pre-inverted at the transmitter side, such that the receivers are able to detect the desired symbols. A practically achievable bit-rate expression is derived for LRP in function of the precoder matrix and integer matrix, tight at high SNR. The LRP design problem is then formulated as the joint optimization of the precoder matrix and integer matrix in order to maximize the weighted sum-rate (WSR) under per-line power constraints.[1]

To tackle this difficult design problem, precoder matrix optimization given a fixed integer matrix is studied. In case of zero-forcing (ZF) precoding, this leads to a gain scaling optimization problem with complex gain scalars (similar to power allocation in linear ZF precoding), for which a successive lower bound maximization method is presented. Additionally, it is established that the achievable sum-rate of this ZF-LRP is upper bounded by the achievable sum-rate of ZF-THP at high SNR. This bound is tight when the (output power normalized) reduced precoder matrix is *unitary* and the integer matrix is *unimodular*. To compute the optimal precoder matrix,[2] on the other hand, an efficient method is developed, by relying on the equivalence between the WSR maximization and the weighted sum of mean square error (MSE) minimization, leading to a locally-optimal MMSE-LRP solution.

Further, a heuristic alternating method is proposed between optimizing the precoder matrix for a fixed integer matrix, and updating the integer matrix based on the lattice reduction of the precoder matrix via the Lenstra-Lenstra-Lovász (LLL) algorithm. Finally, the performance of the proposed LRP schemes is compared with THP schemes, by means of simulations with a measured G.fast cable binder.

B. RELATED WORK

LRP has been introduced for general single-carrier MIMO channels [6]–[9], as a popular low-complexity technique which has been proven to achieve the full diversity order [9]. The original idea of LRP is to precompensate the channel in a suited basis, by decomposing the channel matrix into a reduced one and a unimodular matrix via the LLL algorithm. However, these prior works typically assume equal user rates and consider the bit error rate minimization problem, whereas in this paper the rate maximization problem is studied.

Recently, the rate maximization problem for ZF-LRP has been studied in [10] for downstream G.fast DSL networks. It adopts a closed-form practically achievable bit-rate expression, based on the so-called SNR approximation gap [11]. Subsequently, the authors propose to optimize the bit-rates by introducing a gain scaling with real scalars. The gain scaling problem is approximated in the high SNR regime as a signomial program, which can be locally optimally solved by interior point methods. The approach in this paper differs in that gain scaling with *complex* scalars is considered for ZF-LRP, for which an iterative method is proposed that obtains a local optimum of the *exact* gain scaling problem, and that also the case of MMSE-LRP design is considered. In addition, the combination of lattice reduction and THP in the context of downstream G.fast DSL networks has been considered in [12]. However, this scheme inherits the difficult sequential feedback operation and hardware implementation of THP.

Closely related to LRP are integer-forcing (IF) precoding schemes [13]–[15], where instead of scalar modulo operations, more involved nested *lattice* encoding and decoding and message precoding is used on blocks of symbols. With appropriate lattice construction and the dimensionality of the blocks going up to infinity, IF precoding leads to closed-form theoretically achievable bit-rate expressions [13]. In [15] a (regularized) ZF precoder structure is proposed with arbitrary scaling by a complex diagonal matrix, referred to as diagonally-scaled exact integer-forcing (DIF). A closed-form expression for the SR optimal scaling and integer matrix under a sum-power constraint is established, for the specific case of two users in the high SNR regime. Different from [15], in this paper more than two users and realistic per-line power constraints are considered, such that optimal closed-form expressions are not available.

C. ORGANIZATION AND NOTATION

This paper is organized as follows. Section II introduces the LRP system model for downstream G.fast transmission and derives an achievable bit-rate expression. Section III states the general LRP design problem and proposes an alternating optimization approach. Section IV and Section V address the subproblems corresponding to the precoder optimization given a fixed integer matrix for ZF-LRP and MMSE-LRP, respectively. Section VI presents simulation results for a G.fast cable binder. Finally, Section VII concludes the paper.

Lowercase boldface letters are used to denote vectors and uppercase boldface letters are used to denote matrices. $\mathbb{C}^{m\times n}$ and $\mathbb{Z}^{m\times n}[j] \equiv \mathbb{Z}^{m\times n} + j\mathbb{Z}^{m\times n}$ are used to denote the $m \times n$ dimensional complex and complex integer space, respectively. $\mathbf{I}_A$ is used as the identity matrix of size A, $(\cdot)^T$ as the transpose, $(\cdot)^H$ as the Hermitian transpose, $(\cdot)^*$ as the complex conjugate, $\mathcal{E}\{\cdot\}$ as expectation, $\mathrm{Re}(\cdot)$ as real part, $[\mathbf{X}]_{ij}$ as the i,j-th element of $\mathbf{X}$, $[x]_a^b$ as $\min(\max(x,a),b)$, $\mathrm{Tr}\{\cdot\}$ as the trace, $\det(\cdot)$ as matrix determinant, and $|\cdot|$ as scalar absolute value.

II. LRP FOR DOWNSTREAM G.fast TRANSMISSION

A. DOWNSTREAM G.fast TRANSMISSION MODEL

Downstream G.fast transmission in a cable binder with N lines or users is considered. Assuming standard synchronous DMT modulation with a sufficiently long cyclic prefix, the transmission is modeled independently on each tone

[1] Akin to per-antenna power constraints in the wireless field.

[2] Optimal precoding is sometimes also referred to as minimum mean squared error (MMSE) precoding.

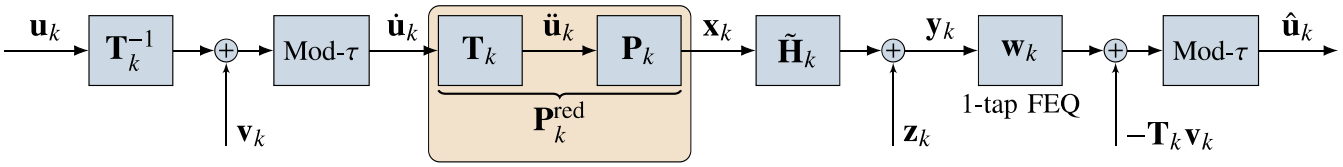


FIGURE 1. A block diagram of the LRP scheme on tone k.

(or frequency sub-carrier) $k = [1, \ldots, K]$ as

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{z}_k. \tag{1}$$

$\mathbf{x}_k \triangleq [x_k^1, \ldots, x_k^N]^T$ is the transmit vector on tone k, with x_k^n the signal transmitted on line n. $\mathbf{y}_k \triangleq [y_k^1, \ldots, y_k^N]^T$ is the receive vector on tone k, with y_k^n the signal received by user n. $\mathbf{z}_k \triangleq [z_k^1, \ldots, z_k^N]^T$ is the vector of uncorrelated additive noise signals on tone k, with $\sigma_k \triangleq \mathcal{E}\{|z_k^n|^2\}$ denoting the noise PSD. $\mathbf{H}_k \triangleq [h_k^{n,m}]$ denotes the $N \times N$ channel matrix on tone k. The diagonal elements of $\mathbf{H}_k$ contain the direct channels whereas the off-diagonal elements contain the crosstalk channels. Perfect knowledge of the channel matrices is assumed. In G.fast, per-line spectral mask and total power constraints are included:

$$\mathcal{E}\{|x_k^l|^2\} \leq P_k^{\text{mask}}, \quad \forall k, l, \tag{2}$$

$$\sum_k \mathcal{E}\{|x_k^l|^2\} \leq P^{\text{total}}, \quad \forall l. \tag{3}$$

For ease of representation, the per-tone case is adopted and the tone index is dropped in the remainder of paper. Tone indices are briefly re-introduced in Appendix B in order to show how the total power constraints in (3) can be handled algorithm-wise.

B. TRANSMITTER AND RECEIVER PROCESSING

In standard LP, the transmit vector $\mathbf{x}$ is formed by

$$\mathbf{x} = \mathbf{P}\mathbf{u} \tag{4}$$

where $\mathbf{P}$ is the $N \times N$ precoder matrix and $\mathbf{u} \triangleq [u^1, \ldots, u^N]$ is the data vector. The data vector consists of QAM symbols with possibly different constellation sizes for each user, and is assumed to be independently and identically distributed (i.i.d.) with unit power, i.e., $\mathcal{E}\{\mathbf{u}\mathbf{u}^H\} = \mathbf{I}_N$.

The main idea of LRP, as shown in Fig. 1, is then to precede the precoder matrix by a (complex) integer matrix $\mathbf{T}$, leading to a "reduced" precoder matrix

$$\mathbf{P}^{\text{red}} = \mathbf{P}\mathbf{T}. \tag{5}$$

It is required that $\mathbf{T} \in \mathbb{Z}[j]^{N\times N}$ is invertible (i.e. of full-rank). In case $|\det(\mathbf{T})| = 1$, it holds that $\mathbf{T}$ is unimodular and $\mathbf{T}^{-1}$ is also complex integer. The overall goal is to improve the mathematical properties of $\mathbf{P}^{\text{red}}$ over $\mathbf{P}$ (e.g. a reduction of the orthogonality defect and condition number), and hence to achieve a performance gain in terms of a reduced transmit power penalty.

The multiplication with $\mathbf{T}$ has to be compensated, which in downstream transmission, without co-located receivers, may be implemented at the transmitter by multiplying the data vector with $\mathbf{T}^{-1}$, i.e, $\dot{\mathbf{u}} = \mathbf{T}^{-1}\mathbf{u}$. Unfortunately, this may increases the power of $\dot{\mathbf{u}}$ (and thus of $\mathbf{x}$). To deal with this issue, the multiplication is followed by a scalar (component-wise) non-linear modulo operation to bound the value of the transmit signal, as also used in THP [16], [17], and the addition of a so-called dither vector $\mathbf{v} \triangleq [v_1, \ldots, v_N]^T$ (which purpose is explained below), i.e.,

$$\dot{\mathbf{u}} = \left[\mathbf{T}^{-1}\mathbf{u} + \mathbf{v}\right] \text{mod-}\tau. \tag{6}$$

The modulo operation in (6) is applied to both dimensions of the QAM constellation (corresponding to a complex x)

$$[x] \text{ mod-}\tau \triangleq [\text{Re}(x)] \text{ mod-}\tau + j\,[\text{Im}(x)] \text{ mod-}\tau \tag{7}$$

with for real y

$$[y] \text{ mod-}\tau \triangleq y - \tau\left\lfloor \frac{y + \tau/2}{\tau} \right\rfloor, \quad y \in \mathbb{R} \tag{8}$$

and $\lfloor\cdot\rfloor$ denoting the floor function. The complex range of mod-τ is denoted by $\Omega_\tau = \{a + jb : a, b \in [-\tau/2, \tau/2)\}$, i.e., $[x]$ mod-$\tau \in \Omega_\tau$ for all complex x. The base τ of the modulo operation, which is the same for all the users, needs to be chosen such that Ω_τ encloses the QAM constellations of all users. The margin between the boundary $\tau/2$ and any point at the edge of a constellation should be at least d where $2d$ is the minimum distance[3] between adjacent constellation points for square and cross-shaped QAM constellations (i.e. d corresponds to the constellation spacing); otherwise, there will be information loss. However, smaller margins yield larger power savings in $\dot{\mathbf{u}}$. Consequently, a valid choice for τ is

$$\tau = \max_n \tilde{\tau}_n \tag{9}$$

where $\tilde{\tau}_n$ is the (1-dim.) constellation boundary of user n. The values for $\tilde{\tau}_n$ can be derived in function of the constellation size $M = 2^b$ and scaling d (see e.g. [18] for the constellations adopted in G.fast [2]).

The modulo operation in (6) may also be seen as the addition of a scaled complex integer perturbation vector $\mathbf{d} \in \tau\mathbb{Z}^{N\times 1}[j]$, i.e.,

$$\dot{\mathbf{u}} \triangleq \mathbf{T}^{-1}\mathbf{u} + \mathbf{v} + \mathbf{d}. \tag{10}$$

Further, the auxiliary signal vector $\ddot{\mathbf{u}} \triangleq \mathbf{T}\dot{\mathbf{u}} = \mathbf{u} + \mathbf{T}\mathbf{v} + \mathbf{T}\mathbf{d}$ is introduced, where $\mathbf{T}\mathbf{d} \in \tau\mathbb{Z}^{N\times 1}[j]$ since $\mathbf{T}$ is a complex

[3]For the 2-QAM and 8-QAM constellations specified in G.fast [2] the minimum distance is actually slightly larger and equals $2\sqrt{2}d$.

integer matrix. Now, the transmit vector $\mathbf{x}$ can be expressed as follows:

$$\mathbf{x} = \mathbf{P}\ddot{\mathbf{u}} = \mathbf{PT}\dot{\mathbf{u}} = \mathbf{P}\left(\mathbf{u} + \mathbf{Tv} + \mathbf{Td}\right), \tag{11}$$

which is to be compared with (4) for the LP scheme.

Let us then explain the purpose of the dither vector $\mathbf{v}$ in (6). Proposed in [19], [20], dithering is a common randomization technique, used here to assure that the power of $\mathbf{x}$ exactly meets the power constraint for every input distribution of $\mathbf{u}$, and to render the estimation error, defined below, (statistically) independent of $\mathbf{u}$. The dither vector is i.i.d. with uniform distribution over Ω_τ and independent of $\mathbf{u}$. It is added to the data vector at the transmitter and subtracted at the receiver, and should thus be known at both the transmitter and receiver (e.g. through the seed of a pseudo-random number generator). Based on dithering, the following proposition can be considered (as a generalization of [19, Lemma 1]):

Proposition 1: If the components of the i.i.d. dither vector $\mathbf{v}$ are uniformly distributed over Ω_τ with unit power and independent of $\mathbf{u}$, then the components of $\dot{\mathbf{u}} = \left[\mathbf{T}^{-1}\mathbf{u} + \mathbf{v}\right] \text{mod-}\tau$ are also independently and uniformly distributed over Ω_τ with unit power, regardless the distribution of $\mathbf{u}$. That is,

$$\mathcal{E}\{\dot{\mathbf{u}}\dot{\mathbf{u}}^H\} = \mathbf{I}_N \quad \text{and} \quad \mathcal{E}\{\ddot{\mathbf{u}}\ddot{\mathbf{u}}^H\} = \mathbf{T}\mathbf{T}^H. \tag{12}$$

Further, the components of $\mathbf{u}$ and $\dot{\mathbf{u}}$ are mutually independent.

Proof: See the proof in Appendix A. ∎

Notice that, as in [19], [20], random dithering does not increase capacity of the LRP scheme, but simplifies the analysis of the theoretical maximum achievable bit-rate in Section II-C.

Finally, the receiver of user n consists of a per-tone (complex) one-tap frequency domain equalizer (FEQ) followed by a subtraction of the dither vector and a second modulo operation to cancel out the first modulo operation, i.e.,

$$\begin{aligned}
\hat{u}_n &= \left[w_n y_n - [\mathbf{Tv}]_n\right] \text{mod-}\tau && (13)\\
&= \Big[\ddot{u}_n + \underbrace{(w_n y_n - \ddot{u}_n)}_{e_n} - [\mathbf{Tv}]_n\Big] \text{mod-}\tau && (14)\\
&= \left[u_n + [\mathbf{Td}]_n + e_n\right] \text{mod-}\tau\\
&= \left[u_n + e_n\right] \text{mod-}\tau, && (15)
\end{aligned}$$

where e_n is the estimation error consisting of the residual crosstalk interference and noise. From equation (15) it follows that a common base-τ for all users is indeed needed for the modulo operations in (6), instead of using a separate base-τ_n for each user n. In case of the latter, the perturbation term $[\mathbf{Td}]_n$ at the receiver of user n corresponds to a sum of multiple integers with different scalings $\{\tau_m\}$ which cannot be completely removed by a single mod-τ_n operation.

C. ACHIEVABLE BIT-RATE EXPRESSION

The key performance metric considered in this paper is the capacity in terms of the maximum achievable bit-rate. Assume that the complex data symbol u_n is uniformly distributed over Ω_τ with unit variance (i.e. with $\tau^2 = 6$ since the variance of a uniformly distributed variable in Ω_τ equals $\tau^2/6$) and denote the effective noise power as $\varepsilon_n \triangleq \mathcal{E}\{|e_n|^2\}$, where e_n is defined in (14). Then the maximum achievable bit-rate of user n (in bits/channel use) obtained by the LRP scheme can be shown to be lower bounded by:

$$\begin{aligned}
I(u_n; \hat{u}_n) &= h(\hat{u}_n) - h(\hat{u}_n|u_n)\\
&= 2\log_2(\tau) - h([e_n] \text{ mod-}\tau\,|u_n)\\
&= 2\log_2(\tau) - h([e_n] \text{ mod-}\tau)\\
&\geq 2\log_2(\tau) - \log_2(\pi e \varepsilon_n)\\
&= \log_2(1/\varepsilon_n) - \log_2(\pi e/6) && (16)
\end{aligned}$$

where $h(\cdot)$ denotes the differential entropy. The second and third equality hold since u_n is uniformly distributed over Ω_τ and independent of e_n, such that $\hat{u}_n$ is also uniformly distributed over Ω_τ (which is the maximum differential entropy of a random variable in Ω_τ). The inequality holds since $h([e_n] \text{ mod-}\tau)$ is upper bounded by the differential entropy of a circular symmetric Gaussian random variable with variance ε_n. The constant factor $1/2\log_2(\pi e/6)$ is the well-known shaping loss of 0.255 bits per dimension at high SNR. Note that IF precoding [13]–[15] is able to recover this shaping loss by using message precoding and nested lattice encoding and decoding on blocks of symbols, together with appropriate lattice construction and the dimensionality of the blocks going up to infinity (instead of using scalar per-symbol modulo operations). In addition, (16) is maximized by using a FEQ that minimizes the effective noise power ε_n, described by the following MMSE equalizer and MMSE expression

$$\begin{aligned}
w_n^{\text{opt}} &\triangleq \frac{\mathcal{E}\{y_n^* \ddot{u}_n\}}{\mathcal{E}\{|y_n|^2\}} = \frac{\sum_m \left[\mathbf{HP}\right]_{nm}^* \left[\mathbf{TT}^H\right]_{nm}}{\sum_m \sum_p \left[\mathbf{HP}\right]_{nm} \left[\mathbf{TT}^H\right]_{mp} \left[\mathbf{HP}\right]_{np}^* + \sigma}\\
&= \frac{\left[\mathbf{HPTT}^H\right]_{nn}^*}{\left[\mathbf{HPTT}^H\mathbf{P}^H\mathbf{H}^H + \sigma\mathbf{I}_N\right]_{nn}} && (17)\\
\varepsilon_n^{\text{opt}} &\triangleq \mathcal{E}\left\{\left|\ddot{u}_n - w_n^{\text{opt}} y_n\right|^2\right\}\\
&= \left[\mathbf{TT}^H\right]_{nn} - \frac{\left|\left[\mathbf{HPTT}^H\right]_{nn}\right|^2}{\left[\mathbf{HPTT}^H\mathbf{P}^H\mathbf{H}^H + \sigma\mathbf{I}_N\right]_{nn}}. && (18)
\end{aligned}$$

Although (16) provides a theoretical achievability, the SNR approximation gap is adopted for LRP in this paper, which is commonly used in DSL networks [11]. It guarantees practical achievability for the case that QAM constellations, practical coding schemes and noise margins are used. This practically achievable bit-rate is given by

$$b_n = \left\lfloor \log_2\left(1 + \frac{\text{SNR}_n(\mathbf{P}, \mathbf{T})}{\Gamma}\right)\right\rfloor^{b^{\max}} \tag{19}$$

where $\text{SNR}_n(\mathbf{P}, \mathbf{T})$ is the receiver SNR, and Γ is the SNR approximation gap or SNR gap. For uncoded square QAM constellations, the SNR gap is closely approximated to 9.58 dB for the G.fast target bit error rate (BER) of 10^{-7} [21]. When considering trellis coded modulation, a coding gain $\Gamma_c = 5.2$ dB may be subtracted from Γ_{unc} [22]. Additionally, in DSL a 6 dB noise margin is added. A maximum bit loading

(bit cap) $b^{\max}$ is also imposed. The total data-rate across all tones k is

$$R_n = f_s \sum_k b_k^n \tag{20}$$

where f_s is the DMT symbol rate.

To be able to use (19), it is necessary to neglect the modulo operation at the receiver. This approximation is tight at high SNR values, because in low-BER systems, where $\Pr(e_n > d)$ is very small, it holds for fairly large constellation sizes in (13) that

$$[u_n + e_n] \bmod\text{-}\tau \approx u_n + e_n, \quad \forall n. \tag{21}$$

As a result, the LRP scheme can be viewed as a set of N parallel AGWN channels, wherefore the effective noise power ε_n corresponds to a ZF-FEQ assumption,[4] i.e.,

$$w_n = \frac{\mathcal{E}\{|\ddot{u}_n|^2\}}{\mathcal{E}\{y_n \ddot{u}_n^*\}} = \frac{\left[\mathbf{T}\mathbf{T}^H\right]_{nn}}{\left[\mathbf{H}\mathbf{P}\mathbf{T}\mathbf{T}^H\right]_{nn}} \tag{22}$$

and

$$e_n = \frac{\left[\mathbf{T}\mathbf{T}^H\right]_{nn}}{\left[\mathbf{H}\mathbf{P}\mathbf{T}\mathbf{T}^H\right]_{nn}} \left(\sum_m [\mathbf{H}\mathbf{P}]_{nm} \ddot{u}_m + z_n \right) - \ddot{u}_n, \tag{23}$$

such that $\mathrm{SNR}_n(\mathbf{P}, \mathbf{T}) \triangleq \mathcal{E}\{|u_n|^2\} / \mathcal{E}\{|e_n|^2\}$ is given by

$$\frac{\left|\left[\mathbf{H}\mathbf{P}\mathbf{T}\mathbf{T}^H\right]_{nn}\right|^2 \Big/ \left[\mathbf{T}\mathbf{T}^H\right]_{nn}}{\left[\mathbf{T}\mathbf{T}^H\right]_{nn} \left(\left[\mathbf{H}\mathbf{P}\mathbf{T}\mathbf{T}^H\mathbf{P}^H\mathbf{H}^H\right]_{nn} + \sigma\right) - \left|\left[\mathbf{H}\mathbf{P}\mathbf{T}\mathbf{T}^H\right]_{nn}\right|^2}. \tag{24}$$

Notice that (24) simplifies to the well-known SNR expression for LP schemes when $\mathbf{T}$ is the identity matrix.

Two losses are however ignored in (19) arising from the non-linear modulo operations. Fortunately, since these losses are only significant for low-SNR channels (with small constellation sizes), they have a low impact on the total data-rates aggregated over the full G.fast spectrum, which are dominated by high-SNR channels.

A first loss is the so-called precoding loss resulting from the modulo operation at the transmitter, such that $\dot{u}_n$ is uniformly distributed over Ω_τ, having a slightly larger variance than the QAM distributed u_n (see e.g. [23]). Note that due to the common modulo-base τ in (9) for all users, this power loss is determined by the user with the smallest constellation, in contrast to THP where per-user modulo-bases τ_n are used [18].

A second loss is the neglected modulo loss in (21) entailing in practice a wrap-around effect at the edges of the QAM constellation where received symbols may be shifted to the other side of the constellation, increasing the average number of nearest neighbors. This loss is quantified in [22] for all constellations and channel coding schemes included in G.fast [2].

[4]Notice that in a true AGWN channel without a modulo operation at the receiver, the use of a MMSE-FEQ over a ZF-FEQ provides no capacity gain.

III. GENERAL LRP DESIGN

The general problem statement corresponds to a joint design of the precoder matrix $\mathbf{P}$ and integer matrix $\mathbf{T}$ in order to maximize the achievable WSR under per-line power constraints

$$\begin{aligned} \underset{\mathbf{P}\in\mathbb{C}^{N\times N}, \mathbf{T}\in\mathbb{Z}^{N\times N}[j]}{\text{maximize}} \quad & \sum_n \alpha_n b_n \\ \text{s.t.} \quad & \sum_m |[\mathbf{P}\mathbf{T}]_{lm}|^2 \le P^{\text{mask}}, \quad \forall l, \\ & \text{Rank}(\mathbf{T}) = N \end{aligned} \tag{25}$$

where α_n denotes the weight of user n and $l \in [1, \cdots, N]$ the line index. The full-rank constraint guarantees that $\mathbf{T}$ is invertible and $|\det(\mathbf{T})| \ge 1$, which is in fact a relaxation of the unimodularity constraint (i.e. $|\det(\mathbf{T})| = 1$). The LRP scheme is able to handle non-unimodular integer $\mathbf{T}$ matrices as well, since the bit-rate expression (19) is valid as long as $\dot{\mathbf{u}}$ is uniformly distributed and independent of e_n (which is always the case if dithering is used).

Being a mixed-integer non-convex problem, the joint design of $\mathbf{P}$ and $\mathbf{T}$ in (25) is a difficult task. Hence, it is appropriate to focus on solving two subproblems instead. The first subproblem amounts to optimizing the precoder matrix $\mathbf{P}$ given a fixed integer matrix $\mathbf{T}$. This subproblem will be addressed in Section IV and V for ZF-LRP and MMSE-LRP, respectively. The second subproblem corresponds to optimizing $\mathbf{T}$ given a fixed $\mathbf{P}$, i.e.,

$$\begin{aligned} \underset{\mathbf{T}\in\mathbb{Z}^{N\times N}[j]}{\text{maximize}} \quad & \sum_n \alpha_n b_n \\ \text{s.t.} \quad & \sum_m |[\mathbf{P}\mathbf{T}]_{lm}|^2 \le P^{\text{mask}}, \quad \forall l \\ & \text{Rank}(\mathbf{T}) = N. \end{aligned} \tag{26}$$

Unfortunately, due to its integer nature, solving (26) is still a difficult task. As a consequence, considering the large number of tones ($K = 4000$) in the G.fast spectrum, an efficient low-complexity solution of (26) is used, by means of the well-known (complex) LLL algorithm [24], [25] for lattice basis reduction, which produces a unimodular $\mathbf{T}$ matrix. Note that for the particular case of a ZF precoder with flat gain scaling and a sum-power constraint, (26) is equivalent to designing a $\mathbf{T}$ that minimizes the Frobenius norm of $\mathbf{P}\mathbf{T}$, corresponding to a successive minima problem for which an optimal algorithm is provided in [7].

Alternating between the two subproblems may significantly improve performance. This is motivated by the fact that an integer matrix $\mathbf{T}$ reducing $\mathbf{P}$, is not necessarily a good reducer for $\mathbf{P}'$ which has been optimized for $\mathbf{T}$. However, since the LLL algorithm is merely a suboptimal heuristic for solving (26), such an alternating algorithm does not always feature a monotonically increasing objective value. This is pragmatically handled here by limiting the number of iterations and retaining the $\{\mathbf{P}, \mathbf{T}\}$-pair that achieves the highest WSR. Nevertheless, as will be shown in Section VI, this approach proves to be effective in G.fast DSL networks.

IV. ZF-LRP DESIGN GIVEN A FIXED T

In this section ZF-LRP design is studied given a fixed integer $\mathbf{T}$ matrix. First, a successive lower bound maximization method for complex gain scaling is proposed, similar to power allocation in linear ZF precoding design. Second, it is established that the achievable ZF-LRP sum-rate is always upper bounded by the achievable ZF-THP sum-rate at high SNR values, leading to a sum-rate gap between both precoding schemes. This sum-rate gap is zero solely for the case that the (output power normalized) reduced ZF precoder is exactly unitary and $\mathbf{T}$ is unimodular.

A. GAIN SCALING OPTIMIZATION

The ZF criterion leads to the following precoder matrix structure[5]

$$\mathbf{P} = \mathbf{H}^{-1}\mathbf{S} \tag{27}$$

where $\mathbf{S} \triangleq \text{diag}\{\mathbf{s}\}$ is a complex diagonal gain scaling matrix with $\mathbf{s} \triangleq [s_1, \ldots, s_N]^T$, offering some degree of freedom that can be exploited to maximize performance. The corresponding gain scaling optimization problem is then formulated as:

$$\underset{\mathbf{s}\in\mathcal{S}}{\text{maximize}}\ f(\mathbf{s}) \tag{28a}$$

$$\text{s.t.} \sum_m \left|\left[\mathbf{H}^{-1}\mathbf{S}\mathbf{T}\right]_{lm}\right|^2 \leq P^{\text{mask}}, \quad \forall l, \tag{28b}$$

with

$$f(\mathbf{s}) = \sum_n \alpha_n \log_2\left(1 + \frac{|s_n|^2}{\Gamma\sigma}\right), \tag{29}$$

where $\mathcal{S} \triangleq \left\{\mathbf{s} \in \mathbb{C}^{N\times 1} \middle| |s_n| \leq \sqrt{\Gamma\sigma(2^{b^{\max}}-1)}, \forall n\right\}$ denotes the convex and compact set of gain scalars that satisfy the maximum bit loading constraint, which is translated here into a maximum magnitude constraint.

Fortunately, (28b) is a convex quadratic constraint. Defining $\mathbf{P}^{\text{red}} \triangleq \mathbf{H}^{-1}\mathbf{S}\mathbf{T}$ and $\tilde{h}_{nm} \triangleq \left[\mathbf{H}^{-1}\right]_{nm}$ and $t_{nm} \triangleq [\mathbf{T}]_{nm}$ yields

$$\begin{aligned}\left|\left[\mathbf{P}^{\text{red}}\right]_{lm}\right|^2 &= \sum_i \sum_j \tilde{h}_{li}\tilde{h}_{lj}^* s_i s_j^* t_{im} t_{jm}^* \\ &= \sum_i \left|\tilde{h}_{li} s_i t_{im}\right|^2 + \sum_i \sum_{j\neq i} s_i s_j^* \tilde{h}_{li}\tilde{h}_{lj}^* t_{im} t_{jm}^*,\end{aligned} \tag{30}$$

such that the transmit power on line l may be re-written as $\sum_m \left|\left[\mathbf{P}^{\text{red}}\right]_{lm}\right|^2 = \mathbf{s}^H\mathbf{A}_l\mathbf{s}$ with $\mathbf{A}_l$ given by

$$[\mathbf{A}_l]_{ji} = \begin{cases} \sum_m \left|\tilde{h}_{li} t_{im}\right|^2 & \text{if } i = j \\ \sum_m \tilde{h}_{li}\tilde{h}_{lj}^* t_{im} t_{jm}^* & \text{if } i \neq j. \end{cases} \tag{31}$$

Since $\sum_m \left|\left[\mathbf{P}^{\text{red}}\right]_{lm}\right|^2$ always produces a positive value, it follows that $\mathbf{s}^H\mathbf{A}_l\mathbf{s} > 0$ for any non-zero $\mathbf{s}$ vector, which means the $\{\mathbf{A}_l\}$-matrices are positive definite by design.

[5]This precoder structure is similar to the "diagonally-scaled integer forcing" precoder of [15].

Consequently, (28a) may be equivalently reformulated as

$$\underset{\mathbf{s}\in\mathcal{S}}{\text{maximize}}\ f(\mathbf{s}) \tag{32a}$$

$$\text{s.t. } \mathbf{s}^H\mathbf{A}_l\mathbf{s} \leq P^{\text{mask}}, \quad \forall l. \tag{32b}$$

Unfortunately, the objective function $f(\mathbf{s})$ is non-concave due to the absolute squared gain scalars $|s_n|^2$, making (32) a non-convex problem in general. Observe that when $\mathbf{T} = \mathbf{I}_N$ (i.e. the case of ZF-LP), the $\{\mathbf{A}_l\}$-matrices are diagonal and (32b) is solely in function of $|s_n|^2$. Hence, in this case the $|s_n|^2, \forall n$ may be taken directly as optimization variables such that (28a) and (32) reduce to a standard convex power allocation problem with linear power constraints, solvable by means of the well-known water-filling method.

To solve (32) for any $\mathbf{T}$ matrix, it is used that $|s_n - \bar{s}_n|^2 \geq 0$ and thus $|s_n|^2 \geq 2\cdot\text{Re}(\bar{s}_n^* s_n) - |\bar{s}_n|^2$ for any complex scalar $\bar{s}_n$. In combination with $\log_2^+(x)$ being monotonically increasing in x, this leads to the following concave function[6]

$$g(\mathbf{s}|\bar{\mathbf{s}}) = \sum_n \alpha_n \log_2^+\left(1 + \frac{2\cdot\text{Re}(\bar{s}_n^* s_n) - |\bar{s}_n|^2}{\Gamma\sigma}\right) \tag{33}$$

which is a tight lower bound for $f(\mathbf{s})$ at any point $\bar{\mathbf{s}} \in \mathbb{C}^{N\times 1}$:

$$g(\mathbf{s}|\bar{\mathbf{s}}) \leq f(\mathbf{s}) \text{ for any } \mathbf{s} \in \mathbb{C}^{N\times 1} \tag{34}$$

$$g(\bar{\mathbf{s}}|\bar{\mathbf{s}}) = f(\bar{\mathbf{s}}). \tag{35}$$

Hence, based on (33) a successive lower bound maximization method[7] [26] is proposed, which generates a sequence of iterates $\mathbf{s}^{(t)}$ with non-decreasing objective values $f(\mathbf{s}^{(t)})$, i.e.,

$$f\left(\mathbf{s}^{(t)}\right) \geq g\left(\mathbf{s}^{(t)}|\mathbf{s}^{(t-1)}\right) \geq g\left(\mathbf{s}^{(t-1)}|\mathbf{s}^{(t-1)}\right) = f\left(\mathbf{s}^{(t-1)}\right) \quad \text{for all } t = 1, 2, \ldots \tag{36}$$

where $\mathbf{s}^{(t)}$ is the solution of

$$\underset{\mathbf{s}\in\mathcal{S}}{\text{maximize}}\ g\left(\mathbf{s}|\mathbf{s}^{(t-1)}\right) \tag{37a}$$

$$\text{s.t. } \mathbf{s}^T\mathbf{A}_l\mathbf{s} \leq P^{\text{mask}}, \quad \forall l. \tag{37b}$$

Although (37) is convex, it is still a generic non-linear program due to the sum-of-log-function (33) as objective. However, (37) may be transformed into an equivalent second-order cone program (SOCP), which is in general more efficiently solved by standard solvers such as e.g. MOSEK [27] of the optimization tool CVX [28]. This transformation is based on re-writing the objective (37a) as a geometric product (or mean)

$$\prod_n \left(\Gamma\sigma + 2\cdot\text{Re}\left(\left(\bar{s}_n^{(t-1)}\right)^* s_n\right) - \left|\bar{s}_n^{(t-1)}\right|^2\right)^{\tilde{\alpha}_n} \tag{38}$$

where the scaling $\tilde{\alpha}_n \triangleq \alpha_n / \max_m(\alpha_m)$ ensures $0 \leq \tilde{\alpha}_n \leq 1, \forall n$, such that $\prod_n (x_n)^{\tilde{\alpha}_n}$ is always concave for positive x_n [29]. In turn, the geometric product can be replaced by

[6]Here $\log_2^+(x)$ denotes the continuous-value extension of $\log_2(x)$ to negative x, with $\log_2^+(x) = -\infty$ if $x \leq 0$ and $\log_2^+(x) = \log_2(x)$ if $x > 0$.

[7]Also known as the Majorization-Minimization (MM) method.

Algorithm 1 A Successive Lower bound Maximization Method for Solving Problem (32)

Find a feasible point $\mathbf{s}^{(0)}$
Set $t = 0$
repeat
  Set $t = t + 1$
  Obtain $\mathbf{s}^{(t)}$ by solving problem (37) with $g(\mathbf{s}|\mathbf{s}^{(t-1)})$.
until $f(\mathbf{s}^{(t)}) - f(\mathbf{s}^{(t-1)}) < \epsilon$

a system of SOC constraints [30], which may be implemented in Matlab by using the *geo_mean* function of the CVX tool.

The complete algorithm (summarized in Alg. 1) is guaranteed to converge to a stationary solution as outlined in Theorem 2 below. Further, solving the (transformed) SOCP (37) every outer iteration in Alg. 1 with the interior-point method requires worst-case $\mathcal{O}(\sqrt{N}\log(1/\epsilon))$ iterations up to accuracy ϵ, with a per-iteration complexity of approximately $\mathcal{O}(N^2\sum_n N + N^3)$, amounting to forming and solving the Newton system [30]. Hence, the total computational complexity of Alg. 1 is $\mathcal{O}(I_o N^{4.5}\log(1/\epsilon))$, where I_o is the number of outer iterations. Additionally, the generalization of Alg. 1 to the case with active total per-line power constraints is provided in Appendix B.

Theorem 2: The sequence of iterates $\{\mathbf{s}^{(t)}\}$ generated by Alg. 1 is guaranteed to converge to the set of stationary points of (32), and equivalently of (28a).

Proof: It is already established that $g(\mathbf{s}|\bar{\mathbf{s}})$ is a continuous tight lower bound of the objective $f(\mathbf{s})$ in (34) and (35); and that the sequence of generated objective values $f(\mathbf{s}^{(t)})$ is monotonically non-decreasing in (36). In addition, it can be verified that $g(\mathbf{s}|\bar{\mathbf{s}})$ has the same first-order derivative as the original objective $f(\mathbf{s})$ at the point where the lower bound is tight (i.e in point $\mathbf{s} = \bar{\mathbf{s}}$). Hence, in combination with the iterates $\{\mathbf{s}^{(t)}\}$ lying in a closed and bounded set, convergence of $\{\mathbf{s}^{(t)}\}$ can be established to the set of stationary points of (32) [26]. It is remarked that the results in [26] are also valid for the case with complex variables. ■

B. ACHIEVABLE SUM-RATE GAP WITH ZF-THP

In this section, it will be established that the achievable sum-rate of ZF-LRP is always smaller than the achievable sum-rate of ZF-THP at high SNR values, leading to a sum-rate gap between both precoding schemes. Recall that ZF-LRP uses the reduced precoder matrix $\mathbf{P}^{\text{red}} = \mathbf{H}^{-1}\mathbf{S}\mathbf{T}$ where $\mathbf{T}$ is a (possibly) non-unimodular integer matrix with $|\det(\mathbf{T})| \geq 1$. To satisfy the per-line power constraint P, the scaling matrix $\mathbf{S}$ has to be chosen such that the row norms of $\mathbf{P}^{\text{red}}$ are less than unity (assuming that $\mathbf{S}$ is normalized by $\sqrt{P}$). Consequently, Hadamard's inequality yields $|\det(\mathbf{P}^{\text{red}})| \leq 1$. In turn, this yields $|\det(\mathbf{S})| = \left|\det(\mathbf{P}^{\text{red}})\right| \cdot |\det(\mathbf{H})| \cdot |\det(\mathbf{T})|^{-1} \leq |\det(\mathbf{H})|$. The achievable sum-rate of ZF-LRP is approximated as follows:

$$R_{\text{LRP}} = \sum_n \log_2\left(1 + \frac{|s_n|^2 P}{\Gamma\sigma}\right)$$
$$= \log_2\left(\prod_n \left(1/|s_n|^2 + P/\Gamma\sigma\right)\prod_m |s_m|^2\right) \quad (39a)$$
$$\approx N\log_2\left(\frac{P}{\Gamma\sigma}\right) + 2\log_2|\det(\mathbf{S})|\,. \quad (39b)$$

where the approximation in (39b) is valid at high SNR values with $|s_n|^2 \gg \Gamma\sigma/P$.

ZF-THP [4], on the other hand, is based on the QR decomposition of the conjugate transpose of the channel matrix, i.e., $\mathbf{H}^H \overset{\text{qr}}{=} \mathbf{Q}\mathbf{R}$. The implementation then combines a feedforward (orthogonal) precoder matrix $\mathbf{P} = \mathbf{Q}\tilde{\mathbf{S}}$, with $\tilde{\mathbf{S}}$ a diagonal scaling matrix, that has to be designed, together with a sequential feedback operation. Also here the scaling matrix $\tilde{\mathbf{S}}$ has to be chosen such that the row norms of the feedforward precoder matrix $\mathbf{P}$ are less than unity (assuming that $\tilde{\mathbf{S}}$ is normalized by $\sqrt{P}$). This yields by Hadamard's inequality that $|\det(\mathbf{P})| = \left|\det(\tilde{\mathbf{S}})\right| = \prod_n |\tilde{s}_n| \leq 1$. In addition, it holds that $|\det(\mathbf{H})| = |\det(\mathbf{R}^H)| = \prod_n |r_{nn}|$ where r_{nn} are the diagonal values of $\mathbf{R}^H$. This leads to the following achievable sum-rate for ZF-THP [4]:

$$R_{\text{THP}} = \sum_n \log_2\left(1 + \frac{|r_{nn}|^2|\tilde{s}_n|^2 P}{\Gamma\sigma}\right)$$
$$= \log_2\left(\prod_n \left(1/|r_{nn}|^2|\tilde{s}_n|^2 + P/\Gamma\sigma\right)\prod_m |r_{mm}|^2|\tilde{s}_m|^2\right) \quad (40a)$$
$$\overset{(i)}{\approx} N\log_2\left(\frac{P}{\Gamma\sigma}\right) + 2\log_2|\det(\mathbf{H})| + 2\log_2\left|\det\left(\tilde{\mathbf{S}}\right)\right|$$
$$\overset{(ii)}{=} N\log_2\left(\frac{P}{\Gamma\sigma}\right) + 2\log_2|\det(\mathbf{H})|\,, \quad (40b)$$

where the approximation in (i) is valid at high SNR values with $|r_{nn}|^2|\tilde{s}_n|^2 \gg \Gamma\sigma/P$, and (ii) is due to the fact that flat gain scaling (i.e. setting $|\tilde{s}_n| = 1, \forall n$) is optimal for ZF-THP at high SNR values. However, for low/moderate SNR values, some spread in the $1/|r_{nn}|^2|\tilde{s}_n|^2 + P/\Gamma\sigma$ factors may counter the loss of $\prod_n |\tilde{s}_n|$ being smaller than unity.

Subtracting (39b) from (40b) leads thus to the achievable sum-rate gap ΔR between the ZF-THP and ZF-LRP, approximated at high SNR values by

$$\Delta R \approx 2\log_2\frac{|\det(\mathbf{H})|}{|\det(\mathbf{S})|} = 2\log_2|\det(\mathbf{T})| - 2\log_2\left|\det(\mathbf{P}^{\text{red}})\right| \geq 0. \quad (41)$$

In order for this bound to be tight: (1) the (output power normalized) reduced precoder matrix $\mathbf{P}^{\text{red}}$ needs to have orthogonal rows with unit norm by Hadamard's inequality, corresponding to a *unitary* matrix (such that $|\det(\mathbf{P}^{\text{red}})| = 1$); and (2) the $\mathbf{T}$ matrix has to be *unimodular* (such that

$|\det(\mathbf{T})| = 1$). A unimodular $\mathbf{T}$ matrix leads to $\prod_n |s_n| = |\det(\mathbf{S})| \leq |\det(\mathbf{H})| = \prod_n |r_{nn}|$ being tight when $\mathbf{P}^{\text{red}}$ is unitary.

By consequence, if $\mathbf{P}^{\text{red}}$ is not exactly orthogonal, some gain can be expected from lattice reducing the ZF precoder $\mathbf{H}^{-1}\mathbf{S}$, in an attempt to further improve the orthogonality of $\mathbf{P}^{\text{red}}$. This motivates the heuristic method proposed in Section III that alternates between lattice reduction of $\mathbf{H}^{-1}\mathbf{S}$ and gain scaling optimization of $\mathbf{S}$ given $\mathbf{T}$. In addition, if instead of e.g. the LLL algorithm which results in $|\det(\mathbf{T})| = 1$, a non-unimodular lattice reduction algorithm is used (see e.g. [7]), the loss from $|\det(\mathbf{T})| > 1$ should be compensated by the extra orthogonality gain in $|\det(\mathbf{P}^{\text{red}})|$.

Finally, it is pointed out that these conclusions are in line with an information-theoretic point of view, since THP can be viewed as a practical implementation of ''dirty paper coding'', which is the capacity-achieving transmission scheme for the broadcast channel [31].

V. MMSE-LRP DESIGN GIVEN A FIXED T

Instead of considering a ZF precoder matrix with complex gain scaling optimization, the goal in this section is to optimally design the entire precoder matrix given a fixed integer $\mathbf{T}$ matrix. The MMSE-LRP design is formulated as the following WSR maximization problem:

$$\underset{\mathbf{P}}{\text{maximize}} \sum_n \alpha_n \log_2\left(1 + \frac{\text{SNR}_n(\mathbf{P},\mathbf{T})}{\Gamma}\right) \quad (42a)$$

$$\text{s.t.} \left[\mathbf{P}\mathbf{T}\mathbf{T}^H\mathbf{P}^H\right]_{ll} \leq P^{\text{mask}}, \quad \forall l \quad (42b)$$

where $\text{SNR}_n(\mathbf{P},\mathbf{T})$ is defined in (24) and (42b) is a convex quadratic constraint. Unfortunately, (42) is a difficult problem, with a non-convex objective function even when $\mathbf{T} = \mathbf{I}_N$ (i.e. the case of linear MMSE precoding).

A feasible approach to solve (42) is to approximate the objective by using the theoretically achievable bit-rate expression defined in (16). Dropping the constant shaping loss in (16), it is actually equivalent to the achievable bit-rate of IF precoding schemes [14], [15] in the broadcast channel. Assuming optimal MMSE-FEQs [see (17) and (18)], the achievable WSR is

$$\begin{aligned} C &= \sum_n \alpha_n \log_2\left(\frac{1}{\varepsilon_n^{\text{opt}}}\right) \\ &= \sum_n \alpha_n \log_2\left(\frac{1}{\left[\mathbf{T}\mathbf{T}^H\right]_{nn}} + \text{SNR}_n(\mathbf{P},\mathbf{T})\right), \end{aligned} \quad (43)$$

leading to the following WSR maximization problem:

$$\begin{aligned} &\underset{\mathbf{P}}{\text{maximize}}\ C \\ &\text{s.t.} \left[\mathbf{P}\mathbf{T}\mathbf{T}^H\mathbf{P}^H\right]_{ll} \leq P^{\text{mask}}, \quad \forall l. \end{aligned} \quad (44)$$

Note that for high SNR values problem (42) and (44) are equivalent, meaning that the optimum precoder $\mathbf{P}$ will be identical for both problems. Further, the maximum bit cap constraints are omitted in (42) and (44) because these are non-convex in general for MMSE precoding.[8]

Nevertheless the approximate problem (44) remains non-convex, its main advantage is that it may be reformulated as an equivalent WMMSE problem and then solved by a block coordinate descend method leading eventually to a stationary point of (44). The equivalence between WSR maximization and WMMSE minimization has been established first for LP in the MIMO broadcast channel [32] and later extended to the MIMO interference channel in [33]. In addition, it can be readily extended to problem (44) for IF precoding in the broadcast channel as well, as outlined below:

Proposition 3: Define $\boldsymbol{\mu} \triangleq [\mu_1, \ldots, \mu_N]$ with μ_n the MSE weight for user n, and $\mathbf{w} \triangleq [w_1, \ldots, w_N]$ with w_n the equalizer tap for user n, and ε_n as the corresponding MSE for user n given by

$$\begin{aligned} \varepsilon_n \triangleq \mathcal{E}\left\{|\ddot{u}_n - w_n y_n|^2\right\} &= \left[\mathbf{T}\mathbf{T}^H\right]_{nn} - w_n^*\left[\mathbf{H}\mathbf{P}\mathbf{T}\mathbf{T}^H\right]_{nn}^* \\ &\quad - w_n\left[\mathbf{H}\mathbf{P}\mathbf{T}\mathbf{T}^H\right]_{nn} + w_n w_n^*\left[\mathbf{H}\mathbf{P}\mathbf{T}\mathbf{T}^H\mathbf{P}^H\mathbf{H}^H + \sigma\mathbf{I}_N\right]_{nn}. \end{aligned} \quad (45)$$

Then the following problem

$$\underset{\boldsymbol{\mu},\mathbf{w},\mathbf{P}}{\text{minimize}} \sum_n \alpha_n\left(\mu_n\varepsilon_n - \log(\mu_n) - 1\right) \quad (46a)$$

$$\text{s.t.} \left[\mathbf{P}\mathbf{T}\mathbf{T}^H\mathbf{P}^H\right]_{ll} \leq P^{\text{mask}}, \quad \forall l \quad (46b)$$

is equivalent to the WSR maximization problem (44), such that the optimum precoder matrix $\mathbf{P}$ for (44) and (46) will be identical. This equivalence can be shown by checking the first-order conditions for $\boldsymbol{\mu}$ and $\mathbf{w}$ [see (47)], substituting them in (46) and making some straightforward simplifications.

The essence of the equivalent WMMSE problem (46) is that the objective function is convex in each of the optimization variables when the other variables are fixed. This allows the use of a block coordinate descent method by successively updating the variables to solve (46) [33]. For fixed $\mathbf{P}$, the first order conditions of w_n and μ_n lead to the following update expressions for w_n and μ_n:

$$w_n^{\text{opt}} = \text{r.h.s. of (17)} \quad \text{and} \quad \mu_n^{\text{opt}} = \varepsilon_n^{-1}, \quad \forall n. \quad (47)$$

For fixed $\boldsymbol{\mu}$ and $\mathbf{w}$ on the other hand, (46) reduces to the following convex problem:

$$\begin{aligned} \underset{\mathbf{P}}{\text{minimize}}\ &\text{Tr}\left\{\mathbf{D}\mathbf{W}\mathbf{H}\mathbf{P}\mathbf{T}\mathbf{T}^H\mathbf{P}^H\mathbf{H}^H\mathbf{W}^H\right\} \\ &\quad -2\text{Re}\left(\text{Tr}\left\{\mathbf{D}\mathbf{W}\mathbf{H}\mathbf{P}\mathbf{T}\mathbf{T}^H\right\}\right) \quad (48a) \\ \text{s.t.}\ &\left[\mathbf{P}\mathbf{T}\mathbf{T}^H\mathbf{P}^H\right]_{ll} \leq P^{\text{mask}}, \quad \forall l, \quad (48b) \end{aligned}$$

where $\mathbf{W} = \text{diag}\{\mathbf{w}\}$ and $\mathbf{D} = \text{diag}\{\{\mu_n\alpha_n\}_{n=1}^N\}$. As strong duality holds in problem (48), Lagrange dual decomposition

[8] In G.fast, bit cap constraints are typically only active for low frequency tones with diagonally dominant channels. Hence, for these tones, the ZF precoder with gain scaling optimization is near-optimal.

Algorithm 2 WSR Maximization for MMSE-LRP

Find a feasible precoder $\mathbf{P}$
repeat
  Update the FEQs $\{w_n\}_{n=1}^N$ and MSE $\{e_n\}_{n=1}^N$ according to (17) and (45), respectively
  Update the MSE weights $\{\mu_n\}_{n=1}^N$ using (47)
  repeat
    Set/update$\{\lambda_n\}_{n=1}^N$ using the subgrad. method
    Update the MMSE precoder $\mathbf{P}$ with (49)
  until $\lambda_n \left|\left[\mathbf{PTT}^H\mathbf{P}^H\right]_{nn} - P^{\text{mask}}\right| < \epsilon \quad \forall n$
until convergence

TABLE 1. Performance comparison between various NLP schemes for the G.fast 212 MHz profile in a 10 × 80 m cable binder.

	Precoding scheme	Mean-Rate [Mbps]	Mean-Rate [%]	Min-Rate [Mbps]	Min-Rate [%]
LRP	ZF-LRP-RNS	1213	86.5	1213	89.0
	ZF-LRP [10]	1263	90.0	1227	90.0
	ZF-LRP (Alg. 1)	1281	91.4	1249	91.6
	ZF-LRP-AO	1319	94.1	1260	92.4
	MMSE-LRP	1287	91.8	1255	92.1
	MMSE-LRP-AO	1330	94.9	1248	91.6
THP	ZF-THP Max-Min	1369	97.6	1364	100
	ZF-THP Max-Sum	1397	99.6	1213	88.9
	MMSE-THP	1402	100	1200	87.9

can be used in order to obtain an optimal closed-form solution [33], i.e.,

$$\mathbf{P}^{\text{opt}} = \left(\mathbf{H}^H\mathbf{W}^H\mathbf{DWH} + \text{diag}\{\boldsymbol{\lambda}\}\right)^{-1}\mathbf{H}^H\mathbf{W}^H\mathbf{D}, \quad (49)$$

where $\boldsymbol{\lambda} \triangleq [\lambda_1, \cdots, \lambda_N]$ are the Lagrange multipliers, which should be chosen such that the per-line power constraints (48b) are either tight or inactive. These Lagrange multipliers may be found using e.g. standard subgradient search with

$$\lambda_n = \left[\lambda_n + \delta\left(\left[\mathbf{PTT}^H\mathbf{P}^H\right]_{nn} - P^{\text{mask}}\right)\right]^+, \quad \forall n, \quad (50)$$

and δ being a pre-defined step size, or by using the ellipsoid method [34].

The block coordinate descent method is summarized in Alg. 2. When $\mathbf{T}$ is the identity matrix (i.e. the case of LP), problem (42) and (44) become equivalent (for zero SNR gap), and Alg. 2 reduces to the original WMMSE algorithm for the broadcast channel. In addition, Alg. 2 may be generalized to the case with active total per-line power constraints, see (3), by re-introducing the tone indices and including additional Lagrange multipliers $\boldsymbol{\theta} \triangleq [\theta_1, \cdots, \theta_N]$ in (49). Further details are omitted for brevity.

VI. G.fast CABLE BINDER SIMULATION

In this section, a cable binder is simulated consisting of 10 lines of 80 m for the downstream G.fast 212 MHz profile. The channel matrices have been obtained by measurements. The observed crosstalk levels in this particular cable binder are rather high compared to other reported G.fast measurements. Following the G.fast recommendation [2], the ATP constraints are set to 8 dBm while the per-tone PSD spectral masks are obtained from [35] ranging from −65 dBm/Hz to −79 dBm/Hz. The total SNR gap Γ is 10.37 dB and the tone spacing Δ_f is 51.75 kHz. The noise PSD is assumed to be −140 dBm/Hz. The symbol rate is 48 kHz and the bitcap is 14. Only sum-rate optimization with $\{\alpha_n = 1\}_{n=1}^N$ is considered. In these simulations, the ATP constraints are observed to be always inactive, due to the per-tone spectral mask and maximum bit loading constraints. For this cable binder, various NLP schemes are evaluated:

- *ZF-LRP-RNS*: represents the baseline LRP scheme with ZF precoding and a flat row norm scaling (RNS). The precoder matrix is hence $\mathbf{P} = \mathbf{H}^{-1}s$, where $\mathbf{T} = \text{LLL}(\mathbf{H}^{-1})$ is obtained by the lattice reduction of $\mathbf{H}^{-1}$

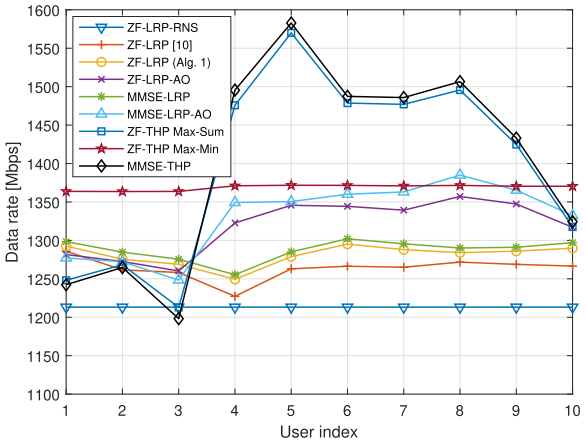


FIGURE 2. User data-rates of various NLP schemes in a 10-user cable binder with the G.fast 212 MHz profile.

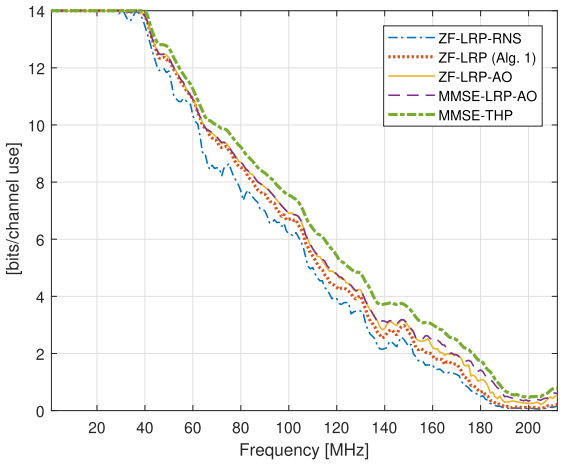
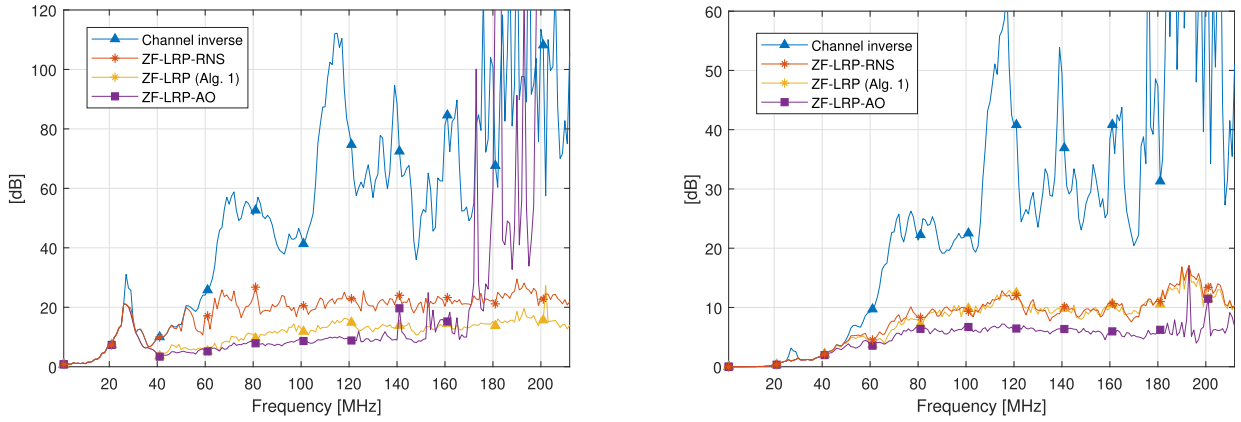


FIGURE 3. Mean-user bit-rates across the frequency range in a 10-user cable binder with the G.fast 212 MHz profile. The values are smoothened by averaging over 1 MHz frequency bins.



(a) Factor ψ (b) Othogonality Defect

FIGURE 4. Factor ψ and the orthogonality effect across the frequency range. Channel inverse corresponds to setting **T** to the identity matrix together with RNS. The values are smoothened by averaging over 1 MHz frequency bins. (a) Factor ψ. (b) Othogonality defect.

via the LLL algorithm and s is set to the square-root of

$$\left[\frac{P^{\text{mask}}}{\max_l \sum_m \left|\left[\mathbf{H}^{-1}\mathbf{T}\right]_{lm}\right|^2}\right]^{\Gamma\sigma(2^{b^{\max}}-1)}. \tag{51}$$

- *ZF-LRP (Alg. 1)*: uses $\mathbf{P} = \mathbf{H}^{-1}\mathbf{S}$ with $\mathbf{T} = \text{LLL}(\mathbf{H}^{-1})$ and the scaling matrix $\mathbf{S}$ obtained by means of Alg. 1.
- *ZF-LRP* [10]: is the ZF-LRP approach proposed in [10], by means of Alg. 1 with a *real* gain scaling matrix $\mathbf{S}$ and a high SNR approximation, and $\mathbf{T} = \text{LLL}(\mathbf{H}^{-1})$.
- *ZF-LRP-AO*: is the final ZF-LRP scheme, as proposed in Section III, which uses alternating optimization (AO) between gain scaling optimization of $\mathbf{S}$ for fixed $\mathbf{T}$ (using Alg. 1) and lattice reduction of $\mathbf{T} = \text{LLL}(\mathbf{H}^{-1}\mathbf{S})$.
- *MMSE-LRP*: uses the optimized MMSE precoder matrix obtained by means of Alg. 2 given $\mathbf{T} = \text{LLL}(\mathbf{H}^{-1})$.
- *MMSE-LRP-AO*: alternates between MMSE precoder matrix optimization and lattice reduction.
- *ZF-THP Max-Min*: corresponds to α-fair QRD-based THP with joint power allocation and user ordering [36]. A large α is used here (i.e. min-rate maximization)
- *ZF-THP Max-Sum*: is α-fair QRD-based THP with $\alpha = 0$ (i.e. sum-rate maximization).
- *MMSE-THP*: is provided by the BC-DSB-NLP [37] together with the same user ordering of ZF-THP Max-Sum.

Foremost among the numerical results, summarized in Table 1 and Fig. 2, is that LRP consistently is outperformed by THP, both in terms of performance and fairness (i.e. in terms of multi-tone sum-rate and min-rate, respectively). This is expected from the results in Section IV-B, where the achievable ZF-LRP sum-rate is shown to be upper bounded at high SNR by the achievable ZF-THP sum-rate. Additionally, ZF-THP in combination with joint fair power allocation and user ordering [36] is able to achieve a higher minimum user rate than LRP. However, as mentioned in Section I, the main disadvantage of THP is the required sequential feedback operation proportional to the number of users, which complicates hardware implementation. In addition, LRP typically has a natural (more or less) fair user rate distribution, while THP without fairness aware user encoding ordering optimization typically results in a large user rate spread. As expected, the proposed ZF-LRP (Alg. 1) with complex scaling achieves a higher sum-rate than ZF-LRP [10] with real scaling, and significantly outperforms ZF-LRP-RNS. It is stressed that although the total bit-rate gains summed over all frequency tones might be rather small, the relative bit-rate gains on an individual tone can be quite significant (see Fig. 3).

To illustrate the joint design of $\mathbf{S}$ and $\mathbf{T}$ for ZF precoding, the factor $\psi \triangleq \prod_n |r_{nn}|/|\tilde{s}_n| \geq 1$ together with the so-called orthogonality defect[9] [38] of the ZF precoders across the frequency range is shown in Fig. 4. The former determines the sum-rate gap (41) with ZF-THP at high SNR, whereas the latter is a measure for matrix orthogonality. Fig. 4 shows that the lattice reduction of $\mathbf{T} = \text{LLL}(\mathbf{H}^{-1})$ substantially reduces the orthogonality defect of the channel inverse, leading to a significant sum-rate-gain for ZF-LRP-RNS. Additional scaling of the ZF precoder by optimizing $\mathbf{S}$ (i.e. ZF-LRP) then further reduces factor ψ, without significantly improving the orthogonality defect. It thus seems that gain scaling optimization primarily leads to the inequality $\prod_n |\tilde{s}_n| \leq \prod_n |r_{nn}|$ being as tight as possible given a certain $\mathbf{T}$, which maximizes the sum-rate at high SNR. Further improving the orthogonality of the reduced and scaled ZF precoder (and hence the sum-rate at high SNR) is the goal of ZF-LRP-AO, which alternates between gain scaling optimization of $\mathbf{S}$ and lattice reduction of $\mathbf{T} = \text{LLL}(\mathbf{H}^{-1}\mathbf{S})$. It results in a further reduction of both the factor ψ and the orthogonality defect, which illustrates the potential of joint gain scaling and lattice reduction design (i.e. to jointly optimize $\mathbf{S}$ and $\mathbf{T}$). Notice that some users may be allocated low or zero $|s_n|$ magnitudes at high

[9]The orthogonality defect of the ZF precoder $\tilde{\mathbf{P}} \triangleq \mathbf{H}^{-1}\mathbf{S}\mathbf{T}$ with $\tilde{\mathbf{P}} = [\mathbf{p}_1, \ldots, \mathbf{p}_N]$ is computed by $\prod_i \|\mathbf{p}_i\|/\sqrt{|\det(\tilde{\mathbf{P}}^H\tilde{\mathbf{P}})|}$, and with any zero columns of $\tilde{\mathbf{P}}$ removed.

frequency tones with low SNR, which blows up factor ψ for ZF-LRP-AO.

VII. CONCLUSION

In this paper, the potential of LRP for downstream transmission in G.fast DSL networks has been studied, as an NLP alternative for THP with a reduced implementation complexity. A practically achievable bit-rate expression has been proposed for LRP, in order to cast the design problem as a WSR maximization in function of a precoder matrix and an integer matrix. For a fixed integer matrix, ZF precoding optimization reduces to a gain scaling optimization, for which a successive lower bound maximization method has been presented. Additionally, it has been established that the achievable ZF-LRP sum-rate is upper bounded by the achievable ZF-THP sum-rate at high SNR. For MMSE precoding optimization an efficient method has been proposed as well, by relying on the equivalence between the WSR maximization and the WMMSE minimization. The LRP design methods may be improved by alternating between precoder matrix optimization for a fixed integer matrix, and updating the integer matrix based on the lattice reduction of the precoder matrix. Simulations of a measured G.fast cable binder have been provided to compare the performance of the proposed LRP schemes with THP.

APPENDIX A
PROOF OF PROPOSITION 1

First, the property $\text{mod}(a+b) = \text{mod}(\text{mod}(a) + \text{mod}(b))$ is used to observe that for every user n:

$$\dot{u}_n = \left[X_n + v_n\right] \text{mod-}\tau, \tag{52}$$

where $X_n \triangleq \left[\sum_m [\mathbf{T}^{-1}]_{nm} u_m\right] \text{mod-}\tau$ forms a random variable in Ω_τ, which is independent of the uniform dither variable v_n, due to the fact that (measurable) functions of independent variables are independent variables themselves (see e.g. [39, Th. 2.1.6]). See that $\dot{u}_n$ is uniformly distributed over Ω_τ for any fixed value of $X_n = x \in \Omega_\tau$. Since this holds for any value x, it holds for any combination of values as well, meaning that $\dot{u}_n$ is uniformly distributed over Ω_τ. Further, the probability density of $\dot{u}_n$ is constant over Ω_τ for any fixed value of the variables $\{u_1, \ldots, u_N\}$, i.e. knowing all the variables $\{u_1, \ldots, u_N\}$ gives no information about $\dot{u}_n$ due to the randomizing dither variable v_n. This means that $\dot{u}_n$ is mutually independent of $\{u_1, \ldots, u_N\}$. Moreover, since the dither variables $\{v_1, \ldots, v_N\}$ are independently distributed across the users, the variables $\{\dot{u}_1, \ldots, \dot{u}_N\}$ can also shown to be independently distributed, by checking that for every n the conditional probability of $\dot{u}_n$ for fixed $\{\dot{u}_1, \ldots, \dot{u}_{n-1}, \dot{u}_{n+1}, \ldots, \dot{u}_N\}$ is constant over Ω_τ.

APPENDIX B
GENERALIZATION OF ALG. 1 TO THE CASE WITH ACTIVE TOTAL PER-LINE POWER CONSTRAINTS

Alg. 1 can be generalized to the case with active total per-line power constraints [see (3)], which then couple the K convex subproblems in (37) at iteration t. This is formulated as follows by re-introducing the tone indices k:

$$\begin{aligned} &\underset{\{\mathbf{s}_k \in \mathcal{S}_k\}}{\text{maximize}} \sum_k g_k\left(\mathbf{s}_k | \mathbf{s}_k^{(t-1)}\right) \\ &\text{s.t. } \mathbf{s}_k^T \mathbf{A}_{lk} \mathbf{s}_k \leq P_k^{\text{mask}}, \quad \forall l, k \\ &\quad \sum_k \mathbf{s}_k^T \mathbf{A}_{lk} \mathbf{s}_k \leq P^{\text{total}}, \quad \forall l. \end{aligned} \tag{53}$$

Since strong duality holds in (53), it may be decoupled across the tones by using Lagrange dual decomposition. Then, for every fixed set of Lagrange multipliers $\boldsymbol{\theta} \triangleq [\theta_1, \cdots, \theta_N]$, the independent per-tone slave problem for tone k is given by

$$\begin{aligned} h_k(\boldsymbol{\theta}) = &\underset{\mathbf{s}_k \in \mathcal{S}}{\text{maximize}}\ g_k\left(\mathbf{s}_k | \mathbf{s}_k^{(t-1)}\right) - \sum_{l=1}^{N} \theta_n \left(\mathbf{s}_k^T \mathbf{A}_{lk} \mathbf{s}_k - P^{\text{total}}\right) \\ &\text{s.t. } \mathbf{s}_k^T \mathbf{A}_{lk} \mathbf{s}_k \leq P_k^{\text{mask}}, \quad \forall l, \end{aligned} \tag{54}$$

which is again a small-scale convex problem that may be solved by e.g. CVX. The master dual problem of (53) is

$$\underset{\boldsymbol{\theta} \succeq \mathbf{0}}{\text{minimize}} \sum_k h_k(\boldsymbol{\theta}), \tag{55}$$

where the optimal $\boldsymbol{\theta}$ are found with e.g. a standard subgradient search

$$\theta_l = \left[\theta_l + \delta \left(\sum_k \mathbf{s}_k^T \mathbf{A}_{lk} \mathbf{s}_k - P^{\text{total}}\right)\right]^+, \quad \forall l, \tag{56}$$

which is guaranteed to converge if the step size δ is chosen sufficiently small [34].

REFERENCES

[1] W. Lanneer, "Vectoring design optimization in ultra-broadband DSL networks," Ph.D. dissertation, Dept. Elect. Eng., KU Leuven, Leuven, Belgium, Feb. 2019.
[2] *Fast Access to Subscriber Terminals (FAST)–Physical Layer Specification*, document recommendation ITU-T G.9701-Amendment 3, Apr. 2017.
[3] R. Strobel, R. Stolle, and W. Utschick, "Wideband modeling of twisted-pair cables for MIMO applications," in *Proc. IEEE Global Commun. Conf. (GLOBECOM)*, Dec. 2013, pp. 2828–2833.
[4] G. Ginis and J. Cioffi, "A multi-user precoding scheme achieving crosstalk cancellation with application to DSL systems," in *Proc. 34th Asilomar Conf. Signals, Syst. Comput.*, vol. 2, Nov. 2002, pp. 1627–1631.
[5] J. Maes, C. Nuzman, and P. Tsiaflakis, "Sensitivity of nonlinear precoding to imperfect channel state information in G.fast," in *Proc. 24th Eur. Signal Process. Conf. (EUSIPCO)*, Aug. 2016, pp. 290–294.
[6] C. Windpassinger, R. Fischer, and J. Huber, "Lattice-reduction-aided broadcast precoding," *IEEE Trans. Commun.*, vol. 52, no. 12, pp. 2057–2060, Dec. 2004.
[7] S. Stern and R. F. Fischer, "Advanced factorization strategies for lattice-reduction-aided preequalization," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jul. 2016, pp. 1471–1475.
[8] C. Windpassinger and R. Fischer, "Low-complexity near-maximum-likelihood detection and precoding for MIMO systems using lattice reduction," in *Proc. IEEE Inf. Theory Workshop*, Mar. 2004, pp. 345–348.
[9] M. Taherzadeh, A. Mobasher, and A. K. Khandani, "Communication over MIMO broadcast channels using lattice-basis reduction," *IEEE Trans. Inf. Theory*, vol. 53, no. 12, pp. 4567–4582, Dec. 2007.
[10] M. Hekrdla, A. Matera, U. Spagnolini, and W. Wang, "Per-line power controlled lattice-reduction aided zero-forcing precoding for G.fast downstream," in *Proc. IEEE Global Commun. Conf. (GLOBECOM)*, Dec. 2016, pp. 1–6.
[11] T. Starr, J. M. Cioffi, and P. J. Silverman, *Understanding Digital Subscriber Line Technology*. Upper Saddle River, NJ, USA: Prentice-Hall, 1999.

[12] Y. Zhang, R. Zhang, A. F. Al Rawi, and L. Hanzo, "Approximate perturbation aided lattice encoding (APPLE) for G.fast and beyond," *IEEE Access*, vol. 6, pp. 53438–53451, 2018.
[13] S.-N. Hong and G. Caire, "Compute-and-forward strategies for cooperative distributed antenna systems," *IEEE Trans. Inf. Theory*, vol. 59, no. 9, pp. 5227–5243, Sep. 2013.
[14] W. He, B. Nazer, and S. Shamai, "Uplink-downlink duality for integer-forcing: Effective SINRs and iterative optimization," in *Proc. IEEE 15th Int. Workshop Signal Process. Adv. Wireless Commun. (SPAWC)*, Jun. 2014, pp. 474–478.
[15] D. Silva, G. Pivaro, G. Fraidenraich, and B. Aazhang, "On integer-forcing precoding for the Gaussian MIMO broadcast channel," *IEEE Trans. Wireless Commun.*, vol. 16, no. 7, pp. 4476–4488, Jul. 2017.
[16] M. Tomlinson, "New automatic equaliser employing modulo arithmetic," *Electron. Lett.*, vol. 7, nos. 5–6, p. 138, Mar. 1971.
[17] H. Harashima and H. Miyakawa, "Matched-transmission technique for channels with intersymbol interference," *IEEE Trans. Commun.*, vol. 20, no. 4, pp. 774–780, Aug. 1972.
[18] J. Neckebroeck, "Error correction, precoding and bitloading algorithms in high-speed access networks," Ph.D. dissertation, Dept. Telecommun. Inf. Process., UGent, Ghent, Belgium, 2016. [Online]. Available: https://biblio.ugent.be/publication/8205721/file/8205722.pdf
[19] U. Erez and R. Zamir, "Log (1+ SNR) on the AWGN channel with lattice encoding and decoding," *IEEE Trans. Inf. Theory*, vol. 50, no. 10, pp. 2293–2314, Oct. 2004.
[20] U. Erez, S. Shamai, and R. Zamir, "Capacity and lattice strategies for canceling known interference," *IEEE Trans. Inf. Theory*, vol. 51, no. 11, pp. 3820–3833, Nov. 2005.
[21] S. Taek Chung and A. Goldsmith, "Degrees of freedom in adaptive modulation: A unified view," *IEEE Trans. Commun.*, vol. 49, no. 9, pp. 1561–1571, Sep. 2001.
[22] R. Strobel, A. Barthelme, and W. Utschick, "Implementation aspects of nonlinear precoding for G.fast—Coding and legacy receivers," in *Proc. 25th Eur. Signal Process. Conf. (EUSIPCO)*, Aug. 2017, pp. 111–115.
[23] J. Neckebroek, M. Moeneclaey, W. Coomans, M. Guenach, P. Tsiaflakis, R. B. Moraes, and J. Maes, "Novel bitloading algorithms for coded G.fast DSL transmission with linear and nonlinear precoding," in *Proc. IEEE Int. Conf. Commun.(ICC)*, Jun. 2015, pp. 945–951.
[24] A. K. Lenstra, H. W. Lenstra, and L. Lovász, "Factoring polynomials with rational coefficients," *Math. Annalen*, vol. 261, no. 4, pp. 515–534, Dec. 1982, doi: 10.1007/BF01457454.
[25] Y. Hung Gan, C. Ling, and W. Ho Mow, "Complex lattice reduction algorithm for low-complexity full-diversity MIMO detection," *IEEE Trans. Signal Process.*, vol. 57, no. 7, pp. 2701–2710, Jul. 2009.
[26] M. Razaviyayn, M. Hong, and Z.-Q. Luo, "A unified convergence analysis of block successive minimization methods for nonsmooth optimization," *SIAM J. Optim.*, vol. 23, no. 2, pp. 1126–1153, Jan. 2013, doi: 10.1137/120891009.
[27] Mosek ApS, Copenhagen, Denmark. (2017). *The MOSEK Optimization Toolbox for MATLAB Manual, Version 8.1*. [Online]. Available: https://docs.mosek.com/8.1/toolbox/index.html
[28] M. Grant and S. Boyd. (Mar. 2014). *CVX: MATLAB Software for Disciplined Convex Programming, Version 2.1*. [Online]. Available: http://cvxr.com/cvx/citing/
[29] S. Boyd and L. Vandenberghe, *Convex optimization*. New York, NY, USA: Cambridge Univ. Press, 2004.
[30] M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret, "Applications of second-order cone programming," *Linear Algebra Appl.*, vol. 284, nos. 1–3, pp. 193–228, Nov. 1998.
[31] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates, and sum-rate capacity of Gaussian MIMO broadcast channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2658–2668, Oct. 2003.
[32] S. S. Christensen, R. Agarwal, E. De Carvalho, and J. M. Cioffi, "Weighted sum-rate maximization using weighted MMSE for MIMO-BC beamforming design," *IEEE Trans. Wireless Commun.*, vol. 7, no. 12, pp. 4792–4799, Dec. 2008.
[33] Q. Shi, M. Razaviyayn, Z.-Q. Luo, and C. He, "An iteratively weighted MMSE approach to distributed sum-utility maximization for a MIMO interfering broadcast channel," *IEEE Trans. Signal Process.*, vol. 59, no. 9, pp. 4331–4340, Sep. 2011.
[34] W. Yu and R. Lui, "Dual methods for nonconvex spectrum optimization of multicarrier systems," *IEEE Trans. Commun.*, vol. 54, no. 7, pp. 1310–1322, Jul. 2006.
[35] *Fast Access to Subscriber Terminals (FAST)—Power Spectral Density Specification*, document Recommendation ITU-T G.9700, Apr. 2014.
[36] W. Lanneer, P. Tsiaflakis, J. Maes, and M. Moonen, "α-fair dynamic spectrum management for QRD-based precoding with user encoding ordering in downstream G.fast transmission," *IEEE Trans. Commun.*, vol. 67, no. 4, pp. 2939–2950, Apr. 2019.
[37] W. Lanneer, P. Tsiaflakis, J. Maes, and M. Moonen, "Linear and nonlinear precoding based dynamic spectrum management for downstream vectored G.fast transmission," *IEEE Trans. Commun.*, vol. 65, no. 3, pp. 1247–1259, Mar. 2017.
[38] P. Q. Nguyen, "Lattice reduction," *Encyclopedia Cryptography Security*, 4th ed. Cambridge, U.K.: Cambridge Univ. Press, Aug. 2010, pp. 346–347.
[39] R. Durrett, *Probability: Theory Examples*, 4th ed. Cambridge, U.K.: Cambridge Univ. Press, 2010.



WOUTER LANNEER received the M.Sc. and Ph.D. degrees in electrical engineering from KU Leuven, Belgium, in 2014 and 2019, respectively. He was a Visiting Researcher at the University of Toronto, in 2018. In 2019, he joined the Fixed Networks Department, Nokia Bell Labs, Antwerp, Belgium, where he currently works on next-generation copper and optical access technologies. His research interests include MIMO communications, information theory, as well as signal processing and optimization.



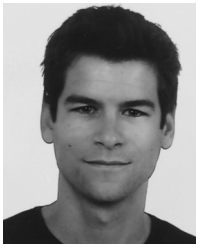
CARL NUZMAN received the bachelor's degrees in electrical engineering and mathematics from the University in Maryland College Park, in 1996, and the Ph.D. degree in electrical engineering from Princeton University, in 2000. He is currently a Bell Labs Fellow and the Head of the Mathematics of Communication Department, Nokia Bell Labs, Murray Hill, NJ, USA. Since joining Bell Labs in 2000, he has performed research in fields of probability, optimization, and dynamical systems, with applications to optical networking, micro electro-mechanical systems, power control, traffic modeling, and vectored digital subscriber line systems.



YANNICK LEFEVRE received the master's degree in engineering sciences from Vrije Universiteit Brussel (VUB), Brussels, Belgium, and Universiteit Gent, Ghent, Belgium, in 2010, and the Ph.D. degree in applied sciences and engineering from VUB, in 2014. He joined Nokia Bell Labs, Antwerp, Belgium, in 2015. As a Research Engineer, he works on next-generation copper and optical access technologies. His research interests include digital signal processing, forward error correction, signal shaping, and modulation. He was a recipient of an Aspirant Grant from the Research Foundation-Flanders (FWO).



PASCHALIS TSIAFLAKIS received the M.Sc. and Ph.D. degrees in electrical engineering from KU Leuven, in 2004 and 2009, respectively. He has further conducted research at Princeton University, UCLA, Tsinghua University, and UC Louvain. Since 2013, he joined Nokia Bell Labs, where his main activities focus on research and innovation, contributing to standardization bodies, and driving innovation into next-generation communication products. He has performed research in fields of optimization, signal processing, and machine learning, with applications to wireline and wireless communication systems. He received both a PhD and Postdoc Fellowship of the Research Foundation Flanders (FWO), the Belgian Young ICT Personality Award, in 2010, the Nokia Innovation Award, in 2017, the Nokia Bell Top Inventor Award, in 2019, the Distinguished Member of Technical Staff Title, in 2019, and several IEEE Best Paper Awards.



WERNER COOMANS received the M.Sc. degree in electrical engineering and the Ph.D. degree in electrical engineering from the Vrije Universiteit Brussel, in 2009 and 2013, respectively. He joined Nokia Bell Labs, Antwerp, Belgium, in 2013, where he currently is Head of the Fixed Networks Department. He has performed research in the fields of photonics, dynamical systems, signal processing, and end-to-end system design. He actively contributes to several standardization bodies, such as ITU-T and CableLabs, and received a Ph.D. Fellowship of the Research Foundation Flanders (FWO), in 2009, the 2018 Bell Labs Outstanding Results Award, and the 2019 Edison Patent Award.



MARC MOONEN (Fellow, IEEE) is currently a Full Professor with the Electrical Engineering Department, KU Leuven, where he is heading a research team working in the area of numerical algorithms and signal processing for digital communications, wireless communications, DSL, and audio signal processing. He is a Fellow of the EURASIP (2018). He received the 1994 KU Leuven Research Council Award, the 1997 Alcatel Bell (Belgium) Award (with Piet Vandaele), the 2004 Alcatel Bell (Belgium) Award (with Raphael Cendrillon), and was a 1997 Laureate of the Belgium Royal Academy of Science. He received journal best paper awards from the IEEE TRANSACTIONS ON SIGNAL PROCESSING (with Geert Leus and with Daniele Giacobello) and from the *Elsevier Signal Processing* (with Simon Doclo). He was the Chairman of the IEEE Benelux Signal Processing Chapter (1998–2002), a member of the IEEE Signal Processing Society Technical Committee on Signal Processing for Communications, and the President of the European Association for Signal Processing (EURASIP) (2007–2008 and 2011–2012). He has served as the Editor-in-Chief for the *EURASIP Journal on Applied Signal Processing* (2003–2005), an Area Editor for Feature Articles in the *IEEE Signal Processing Magazine* (2012–2014), and has been a member of the editorial board of *Signal Processing*, the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS II, the *IEEE Signal Processing Magazine*, the *Integration-the VLSI Journal*, the *EURASIP Journal on Wireless Communications and Networking*, and the *EURASIP Journal on Advances in Signal Processing*.