Systems & Control Letters 206 (2025) 106287

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Symmetry structures in linear physical systems

Arjan van der Schaft [a,*], Rodolphe Sepulchre [b,c], Thomas Chaffey [d]

[a] Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, The Netherlands
[b] Department of Electrical Engineering (STADIUS), KU Leuven, Belgium
[c] Department of Engineering (Control Group), University of Cambridge, UK
[d] School of Electrical and Computer Engineering, University of Sydney, Australia

ARTICLE INFO

MSC:
93C05
34A09
37J06

Keywords:
Reciprocity
Hankel operator
Relaxation
Hamiltonian
Lagrangian subspace
Time reversibility

ABSTRACT

Physical systems with symmetry structures arise abundantly in applications, often admit scalable control laws, and are important building blocks in synthesis problems. The present paper focusses on linear reciprocal systems and input–output Hamiltonian systems. Their characterization is studied from an input–output, as well as from a state space point of view. Geometrically both are seen to define Lagrangian subspaces with corresponding generating functionals. Furthermore, the relation of reciprocity and input–output Hamiltonian structure with time reversibility is analyzed.

1. Introduction

In this paper we will be concerned with systems with inputs and outputs that are characterized by some *symmetry structure*. First class of systems to be studied are *reciprocal* systems. The notion of reciprocity of physical systems is going back at least to Green (Green's reciprocity in electrical circuits), Onsager (Onsager's reciprocal relations in thermodynamics), and Maxwell (Maxwell's reciprocal rule in mechanics). The second class of systems concerns *input–output Hamiltonian* systems, which were introduced as a direct extension of classical Hamiltonian dynamics in [1], and further developed in e.g. [2,3]. From an engineering perspective the interest in these systems is motivated both from an *analysis* and *control* point of view, as well as by *synthesis* purposes. In complex engineering systems symmetry structures have been exploited to improve scalability and to obtain controller architectures which are simpler, more robust, and have a clear physical interpretation; see, e.g., [4–6]. The interest in the role of symmetry for synthesis has a long history as well, often motivated by electrical and mechanical synthesis problems. A classical example is electrical network design, utilizing a restricted set of ideal system components (capacitors, inductors, resistors, ⋯), and relying on reciprocity and passivity; see e.g. [7,8]. A new impetus to engineering design is the emerging area of *neuromorphic computing*, which requires the identification of suitable classes of physical components to be used in neuromorphic devices.

While classical electrical network synthesis was for a large part concerned with *linear* system components, in many other applications (e.g., mechanical systems and neuromorphic computing) this emphasis on linear systems does not suffice. On the other hand, the development of a mathematical theory of symmetry structures for linear systems can be expected to pave the way to nonlinear generalizations as well. Hence, in this paper we will concentrate on the *linear* case, and leave their proper nonlinear extensions for future work; see already [9,10] and the references quoted therein.

Reciprocity and input–output Hamiltonian structure can be studied from multiple mathematical angles. From a state space point of view, the external symmetry structure is reflected in the existence of a (pseudo-) inner product structure or a symplectic structure on the state space. From an input–output view, it turns out that reciprocity is best described using the Hankel operator, which is self-adjoint and admits an orthonormal decomposition of its kernel. On the other hand, the external symmetry of an input–output Hamiltonian system turns out to be characterized by its Volterra operator restricted to input functions of compact support on $(-\infty, \infty)$ for which also the output functions have compact support. Geometrically, both cases admit a description in terms of Lagrangian subspaces with corresponding generating functionals. Finally, the paper explores the connection of reciprocity and input–output Hamiltonian structure with *time-reversibility*.

* Corresponding author.
E-mail addresses: a.j.van.der.schaft@rug.nl (A. van der Schaft), rodolphe.sepulchre@kuleuven.be (R. Sepulchre), thomas.chaffey@sydney.edu.au (T. Chaffey).

https://doi.org/10.1016/j.sysconle.2025.106287
Received 19 July 2025; Received in revised form 26 October 2025; Accepted 7 November 2025
Available online 10 November 2025
0167-6911/© 2025 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

The main aim of the paper is to *unify* already existing results by filling in several gaps, thereby providing new insights. Specific novel contributions include Proposition 2.3, the formulation in Proposition 2.5 of the graph of the Hankel operator of a reciprocal system as an infinite-dimensional Lagrangian subspace, Proposition 2.9 and its implications, the Lagrangian subspace formulation of the graph of the restricted Volterra operator of an input–output Hamiltonian system, nonnegative input–output Hamiltonian systems, as well as Propositions 4.2 and 4.5 concerning the relationships with time-reversibility.

The outline of the paper is as follows. Section 2 deals with reciprocal systems, derives an orthonormal decomposition of its self-adjoint Hankel operator, and provides the formulation of reciprocal systems as Lagrangian subspaces modeled on $L_2[0,\infty)$. The physical structure of reciprocal passive systems, including relaxation systems, is discussed via their port-Hamiltonian formulation. Furthermore, it is shown how the use of signature matrices yields the extra flexibility to combine reciprocity with losslessness. Section 3 is about input–output Hamiltonian systems. It is shown that the Volterra operator on the time axis $(-\infty,\infty)$, when restricted to input and output functions of compact support, is self-adjoint and defines a Lagrangian subspace. Relations to port-Hamiltonian systems and negative imaginary systems are indicated. Section 4 brings in another form of symmetry, namely time reversibility, and studies its relations to reciprocity and input–output Hamiltonian structure. Section 5 contains conclusions and further outlook. Required background on Lagrangian subspaces and Dirac structures is given in Appendices A and B.

Notation Throughout this paper we consider finite-dimensional linear time-invariant systems Σ given in the standard input-state-output form

$$\Sigma: \quad \begin{array}{rcll} \dot{x} & = & Ax+Bu, & x\in\mathcal{X}=\mathbb{R}^n,\ u\in\mathcal{U}=\mathbb{R}^m \\ y & = & Cx+Du, & y\in\mathcal{Y}=\mathbb{R}^m \end{array} \tag{1}$$

with equal number of inputs and outputs. The *impulse response matrix*, respectively *transfer matrix*, of Σ will be denoted by

$$W(t-\tau) := Ce^{A(t-\tau)}B + D\delta(t-\tau), \quad K(s) := C(Is-A)^{-1}B + D. \tag{2}$$

2. Reciprocal systems

2.1. Basic definitions and properties

First class of systems under consideration are *reciprocal systems*. Recall that a *signature matrix* σ is a diagonal matrix with elements $+1$ and -1 on the diagonal. Hence $\sigma^2 = I$, and thus the linear mapping corresponding to σ is an *involution*. Conversely, any involution $\sigma : \mathcal{F}\to\mathcal{F}$ for some finite-dimensional linear space $\mathcal{F}$ can be transformed into a signature matrix by a similarity transformation. The relevance of signature matrices will become clear by discussing the relation with *hybrid* impedance/admittance descriptions in electrical network theory in Remark 2.2, as well with the representation of Lagrangian subspaces in Appendix A.

Definition 2.1. A system Σ is *externally reciprocal* with respect to a signature matrix σ if its transfer matrix $K(s)$ satisfies $\sigma K(s) = K^\top(s)\sigma$, or equivalently its impulse response matrix $W(t-\tau)$ satisfies $\sigma W(t-\tau) = W^\top(t-\tau)\sigma$.

Externally reciprocal systems arise abundantly in applications. For example, in electrical networks reciprocal systems that are also *passive* (see Section 2.2) correspond to networks containing capacitors, inductors, resistors, and transformers (but *no* gyrators); cf. [7,8,11].

Remark 2.2. External reciprocity amounts to *symmetry* of the input–output behavior of Σ in the following sense. Consider first $\sigma = I_m$. Let the system be initialized at $x(-\infty) = 0$. Apply an input function on $(-\infty,\infty)$ with all components equal to zero, except for the i-the component $u_i(\cdot) = v(\cdot)$, and observe the jth output component $y_j(\cdot)$. Next apply an input function with all components equal to zero, except for the j-the component $u_j(\cdot)$, where $u_j(\cdot) = v(\cdot)$ is the same scalar function as before. Then the observed ith output component $y_i(\cdot)$ is *equal* to $y_j(\cdot)$, and this symmetry property should hold for all $i,j = 1,\dots,m$. The appearance of signature matrices σ can be explained by a simple example like a resistor electrical network. Assume the (in this case *static*) relation between the vector of voltage potentials ψ and the vector of currents J at the terminals of the network is $J = R\psi$ for some matrix R. Then $R = R^\top$, i.e., reciprocity with respect to $\sigma = I$. Now partition $\psi = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}$ and correspondingly $J = \begin{bmatrix}J_1\\ J_2\end{bmatrix}$, and assume the relation between ψ_1,ψ_2,J_1,J_2 can be also expressed as a map $\begin{bmatrix}J_1\\ \psi_2\end{bmatrix} = S\begin{bmatrix}\psi_1\\ J_2\end{bmatrix}$ for some matrix S. Then it is easily checked that

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2}\end{bmatrix} S = S^\top \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2}\end{bmatrix}, \tag{3}$$

where $n_1 = \dim\psi_1 = \dim J_1, n_2 = \dim\psi_2 = \dim J_2$. This is called a *hybrid input–output representation* in electrical network theory. Same holds for other physical domains. The geometry behind this is explained in Appendix A.

External reciprocity translates to the state space as follows. As shown in the seminal paper [8] a minimal system Σ is externally reciprocal with respect to a signature matrix σ if and only if there exists an invertible matrix $G = G^\top$ (which is unique) such that

$$A^\top G = GA,\ B^\top G = \sigma C,\ \sigma D = D^\top\sigma. \tag{4}$$

Any system Σ satisfying (4) (not necessarily minimal) will be called a *reciprocal system*. The matrix G defines a, possibly indefinite, inner product on the state space $\mathcal{X}$; in the rest of the paper referred to as a *pseudo-inner product*. By defining the symmetric matrix $P := -GA$, any reciprocal system Σ can be written as a *pseudo-gradient system*

$$\begin{array}{rcll} G\dot{x} & = & -Px + C^\top\sigma u & \\ y & = & Cx+Du, & \sigma D = D^\top\sigma, \end{array} \tag{5}$$

with potential function $\frac{1}{2}x^\top Px$; see [10] and the references quoted therein.

The system Σ with inputs u and outputs σy is reciprocal whenever it has the same input–output behavior as its *dual system*[1] defined as

$$\Sigma^d: \quad \begin{array}{rcl} \dot{z} & = & A^\top z + C^\top u^d \\ y^d & = & B^\top z + D^\top u^d \end{array} \tag{6}$$

with inputs $u^d = \sigma u$ and outputs y^d. Indeed, the unique state space isomorphism between Σ and Σ^d is given as $z = Gx$. One verifies, [8,10], that along the solutions of a reciprocal system

$$\frac{d}{dt}x^\top(-t)Gx(t) = -u^\top(-t)\sigma y(t) + \sigma y^\top(-t)u(t), \tag{7}$$

Hence, by considering solutions with $u(t) = 0$ for $t \geq 0$, and integrating over $[0,\infty)$, one obtains $x^\top(-\infty)Gx(\infty) - x^\top(0)Gx(0) = -\int_0^\infty u^\top(-t)\sigma y(t)dt$. Thus if either $x(-\infty)$ or $x(\infty)$ equals zero, then G is determined by

$$x^\top(0)Gx(0) = \int_0^\infty u^\top(-t)\sigma y(t)dt \tag{8}$$

for any $x(0)\in\mathcal{X}$. In particular, if the system is *controllable* then any $x(0)$ can be reached from $x(-\infty) = 0$, implying that G is fully determined by the input–output behavior of the system.

Using (4) the impulse response matrix $W(t-\tau)$ of a reciprocal system Σ can be rewritten as

$$\begin{array}{rcl} \sigma W(t-\tau) & = & B^\top Ge^{At}e^{-A\tau}B + \sigma D\delta(t-\tau) \\ & = & B^\top e^{A^\top t}Ge^{-A\tau}B + D^\top\sigma\delta(t-\tau), \end{array} \tag{9}$$

[1] The *time-reversed* dual state $\hat{z}(t) = z(-t)$ satisfies $\dot{\hat{z}}(t) = -A^\top\hat{z}(t) - C^\top\hat{u}^d(t)$, with $\hat{u}^d(t) = u^d(-t)$. These are the equations of the *adjoint system* to be defined in (33).

making explicit the symmetry of $W(t-\tau)$. This motivates to consider reciprocal systems from the *Hankel* point of view. For any past input function $u_p : (-\infty, 0] \to \mathbb{R}^m$ define the *time-reversed* input $\hat{u}_p : [0,\infty) \to \mathbb{R}^m$ as $\hat{u}_p(t) := u_p(-t), t \in [0,\infty)$. Then the Hankel operator $\mathcal{H} : L_2([0,\infty),\mathbb{R}^m) \to L_2([0,\infty),\mathbb{R}^m)$ is given as the map $\hat{u}_p \mapsto y_f$ given as

$$y_f(t) = \sigma B^\top e^{A^\top t} G \int_0^\infty e^{A\tau} B \hat{u}_p(\tau) d\tau,\ t \in [0,\infty), \tag{10}$$

where $y_f(t), t \in [0,\infty)$, is the future output of Σ resulting from $x(-\infty) = 0$ and the L_2 input $u(t) = u_p(t), t \in (-\infty, 0], u(t) = 0, t \in [0,\infty)$. In order that this map is well-defined (i.e., $y_f(\cdot) \in L_2([0,\infty),\mathbb{R}^m)$) we assume in the rest of this section that A is *Hurwitz*. In particular, this implies $x(\infty) = 0$, and thus by (8) the matrix G is uniquely determined by the input–output behavior (even if Σ is not minimal).

As follows from (10), $\sigma\mathcal{H} = \mathcal{H}^*\sigma$, where * denotes *adjoint*. Hence the signature modified Hankel operator $\sigma\mathcal{H} : L_2([0,\infty),\mathbb{R}^m) \to L_2([0,\infty),\mathbb{R}^m)$ is *self-adjoint*. This implies that there is an associated *quadratic functional*

$$\mathfrak{H} : L_2([0,\infty),\mathbb{R}^m) \to \mathbb{R}, \tag{11}$$

defined as

$$\mathfrak{H}(\hat{u}_p) = \frac{1}{2}\left\langle \hat{u}_p, \sigma\mathcal{H}(\hat{u}_p)\right\rangle = \frac{1}{2}\left(\int_0^\infty e^{At} B u_p(-t) dt\right)^\top G \int_0^\infty e^{A\tau} B u_p(-\tau) d\tau \tag{12}$$

Indeed, see also [12], the *Gâteaux derivative* of $\mathfrak{H}$ equals $\sigma\mathcal{H}$, i.e.,

$$\frac{\delta\mathfrak{H}}{\delta\hat{u}_p}(\hat{u}_p) = \sigma\mathcal{H}(\hat{u}_p). \tag{13}$$

Note as well that $\mathfrak{H}(\hat{u}_p) = \frac{1}{2}x(0)^\top G x(0)$, and thus the quadratic functional $\mathfrak{H}$ factorizes over the state space $\mathcal{X}$. In this sense, $\mathfrak{H}$ is a *memory functional*. Furthermore, instead of splitting $(-\infty,\infty)$ into the past $(-\infty, 0]$ and the future $(0,\infty)$ at time 0, by time-invariance one can also split $(-\infty,\infty) = (-\infty, t] \cup (t,\infty)$ for any present time $t \geq 0$. In this way $\mathfrak{H}$ becomes equal to $\frac{1}{2}x(t)^\top G x(t)$; see [12].

The symmetric kernel of $\sigma\mathcal{H}$, which by (10) is given by $B^\top e^{A^\top t} G e^{A\tau} B$, admits the following *orthonormal decomposition*.

Proposition 2.3. *Consider an n-dimensional system Σ with Hankel operator $\mathcal{H}$. Suppose the system is reciprocal with respect to σ. Then $\sigma\mathcal{H}$ has, apart from zero eigenvalues, n real eigenvalues $\lambda_1,\dots,\lambda_n$ given as the eigenvalues of CG, where $G = G^\top$ is determined by (4), and C is the controllability Gramian $C := \int_0^\infty e^{A\tau} B B^\top e^{A^\top\tau} d\tau$. Furthermore $\lambda_i \neq 0, i = 1,\dots,n$, if and only if the system is controllable, while $\lambda_i > 0, i = 1,\dots,n$, if and only if the system is controllable and $G > 0$. The corresponding eigenfunctions are*

$$\phi_i(t) = B^\top e^{A^\top t} C^{-1} x_i, \quad i = 1,\dots,n, \tag{14}$$

where $x_i, i = 1,\dots,n$, are eigenvectors of CG. By scaling x_i such that $x_i C^{-1} x_i = 1, i = 1,\dots,n$, it follows that $\phi_1,\dots,\phi_n$ is an orthonormal set in $L_2([0,\infty),\mathbb{R}^m)$, and the kernel of $\sigma\mathcal{H}$ decomposes as

$$\sigma W(t+\tau) = B^\top e^{At} G e^{A\tau} B = \sum_{i=1}^n \lambda_i \phi_i(t) \phi_i^\top(\tau) \tag{15}$$

Proof. Since the rank of $\mathcal{H}$ is less than or equal to n, the spectrum of $\sigma\mathcal{H}$ is of the form $\{\lambda_1,\dots,\lambda_n\} \cup \{0\}$, $\lambda_i \in \mathbb{C}$. Since $\sigma\mathcal{H}$ is self-adjoint $\lambda_1,\dots,\lambda_n$ are real. Furthermore, as shown in [13] (see already [14,15] for the SISO case), $\lambda_1,\dots,\lambda_n$ are equal to the eigenvalues of the *cross-Gramian* Z, defined as the unique solution of the Sylvester equation $AZ + ZA = -B\sigma C$. By [16,17] $Z = CG$. One directly verifies that $\phi(t) := B^\top e^{A^\top t} C^{-1} x$ satisfies

$$\sigma\left(\mathcal{H}\phi\right)(t) = B^\top e^{A^\top t} G \left(\int_0^\infty e^{A\tau} B B^\top e^{A^\top\tau} d\tau\right) C^{-1} x = B^\top e^{A^\top t} G x. \tag{16}$$

Hence $\phi_i(t) = B^\top e^{A^\top t} C^{-1} x_i$ is an eigenfunction corresponding to λ_i if and only if $G x_i = \lambda_i C^{-1} x_i$, or equivalently $CG x_i = \lambda_i x_i$. Thus the eigenfunctions ϕ_i corresponding to the eigenvalues $\lambda_1,\dots,\lambda_n$ of $\sigma\mathcal{H}$ are generated by eigenvectors $x_1,\dots,x_n$ of CG corresponding to the same eigenvalues of CG.

Finally, since $\sigma\mathcal{H}$ is self-adjoint the eigenfunctions ϕ_i are *orthogonal* with respect to the $L_2([0,\infty),\mathbb{R}^m)$ inner product. This means that for $i \neq j$ the eigenvectors x_i of $Z = CG$ satisfy

$$0 = \int_0^\infty x_i^\top C^{-1} e^{At} B B^\top e^{A^\top t} C^{-1} x_j dt = x_i^\top C^{-1} x_j \tag{17}$$

By scaling x_i such that $x_i C^{-1} x_i = 1, i = 1,\dots,n$, one concludes that the resulting eigenfunctions $\phi_1,\dots,\phi_n$ satisfy (15). □

Remark 2.4. Obviously Proposition 2.3 has direct consequences for model reduction of a reciprocal system Σ. In fact, there has been extensive work on model reduction of what in this context are also called *symmetric systems* (that is, externally reciprocal systems with $\sigma = I$). Siso systems are trivially symmetric [14]. See [13] and the references therein, also for the observation that the non-zero eigenvalues of the Hankel operator of symmetric systems are equal to the eigenvalues of the cross-Gramian and are real.

External reciprocity can be interpreted from a *geometric* point of view as follows. Consider the Hilbert space $\mathcal{F} := L_2([0,\infty),\mathbb{R}^m)$, and its dual $\mathcal{E} = \mathcal{F}^* = L_2([0,\infty),\mathbb{R}^m)$ (using Riesz' representation theorem). Then $\mathcal{F}\times\mathcal{E}$ is endowed with the canonical *symplectic form* (see Appendix A)

$$\langle\langle (f^a, e^a), (f^b, e^b) \rangle\rangle := \int_0^\infty [e^a(t)^\top f^b(t) - e^b(t)^\top f^a(t)]\, dt, \tag{18}$$

Furthermore, recall from Appendix A the notion of a *Lagrangian subspace*. Then the above characterization of external reciprocity admits the following geometric formulation.

Proposition 2.5. *The system Σ is externally reciprocal with respect to σ if and only if the subspace $\mathcal{L} \subset \mathcal{F}\times\mathcal{E} = L_2([0,\infty),\mathbb{R}^m) \times L_2([0,\infty),\mathbb{R}^m)$ defined as*

$$\mathcal{L} := \{(\hat{u}_p, \sigma y_f) : [0,\infty) \to \mathbb{R}^m \times \mathbb{R}^m \mid \textit{where } y_f(t), t \in [0,\infty),\ \textit{is the output resulting from } u(t) = u_p(t), t \in (-\infty, 0], u(t) = 0, t \in (0,\infty), \textit{and } x(-\infty) = 0\} \tag{19}$$

is a Lagrangian subspace, with generating functional $\mathfrak{H}$ given by (12).

2.2. Reciprocity and passivity; physical examples of reciprocal systems

Physical examples of reciprocal systems are numerous, but often arise in combination with *passivity*; see [8]. Recall that a system Σ is passive if and only if there exists a matrix $Q = Q^\top \geq 0$ satisfying the *dissipation inequality* $\frac{d}{dt}\frac{1}{2}x^\top Q x \leq y^\top u$, or equivalently the Linear Matrix Inequality

$$\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -A & -B \\ C & D \end{bmatrix} + \begin{bmatrix} -A^\top & C^\top \\ -B^\top & D^\top \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \geq 0. \tag{20}$$

The quadratic function $\frac{1}{2}x^\top Q x \geq 0$ is called a *storage function* for the system Σ, and Q is called a storage matrix. The system is called *lossless* if $\frac{d}{dt}\frac{1}{2}x^\top Q x = y^\top u$, in which case (20) is satisfied with equality. If Q is indefinite, then the system is called *cyclo*-passive, respectively *cyclo*-lossless. The following result is well-known; see [10] for a short proof.

Proposition 2.6. *Suppose $Q = Q^\top$ is a solution to (20). Then* $\ker Q$ *is A-invariant and contained in* $\ker C$. *In particular, if the system is observable then necessarily* $\ker Q = 0$.

The storage matrix of a (cyclo-)lossless system is *unique*, but in general a (cyclo-)passive system admits *many* Q satisfying (20). On the other hand, if Q is an invertible solution to (20), and additionally the system is *reciprocal* with respect to G and σ, then, by combining (4) and (20), one verifies that also $Q' := GQ^{-1}G$ is a solution of (20). As shown in [8], see also [18], this implies that there exist Q satisfying (20) *and* $Q = GQ^{-1}G$. Such solutions Q of (20) are called *compatible* with G. Compatible Q define storage functions with a clear physical relevance; cf. [8,10,18].

In general, there may still exist many compatible Q. However, if $G > 0$ then, as shown in [19], there is a *unique* compatible storage matrix, namely $Q = G$. This case amounts to the definition of a *relaxation system*.[2]

Definition 2.7. A *relaxation* system is a passive system that is reciprocal with respect to the identity matrix $\sigma = I$ and with $G > 0$.

Thus relaxation systems Σ are reciprocal systems with inner product $G > 0$ satisfying the passivity dissipation inequality

$$\frac{d}{dt}\frac{1}{2}x^\top Gx \le y^\top u. \tag{21}$$

Remark 2.8. Interestingly, the assumption of A being Hurwitz together with $G > 0$ already *implies* (21), and thus passivity (as was noted in [12]). This follows from writing $GA = G^{\frac{1}{2}}(G^{\frac{1}{2}}AG^{-\frac{1}{2}})G^{\frac{1}{2}}$, and noting that the symmetric matrix $G^{\frac{1}{2}}AG^{-\frac{1}{2}}$ is similar to A, and thus negative definite.

The physical properties of passive reciprocal systems become especially clear in their *port-Hamiltonian* formulation [20]; see also [19]. Consider a reciprocal system Σ, i.e., satisfying (4). Write the system into its pseudo-gradient form (5). Furthermore, let the system be passive, with invertible Q satisfying (20) and $Q = GQ^{-1}G$. For simplicity of exposition let $\sigma = I$; otherwise see [10]. By Lemma 2.2 in [19] there exist coordinates $x = (x_1, x_2)$ in which

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad G = \begin{bmatrix} Q_1 & 0 \\ 0 & -Q_2 \end{bmatrix}. \tag{22}$$

In such coordinates the pseudo-gradient system takes the form, see [19],

$$\begin{aligned} \begin{bmatrix} Q_1 & 0 \\ 0 & -Q_2 \end{bmatrix}\dot{x} &= -Px + \begin{bmatrix} C_1^\top \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} C_1 & 0 \end{bmatrix} x + Du, \quad D = D^\top \ge 0 \end{aligned} \tag{23}$$

Partition the symmetric matrix P correspondingly as $P = \begin{bmatrix} P_1 & P_c \\ P_c^\top & P_2 \end{bmatrix}$. Then passivity implies $P_1 = P_1^\top \ge 0, P_2 = P_2^\top \le 0$. Multiply the second part (corresponding to x_2) of the differential equations in (23) on both sides with a minus sign, to obtain the equivalent system description

$$\begin{aligned} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\dot{x} &= -\begin{bmatrix} P_1 & P_c \\ -P_c^\top & -P_2 \end{bmatrix} x + \begin{bmatrix} C_1^\top \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} C_1 & 0 \end{bmatrix} x + Du \end{aligned} \tag{24}$$

with $P_1 = P_1^\top \ge 0, P_2 = P_2^\top \le 0$. Then in the new coordinates $z = (z_1, z_2)$, where $z_1 = Q_1 x_1$ and $z_2 = Q_2 x_2$, (24) takes the port-Hamiltonian form

$$\begin{aligned} \dot{z} &= \left(\begin{bmatrix} 0 & -P_c \\ P_c^\top & 0 \end{bmatrix} - \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix}\right)\begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix} z + \begin{bmatrix} C_1^\top \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} C_1 & 0 \end{bmatrix}\begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix} z + Du, \quad D = D^\top \ge 0, \end{aligned} \tag{25}$$

[2] The original paper [8] starts with an alternative definition of relaxation systems in terms of complete monotonicity of the impulse response matrix, which then is shown to be equivalent to the definition given here.

with Hamiltonian $\frac{1}{2}z_1^\top Q_1^{-1}z_1 + \frac{1}{2}z_2^\top Q_2^{-1}z_2$.

Hence minimal passive reciprocal systems correspond to systems that can be split into *two energy domains*, corresponding to z_1 and z_2, which are *only* interconnected through the coupling matrix P_c. Physically, z are the energy state variables, while $x = Q^{-1}z$ are the co-energy state variables. For example, in an RLC electrical network with current sources, the components of z_1 are the *charges* of the capacitors, and z_2 the *flux linkages* of the inductors, while x_1 are the voltages across the capacitors and x_2 the currents through the inductors. Furthermore, $\frac{1}{2}x_1^\top P x_1$ is the content function of the combined current-controlled *resistors*, and $\frac{1}{2}x_2^\top P x_2$ is the co-content of the voltage-controlled *conductors*. Finally, P_c is defined by the network topology and possibly the transformers (coupling capacitors to inductors, and thus electric to magnetic energy). As already indicated in Remark 2.2 *signature matrices* σ come into play when considering *hybrid* input–output representations, from *part* of the currents together with a *complementary part* of the voltages, to the rest of the currents and voltages. For RLCT electrical networks (where 'T' stands for transformers) such an hybrid input–output representation always exists; see e.g. [7,11].

In the special case of a relaxation system one has $G = Q$ and thus the system (25) simplifies to

$$\begin{aligned} \dot{z} &= -PG^{-1}z + C^\top u \\ y &= CG^{-1}z + Du, \qquad D = D^\top \ge 0 \end{aligned} \tag{26}$$

involving just one energy domain. For example, an RL electrical network is a relaxation system, with z the flux linkages of the inductors and x the currents through the inductors, while the potential $\frac{1}{2}x^\top Px$ is the content of the current-controlled *resistors*.

2.3. Static reciprocity and losslessness

Roughly speaking, static *lossless* structures correspond to *skew-symmetric* matrices, while static *reciprocal* structures correspond to *symmetric* matrices. In this sense losslessness and reciprocity could be expected to *exclude each other*. The purpose of this subsection is to show that this is *not* necessarily the case if we consider reciprocity *with respect to a signature matrix*.

As discussed above, reciprocity corresponds to Lagrangian subspaces. Similarly, skew-symmetry corresponds to *Dirac structures*; cf. Appendix B. Now recall from [5] that a Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is called *separable* if $\mathcal{D} = \mathcal{K} \times \mathcal{K}^\perp$ for some subspace $\mathcal{K} \subset \mathcal{F}$, where $\mathcal{K}^\perp \subset \mathcal{E} = \mathcal{F}^*$ denotes the orthogonal complement of $\mathcal{K}$ with respect to the duality product between $\mathcal{F}$ and $\mathcal{E}$. Furthermore, as shown in [5], a Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is separable if and only if

$$\langle e_b, f_a \rangle = 0 \tag{27}$$

for all pairs $(f_a, e_a), (f_b, e_b) \in \mathcal{D}$. A typical example of a separable Dirac structure is the space of currents and voltages constrained by Kirchhoff's current and voltage laws. It is separable because Kirchhoff's *current* laws are decoupled from Kirchhoff's *voltage* laws. The expression (27) in this context is Tellegen's theorem $\langle V_1, I_2 \rangle = 0$ for all V_1 satisfying Kirchhoff's voltage laws, and all I_2 satisfying Kirchhoff's current laws (where V_1 and I_2 need not be the *actual* voltages and currents). Since transformers do not mix voltages and currents either, Kirchhoff's current and voltage laws *together with* transformers define separable Dirac structures as well.

We have the following alternative characterization of separability of Dirac structures.

Proposition 2.9. *A Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is separable if and only if it is Lagrangian.*

Proof. Let $\mathcal{D}$ be a separable Dirac structure. Take any two $(f_a, e_a), (f_b, e_b) \in \mathcal{D}$. Then by (27) $\langle e_a, f_b \rangle = 0, \langle e_b, f_a \rangle = 0$. Hence $\langle e_a, f_b \rangle - \langle e_b, f_a \rangle = 0$, and thus $\mathcal{D}$ is Lagrangian. Conversely, let $\mathcal{D}$ be a Dirac structure that is also Lagrangian. Since $\mathcal{D}$ is a Dirac structure $\langle e_a, f_b \rangle +$

$\langle e_b, f_a\rangle = 0$ for any two $(f_a, e_a), (f_b, e_b) \in \mathcal{D}$. Since $\mathcal{D}$ is Lagrangian also $\langle e_a, f_b\rangle - \langle e_b, f_a\rangle = 0$, implying that $\langle e_a, f_b\rangle = 0, \langle e_b, f_a\rangle = 0$. Hence $\mathcal{D}$ is separable. □

From an equational point of view the equivalence between *separability* of $\mathcal{D}$, and $\mathcal{D}$ being *Lagrangian*, amounts to the following. As detailed in Appendix B, for any Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ there exist square matrices F, E, satisfying $FE^\top + EF^\top = 0$, such that $\mathcal{D} = \ker \begin{bmatrix} F & E \end{bmatrix}$. Furthermore, modulo column permutations, one can always split $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$ and correspondingly $E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}$ such that $\begin{bmatrix} F_1 & E_2 \end{bmatrix}$ is invertible. It follows that $\mathcal{D}$ equals the graph of the skew-symmetric matrix $J = -\begin{bmatrix} F_1 & E_2 \end{bmatrix}^{-1} \begin{bmatrix} E_1 & F_2 \end{bmatrix}$. Let now $\mathcal{D}$ be separable, i.e. $\mathcal{D} = \mathcal{K} \times \mathcal{K}^\perp$ for some subspace $\mathcal{K} \subset \mathcal{F}$. Then it follows that modulo row permutations F and E take the form

$$\begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} \bar{F}_1 & \bar{F}_2 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} E_1 & E_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \bar{E}_1 & \bar{E}_2 \end{bmatrix}, \tag{28}$$

where $\mathcal{K} = \ker \begin{bmatrix} \bar{F}_1 & \bar{F}_2 \end{bmatrix}$. Therefore the skew-symmetric matrix J takes the form

$$J = \begin{bmatrix} 0 & M \\ -M^\top & 0 \end{bmatrix}, \tag{29}$$

for some matrix M. Hence its satisfies

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} J = J^\top \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \tag{30}$$

and thus is also *reciprocal* with respect to the signature matrix $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

3. Input–output Hamiltonian systems

In this section we discuss a *second* class of systems with symmetry structure, namely linear *input–output Hamiltonian systems* as initiated in [1,2].

3.1. Definition and relation to other system classes

Definition 3.1. A system Σ is an *input–output (IO) Hamiltonian system* with respect to a signature matrix σ if its transfer matrix $K(s)$ satisfies $\sigma K(s) = K^\top(-s)\sigma$, or equivalently its impulse response matrix $W(t-\tau)$ satisfies $\sigma W(t-\tau) = -W^\top(\tau - t)\sigma$.

As shown in [2], a minimal state space system realization Σ of $K(s)$ is an IO Hamiltonian system with respect to σ if and only if there exists an invertible $\Omega = -\Omega^\top$ (which is unique) such that

$$A^\top\Omega + \Omega A = 0, \quad B^\top\Omega = \sigma C, \quad \sigma D = D^\top\sigma. \tag{31}$$

The matrix Ω is known as a *symplectic form* on the state space $\mathcal{X}$, and there exist (*canonical*) coordinates in which

$$\Omega = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \tag{32}$$

(In particular, the dimension of $\mathcal{X}$ is even.) Equivalently, a minimal state space system Σ, with inputs u and outputs y, is IO Hamiltonian with respect to σ whenever the system with inputs u and outputs σy has the same input–output behavior as its *adjoint system*

$$\Sigma^a: \quad \begin{aligned} \dot{p} &= -A^\top p - C^\top u^a \\ y^a &= B^\top p + D^\top u^a \end{aligned} \tag{33}$$

with inputs $u^a = \sigma u$ and outputs y^a. In fact, the unique state space isomorphism between Σ and Σ^a is given as $p = \Omega x$. In this sense input–output Hamiltonian systems are *self-adjoint*; see [3]. Recall that the adjoint system Σ^a is characterized by the property

$$\frac{d}{dt} p^\top(t)x(t) = (y^a(t))^\top u(t) - (u^a(t))^\top y(t). \tag{34}$$

Using $p = \Omega x$, $\Omega^\top = -\Omega$ and $\sigma^2 = I$, this leads to the following defining identity

$$\frac{d}{dt} x_2^\top(t)\Omega x_1(t) = u_2^\top(t)y_1(t) - y_2^\top(t)u_1(t), \tag{35}$$

for all solution triples $(x_i, u_i, y_i), i = 1, 2$.

Input–output Hamiltonian systems have been introduced as a direct extension to classical Hamiltonian dynamics, illustrated by the following simple example.

Example 3.2. A mass–spring system with mass m, spring constant k, and external force u defines the IO Hamiltonian system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{36}$$

with output $y = q$, where q is the position of the mass and p its momentum.

In general, any mechanical system with collocated inputs (external forces) and outputs (positions) defines an IO Hamiltonian system, as long as there is no internal energy dissipation (e.g., dampers). Furthermore, similar to the electrical network situation, cf. Remark 2.2, signature matrices come into play if the inputs are given by a *part* of the external forces and a *complementary part* of the corresponding positions, while the outputs are the remaining positions and external forces.

Input–output Hamiltonian systems are related to *port-Hamiltonian systems* [6,20] as follows. Consider an input–output Hamiltonian system Σ with $D = 0$, and for clarity of exposition let $\sigma = I$. Define the matrix $Q = \Omega A$, which by (31) and $\Omega = -\Omega^\top$ is symmetric. Then rewrite the equations of the input–output Hamiltonian system as

$$\begin{aligned} \dot{x} &= JQx - JC^\top u \\ y &= Cx, \end{aligned} \tag{37}$$

with skew-symmetric matrix $J := \Omega^{-1}$ (the Poisson structure matrix). Replacing the output y by its *time-derivative* $z = \dot{y} = CJQx - CJC^\top u$ leads to

$$\begin{aligned} \dot{x} &= JQx - JC^\top u \\ z &= CJQx - CJC^\top u \end{aligned} \tag{38}$$

This is a *port-Hamiltonian system* with Hamiltonian $\frac{1}{2}x^\top Qx$, satisfying

$$\frac{d}{dt}\frac{1}{2}x^\top Qx = x^\top Q\left(JQx - JC^\top u\right) = z^\top u, \tag{39}$$

since $x^\top QJQx = 0, u^\top CJC^\top u = 0$. In a mechanical systems context, if y are position variables, then z are *velocities* (and thus $z^\top u$ equals the externally supplied power).

In [21] it has been shown how by *extending* the definition of input–output Hamiltonian systems to systems including *energy dissipation* and with J not necessarily invertible, they become equivalent to *negative imaginary systems* [22,23] and *counter-clockwise input–output systems* [24].

3.2. Volterra operators of input–output Hamiltonian systems

The input–output Hamiltonian symmetry is primarily reflected in the properties of the *Volterra operator* of the system, instead of the Hankel operator as in the reciprocal case. In view of (31) the impulse response matrix of an input–output Hamiltonian system can be rewritten as

$$\begin{aligned} \sigma W(t-\tau) &= B^\top J e^{At} e^{-A\tau} B + \sigma D\delta(t-\tau) \\ &= B^\top e^{-A^\top t} J e^{-A\tau} B + D^\top \sigma\delta(t-\tau). \end{aligned} \tag{40}$$

For simplicity of exposition let us take throughout $D = 0$ and $\sigma = I$. Then consider the input–output map $\mathcal{V}: u(\cdot) \mapsto y(\cdot)$ given by

$$y(t) = B^\top e^{-A^\top t}\Omega \int_{-\infty}^{t} e^{-A\tau} Bu(\tau)d\tau, \tag{41}$$

where we *restrict* to input functions $u \in L_2((-\infty, \infty), \mathbb{R}^m)$ such that (1) u has compact support, (2) the corresponding output y has compact support as well. This second condition is equivalent to

$$\int_\alpha^\beta e^{-A\tau} Bu(\tau)d\tau = 0, \quad \text{for some } \alpha, \beta \in \mathbb{R} \tag{42}$$

(which ensures that the support of y is within (α, β)). Then $\mathcal{V} : \mathcal{S}_u \to \mathcal{S}_y$, where the subspaces $\mathcal{S}_u, \mathcal{S}_y$ of $L_2((-\infty, \infty), \mathbb{R}^m)$ are defined as

$$\begin{aligned} \mathcal{S}_u &:= \{u \in L_2((-\infty, \infty), \mathbb{R}^m) \mid u \text{ has compact support and (42) holds}\} \\ \mathcal{S}_y &:= \{y \in L_2((-\infty, \infty), \mathbb{R}^m) \mid y \text{ has compact support}\} \end{aligned} \tag{43}$$

We see three major differences between the input–output Hamiltonian case and the previous, reciprocal, case: (1) Volterra on $(-\infty, \infty)$ instead of Hankel on $[0, \infty)$, (2) A does not need to be Hurwitz, (3) because of skew-symmetry of Ω the kernel

$$B^\top e^{-A^\top t} \Omega e^{-A\tau} B, \ \tau < t, \qquad 0 \text{ whenever } \tau > t, \tag{44}$$

appears to be skew-symmetric for $t > \tau$, instead of symmetric as in the reciprocal case.

We start from the following proposition proven in [25], specializing a main result of [3] to the linear case.

Proposition 3.3. *Consider an arbitrary signal pair u^a, y^a on $(-\infty, \infty)$ with compact support. Then*

$$\int_{-\infty}^{\infty} (u^a(t))^\top y(t) dt = \int_{-\infty}^{\infty} (y^a(t))^\top u(t) dt \tag{45}$$

for all input–output trajectories (u, y) with compact support of a given system Σ, if and only if (u^a, y^a) is an input–output trajectory of the adjoint system $(-A^\top, -C^\top, B^\top, D^\top)$ of Σ.

This proposition immediately leads to the following characterization of input–output Hamiltonian systems.

Proposition 3.4. *A minimal state space system Σ is input–output Hamiltonian if and only if its input–output behavior of compact support is the same as that of its adjoint system $(-A^\top, -C^\top, B^\top, D^\top)$.*

Proposition 3.3 tells us that for an input–output Hamiltonian system the bilinear form

$$\big((u_1, y_1), (u_2, y_2)\big) := \int_{-\infty}^{\infty} u_1^\top(t) y_2(t) dt \tag{46}$$

is *symmetric* when restricted to the set of input–output trajectories of compact support. Thus although $B^\top e^{-A^\top t} \Omega e^{-A\tau} B$ *looks* skew-symmetric, the bilinear form (46) is actually *symmetric* in view of the constraints (42). Equivalently, the Volterra operator $\mathcal{V} : \mathcal{S}_u \to \mathcal{S}_y$ operator is *self-adjoint*.

Example 3.5. Consider the mass–spring system from Example 3.2; for simplicity with $m = 1$, $k = 1$. Then its impulse response is $W(t, \tau) = \sin(t - \tau)$. The constraints (42) amount to

$$0 = \int_{\alpha}^{\beta} u(\tau) \sin \tau d\tau = \int_{\alpha}^{\beta} u(\tau) \cos \tau d\tau. \tag{47}$$

From a geometric point of view, self-adjointness of the Volterra operator means that the graph of $\mathcal{V} : \mathcal{S}_u \to \mathcal{S}_y$ defines a Lagrangian subspace $\mathcal{L}$ of $L_2((-\infty, \infty), \mathbb{R}^m) \times L_2((-\infty, \infty), \mathbb{R}^m)$, endowed with the symplectic form (see Appendix A)

$$\langle\langle (f^a, e^a), (f^b, e^b) \rangle\rangle := \int_{-\infty}^{\infty} e^a(t)^\top f^b(t) - e^b(t)^\top f^a(t) dt, \tag{48}$$

The generating functional of $\mathcal{L}$ is the quadratic functional $\mathfrak{V} : \mathcal{S}_u \to \mathbb{R}$ defined as

$$\begin{aligned} \mathfrak{V}(u) &:= \tfrac{1}{2} \textstyle\int_{-\infty}^{\infty} u^\top(t) y(t) dt \\ &= \tfrac{1}{2} \textstyle\int_{-\infty}^{\infty} u^\top(t) B^\top e^{-A^\top t} \Omega \left(\int_{-\infty}^{t} e^{-A\tau} B u(\tau) d\tau \right) dt. \end{aligned} \tag{49}$$

Indeed, using symmetry of the bilinear form (46), it follows that the Gâteaux derivative of $\mathfrak{V}$ is the output function y.

3.3. Nonnegative input–output Hamiltonian systems

A special case occurs if the generating functional $\mathfrak{V}$ is *nonnegative*, i.e., $\int_{-\infty}^{\infty} u^\top(t) y(t) dt \geq 0$ for all input–output trajectories $(u(\cdot), y(\cdot))$ of compact support. This is the same as *cyclo-passivity* of the input–output Hamiltonian system. It can be derived from the results in [26] that there exists an indefinite storage function $\frac{1}{2} x^\top W x$, with $W = W^\top$, satisfying

$$\frac{d}{dt} \frac{1}{2} x^\top W x \leq u^\top y. \tag{50}$$

Furthermore, if an invertible W is a solution to the dissipation inequality (50) then $\Omega W^{-1} \Omega$ is a solution as well. Since the set of W satisfying (50) is convex and compact, and the map $W \mapsto \Omega W^{-1} \Omega$ is continuous, it follows from Brouwer's fixed point theorem that there exist W satisfying (50) as well as $W = \Omega W^{-1} \Omega$. (The same argument was used for proving that in the reciprocal case there exists Q satisfying (4) and $Q = GQ^{-1}G$ [18].) As discussed in [26] this, in turn, implies that

$$(\Omega^{-1} W)^\top \Omega (\Omega^{-1} W) = -\Omega, \quad (\Omega^{-1} W)^2 = I \tag{51}$$

Said otherwise, $\Omega^{-1} W$ is an anti-symplectomorphism and an involution. As follows from the results in [27], cf. [26], this means that there exist linear coordinates $x = (q, p)$ in which

$$\Omega = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \tag{52}$$

i.e., the storage function is given by $p^\top q$. Note that this corresponds to the following appealing form of the dissipation inequality

$$\frac{d}{dt} \begin{bmatrix} q^\top & p^\top \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \leq \begin{bmatrix} u^\top & y^\top \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}. \tag{53}$$

Starting from here, see [26] for such developments in the context of spectral density matrices, it follows that nonnegative input–output Hamiltonian systems have transfer matrices $K(s)$ that can be *factorized* as $M(s) M^\top(-s)$ (and conversely).

Example 3.6. Returning to the mass–spring system of Examples 3.2 and 3.5 it can be seen that this system is *not* cyclo-dissipative with respect to the supply rate uy, since its transfer function $K(s) = \frac{1}{s^2+1}$ (for $m = k = 1$) does not admit a factorization. In fact its generating functional $\mathfrak{V}$ is *indefinite*.

4. Connections of reciprocity and input–output Hamiltonian structure to time-reversibility

In this section we explore another type of symmetry, namely *time-reversibility*, in relation with reciprocity and input–output Hamiltonian structure. Again (cyclo-)passivity will play a role as well.

4.1. Reciprocal systems and signed time-reversibility

First, let us investigate the combination of reciprocity with *signed time-reversibility*.

Definition 4.1. A system (1) is signed time-reversible if its transfer matrix $K(s)$ satisfies $K(s) = -K(-s)$.

Equivalently, a system is signed time-reversible if whenever the pair $(u(t), y(t)), t \in \mathbb{R}$, is in the input–output behavior, then so is $(-u(-t), y(t))$, $t \in \mathbb{R}$. ('Signed' because of the minus sign in front of u.) It is shown in [28] that a minimal system $\Sigma = (A, B, C, D)$ is signed time-reversible if and only if there exists an invertible $R : \mathcal{X} \to \mathcal{X}$ (the signed time-reversibility map), which is *unique*, such that

$$RA = -AR, \quad RB = B, \quad C = CR. \tag{54}$$

Furthermore, if R satisfies these equations, then so does R^{-1}, and thus by uniqueness $R^2 = I$; i.e., the signed time-reversibility map R is an *involution*.

Now consider a minimal *reciprocal* system (for simplicity with respect to $\sigma = I$), which is also signed time-reversible. Then its transfer matrix $K(s)$ satisfies $K(s) = -K(-s) = -K^\top(-s)$. This means that the system is also *cyclo-lossless* with a storage function $\frac{1}{2}x^\top Qx$, where $Q = Q^\top$ is invertible, but not necessarily positive definite. Conversely, if a system is cyclo-lossless with storage matrix Q, then (20) with equality yields

$$A^\top Q + QA = 0, \quad B^\top Q = C, \quad D + D^\top = 0 \tag{55}$$

This leads to the following proposition.

Proposition 4.2. *A minimal system Σ that is reciprocal and signed time-reversible is also cyclo-lossless. Conversely, if the system is cyclo-lossless with invertible storage matrix Q, and reciprocal with pseudo-inner product G, then $D = 0$ and $Q = GQ^{-1}G$, while the system is signed time-reversible with signed time-reversibility map $R := Q^{-1}G$.*

Proof. $D + D^\top = 0$ and $D = D^\top$ obviously implies $D = 0$. It is immediately checked that $GQ^{-1}G$ satisfies (55). Hence by uniqueness of Q it follows that $Q = GQ^{-1}G$. Thus $R = Q^{-1}G$ is an involution, and one verifies that R satisfies (54). □

Remark 4.3. Conversely it can be shown that cyclo-passivity together with signed time-reversibility implies reciprocity and cyclo-losslessness; see the corresponding statement for the non-cyclo case in [8].

A strengthened version of Proposition 4.2 is obtained whenever $Q > 0$, i.e., the system is lossless. Indeed, if $Q > 0$ and the system is reciprocal with pseudo-inner product G, then there are coordinates $x = (x_1, x_2)$ such that Q and G take the form (22), and thus $R = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Rewriting the system in pseudo-gradient system form (5) it follows that the system takes the form (compare with (24))

$$\begin{aligned} \begin{bmatrix} Q_1 & 0 \\ 0 & -Q_2 \end{bmatrix} \dot{x} &= -\begin{bmatrix} 0 & P_c \\ P_c^\top & 0 \end{bmatrix} x + \begin{bmatrix} C_1^\top \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} C_1 & 0 \end{bmatrix} x + Du \end{aligned} \tag{56}$$

This is a lossless system with two types of energy $\frac{1}{2}x_1^\top Q_1 x_1$ and $\frac{1}{2}x_2^\top Q_2 x_2$, and power-conserving interconnection structure defined by P_c. An example is provided by LC electrical networks, where x_1 refer to capacitor voltages and x_2 to inductor currents.

Remark 4.4. It follows that *relaxation systems* are typically *not* signed time-reversible. In fact, since $G > 0$, a relaxation system is signed time-reversible if and only if it is of the *integrator* form (and thus energy-conserving)

$$G\dot{x} = C^\top u, \; G = G^\top > 0, \qquad y = Cx. \tag{57}$$

4.2. Input–output Hamiltonian structure, reciprocity, and time-reversibility

Throughout this subsection we take $D = 0$ and $\sigma = I$ for ease of exposition. As noted before, the transfer matrix $K(s)$ of an input–output Hamiltonian system is characterized by the property $K(s) = K^\top(-s)$. On the other hand, reciprocity amounts to $K(s) = K^\top(s)$. Taken together this implies $K(s) = K^\top(-s) = K(-s)$. This last property corresponds to *time-reversibility*: whenever $(u(t), y(t)), t \in \mathbb{R}$, is an input–output trajectory of the system, then so is the time-reversed version $(u(-t), y(-t)), t \in \mathbb{R}$. (Note this is *different* from *signed* time-reversibility, where we needed a minus sign in front of the time-reversed input function.) Similar reasoning as for signed time-reversibility of reciprocal systems implies that among the three notions of (1) input–output Hamiltonian, (2) reciprocity, (3) time-reversibility, *two* of these three notions actually *imply* the third one.

From a *state space* point of view this is seen as follows. For simplicity of exposition let Σ be given by a minimal triple (A, B, C) with $D = 0$. Being *input–output Hamiltonian* corresponds by (31) to the existence of a unique invertible Ω such that

$$A^\top \Omega + \Omega A = 0, \; B^\top \Omega = C, \; \Omega = -\Omega^\top. \tag{58}$$

On the other hand, as seen before, *reciprocity* amounts to the existence of a unique invertible $G = G$ such that

$$A^\top G = GA, \; B^\top G = C, \; G = G^\top. \tag{59}$$

Finally, *time-reversibility* amounts to the existence of a unique invertible R such that

$$RA = -AR, \; RB = -B, \; C = CR, \; R = R^{-1}. \tag{60}$$

(Note the extra minus sign in front of B as compared to the Eqs. (54) defining *signed* time-reversibility.)

Proposition 4.5. *Consider a minimal system (A, B, C). Then*

1. *Suppose the system satisfies (58) and (59). Then $G\Omega^{-1}G = \Omega$, and $R := \Omega^{-1}G$ satisfies (60).*
2. *Suppose the system satisfies (59) and (60). Then $R^\top GR = -G$ and $\Omega := GR$ satisfies (58).*
3. *Suppose the system satisfies (58) and (60). Then $R^\top \Omega R = -\Omega$ and $G := \Omega R$ satisfies (59).*

Hence if the system satisfies two out of the three Eqs. (58), (59), (60), then it also satisfies the third.

Proof. If Ω and G satisfy (58), respectively (59), then also $G\Omega^{-1}G$ satisfies (58). Hence by uniqueness $G\Omega^{-1}G = \Omega$. It is immediately checked that $R := \Omega^{-1}G$ satisfies (60). This proves the first statement. Proof of the other two statements is analogous. □

Now suppose the input–output Hamiltonian system $\Sigma = (A, B, C)$ is *time-reversible* (and therefore also reciprocal). In view of $R^\top \Omega R = -\Omega$ this means, see [27,29], that there exist coordinates $x = (q, p)$ for $\mathcal{X}$ in which

$$\Omega = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \; R = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \text{ and therefore } G = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{61}$$

In such coordinates the system takes the appealing form [29]

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & P \\ -Q & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{B} \end{bmatrix} u \\ y &= \begin{bmatrix} \tilde{B}^\top & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \end{aligned} \tag{62}$$

for matrices $P = P^\top$, $Q = Q^\top$, and $\tilde{B}$. This is the form usually encountered in mechanical systems: inputs are external forces, outputs are collocated positions, and there are no gyroscopic forces (which otherwise would correspond to non-zero diagonal blocks in the A-matrix).

5. Conclusions and outlook

Two symmetry structures for linear systems, namely reciprocity and input–output Hamiltonian structure, have been studied in depth, linking various existing results in the literature, and providing new insights and results. A striking feature is the interplay between state space, input–output, and geometric formulations. Some extensions to the nonlinear case already exist, see e.g. [9,10] for the reciprocal case and [3] for the input–output Hamiltonian case, but many open questions remain. Another challenge is the further use of these system components for synthesis and control complex systems. From a theoretical perspective this calls for a theory of interconnection of systems with symmetry structures.

CRediT authorship contribution statement

Arjan van der Schaft: Writing – original draft, review & editing, Investigation, Formal analysis, Conceptualization. **Rodolphe Sepulchre:** Writing – review & editing, Investigation, Formal analysis, Conceptualization. **Thomas Chaffey:** Writing – review & editing, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This work was supported by funding from the European Research Council under the European Union's Horizon 2020 research and innovation program/ERC Advanced Grant : SpikyControl (7101054323).

Appendix A. Lagrangian subspaces

Let $\mathcal{F}$ be a linear space. Denote by $\mathcal{E} := \mathcal{F}^*$ its dual, with $\langle e, f\rangle$ the duality product between $f \in \mathcal{F}$ and $e \in \mathcal{E}$. Then $\mathcal{F}\times\mathcal{E}$ is endowed with the canonical *symplectic form*

$$\langle (f_a, e_a), (f_b, e_b)\rangle = \langle e_a, f_b\rangle - \langle e_b, f_a\rangle. \tag{A.1}$$

In case $\mathcal{F} = \mathbb{R}^n$ the symplectic form amounts to the matrix

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \tag{A.2}$$

A subspace $\mathcal{L} \subset \mathcal{F}\times\mathcal{E}$ is *Lagrangian* if the symplectic form $\langle\cdot,\cdot\rangle$ is *zero* restricted to $\mathcal{L}$, and furthermore $\mathcal{L}$ is *maximal* with respect to this property. In case $\mathcal{F}$ is an n-dimensional linear space this means $\dim\mathcal{L} = n$. Conversely, any n-dimensional subspace $\mathcal{L}$ on which $\langle\cdot,\cdot\rangle$ is zero is Lagrangian. Let $\mathcal{L} \subset \mathcal{F}\times\mathcal{E}$ be Lagrangian. If there exists a mapping $S : \mathcal{F} \to \mathcal{E}$ such that $\mathcal{L}$ is the graph of S, then S is *symmetric*. Conversely, for any symmetric $S : \mathcal{F} \to \mathcal{E}$ the subspace graph S is Lagrangian. Similarly, if there is a mapping $S : \mathcal{E} \to \mathcal{F}$ such that $\mathcal{L}$ is the graph of S.

If $\mathcal{L}$ can*not* be written as the graph of a map, either from $\mathcal{F}$ to $\mathcal{E}$ or from $\mathcal{E}$ to $\mathcal{F}$, we obtain the following general result in case $\mathcal{F}$ is finite-dimensional. Let $\mathcal{F} = \mathbb{R}^n$. Denote the linear coordinates for $\mathcal{F}$ and $\mathcal{E}$ by $f_1, \dots, f_n$ and $e_1, \dots, e_n$. Then for any Lagrangian subspace $\mathcal{L}$ there exists a *partitioning* $\{1, \dots, n\} = \mathcal{I}_1 \cup \mathcal{I}_2$ such that $\mathcal{L}$ is parametrized by the coordinates $f_i, i \in \mathcal{I}_1$, and $e_i, i \in \mathcal{I}_2$. Write (possibly after joint permutations in f and e)

$$f = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix}, \quad e = \begin{bmatrix} e^1 \\ e^2 \end{bmatrix}, \tag{A.3}$$

where the components of f^1 are $f_i, i \in \mathcal{I}_1$, and of f^2 are $f_i, i \in \mathcal{I}_2$. Same for e^1 and e^2. Thus there exists an $n\times n$ matrix S such that $\mathcal{L}$ is the graph of a mapping from f^1, e^2 to e^1, f^2, i.e.,

$$\mathcal{L} = \left\{ \begin{bmatrix} f^1 \\ f^2 \\ e^1 \\ e^2 \end{bmatrix} \mid \begin{bmatrix} e^1 \\ f^2 \end{bmatrix} = S \begin{bmatrix} f^1 \\ e^2 \end{bmatrix} \right\} \tag{A.4}$$

In this case S satisfies

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix} S = S^\top \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}, \tag{A.5}$$

where n_1, respectively n_2, is the cardinality of $\mathcal{I}_1$, respectively $\mathcal{I}_2$. Conversely, for any S satisfying (A.5) the subspace (A.4) is a Lagrangian subspace. Note that $\sigma = \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}$ is a *signature matrix*, and that S is *reciprocal* with respect to this signature matrix σ. The quadratic function

$$\begin{bmatrix} (f^1)^\top & (e^2)^\top \end{bmatrix} S \begin{bmatrix} f^1 \\ e^2 \end{bmatrix} \tag{A.6}$$

is called a *generating function* of the Lagrangian subspace (A.4).

The definition of a Lagrangian subspace (partly) generalizes to the infinite-dimensional case as follows; see e.g. [30] for further information. Let $\mathcal{F}$ be a Hilbert space and $\mathcal{E} = \mathcal{F}^*$ its dual. Then as before (A.1) defines a symplectic form on the Hilbert space $\mathcal{F}\times\mathcal{E}$. Similarly, a subspace $\mathcal{L} \subset \mathcal{F}\times\mathcal{E}$ is Lagrangian if the symplectic form $\langle\cdot,\cdot\rangle$ is *zero* restricted to $\mathcal{L}$, and furthermore $\mathcal{L}$ is *maximal* with respect to this property. It follows [30] that $\mathcal{L}$ is a *closed* subspace, and that there exists a complementary subspace $\mathcal{L}' \subset \mathcal{F}\times\mathcal{E}$ on which $\langle\cdot,\cdot\rangle$ is also zero. (Hence $\mathcal{F}\times\mathcal{E} = \mathcal{L}\oplus\mathcal{L}'$ and also $\mathcal{L}'$ is Lagrangian.)

Appendix B. Dirac structures

On the other hand, *Dirac structures* can be regarded as 'skew-symmetric' counterparts of Lagrangian subspaces. Instead of (A.1) consider the following bilinear form on $\mathcal{F}\times\mathcal{E}$ (replace $-$ by $+$)

$$[[(f_a, e_a), (f_b, e_b)]] := \langle e_a, f_b\rangle + \langle e_b, f_a\rangle \tag{B.1}$$

A subspace $\mathcal{D} \subset \mathcal{F}\times\mathcal{E}$ is a (constant) *Dirac structure* if the bilinear form $[[\cdot,\cdot]]$ is *zero* restricted to $\mathcal{D}$, and furthermore $\mathcal{D}$ is *maximal* with respect to this property. In case $\mathcal{F}$ is an n-dimensional linear space this means that $\dim\mathcal{D} = n$. Conversely, any n-dimensional subspace $\mathcal{D}$ on which $[[\cdot,\cdot]]$ is zero is a Dirac structure. Let $\mathcal{D} \subset \mathcal{F}\times\mathcal{E}$ be a Dirac structure. If there exists a mapping $Z : \mathcal{F} \to \mathcal{E}$ such that $\mathcal{D}$ is the graph of Z, then Z is *skew-symmetric*. Conversely, for any skew-symmetric $Z : \mathcal{F} \to \mathcal{E}$ the subspace graph Z is a Dirac structure. Analogously to the Lagrangian subspace case (A.4) we have the following result. For any Dirac structure $\mathcal{D} \subset \mathcal{F}\times\mathcal{E}$ there exist square matrices F, E, satisfying $FE^\top + EF^\top = 0$, such that $\mathcal{D} = \ker\begin{bmatrix} F & E \end{bmatrix}$; cf. [20]. Furthermore, cf. [31], modulo column permutations, one can always split $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$ and correspondingly $E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}$ such that $\begin{bmatrix} F_1 & E_2 \end{bmatrix}$ is invertible. Denote by $f = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix}$ and $e = \begin{bmatrix} e^1 \\ e^2 \end{bmatrix}$ the corresponding splitting of the vectors f, e. Then, similarly to (A.4),

$$\mathcal{L} = \left\{ \begin{bmatrix} f^1 \\ f^2 \\ e^1 \\ e^2 \end{bmatrix} \mid \begin{bmatrix} e^1 \\ f^2 \end{bmatrix} = J \begin{bmatrix} f^1 \\ e^2 \end{bmatrix} \right\}, \; J := -\begin{bmatrix} F_1 & E_2 \end{bmatrix}^{-1} \begin{bmatrix} E_1 & F_2 \end{bmatrix}, \tag{B.2}$$

where J is *skew-symmetric* (thus, in contrast with the Lagrangian subspace case, we do not need signature matrices.).

Data availability

No data was used for the research described in the article.

References

[1] R.W. Brockett, Control theory and analytical mechanics, in: C. Martin, R. Hermann (Eds.), in: Geometric Control Theory, Lie Groups: History, Frontiers, and Applications, vol. VII, Math. Sci. Press, Brookline, MA, 1977, pp. 1–46.
[2] A.J. van der Schaft, Hamiltonian dynamics with external forces and observations, Math. Syst. Theory 15 (1982) 145–168.
[3] P.E. Crouch, A.J. van der Schaft, Variational and Hamiltonian control systems, in: Lect. Notes in Control and Information Sciences, vol. 101, Springer-Verlag, Berlin, 1987.
[4] R. Pates, Passive and reciprocal networks, IEEE Control Syst. Mag. 42 (3) (2022) 73–92.
[5] A.J. van der Schaft, B.M. Maschke, Port-Hamiltonian systems on graphs, SIAM J. Control Optim. 51 (2) (2013) 906–937.
[6] A.J. van der Schaft, L_2-Gain and Passivity Techniques in Nonlinear Control, third ed., Springer International, Cham, Switzerland, 2017.

[7] B.D.O. Anderson, S. Vongpanitlerd, Network Analysis and Synthesis: A Modern Systems Theory Approach, Pearson Prentice-Hall, 1973.
[8] J.C. Willems, Dissipative dynamical systems, part II: Linear systems with quadratic supply rates, Arch. Ration. Mech. Anal. 45 (1972) 352–393.
[9] R. Sepulchre, T. Chaffey, F. Forni, Y. Huo, On the relaxation property of nonlinear circuit elements, in: 63rd IEEE Conf. Decision and Control, 2024, pp. 3729–3734.
[10] A.J. van der Schaft, Reciprocity of nonlinear systems, SIAM J. Control Optim. 62 (6) (2024) 3019–3041.
[11] T.H. Hughes, On reciprocal systems and controllability, Automatica 101 (2019) 396–408.
[12] T. Chaffey, H.J. van Waarde, R. Sepulchre, Relaxation systems and cyclic monotonicity, in: 62nd IEEE Conf. Decision and Control, 2023, pp. 1673–1679, see also arXiv:2312.03389.
[13] D.C. Sorensen, A.C. Antoulas, The sylvester equation and approximate balanced reduction, Linear Algebra Appl. 351–352 (2002) 671–700.
[14] K.V. Fernando, H. Nicholson, On the structure of balanced and other principal representations of SISO systems, IEEE Trans. Aut. Control. AC- 28 (2) (1983) 228–231.
[15] K.V. Fernando, H. Nicholson, On the cross-gramian for symmetric MIMO systems, IEEE Trans. Circuits Syst. CAS- 32 (5) (1985) 487–489.
[16] T.C. Ionescu, K. Fujimoto, J.M.A. Scherpen, Singular value analysis of nonlinear symmetric systems, IEEE Trans. Aut. Control. AC-56 (9) (2011) 2073–2086.
[17] J.M.A. Scherpen, A.J. van der Schaft, Balanced model reduction of gradient systems, in: Preprints of the 18th IFAC World Congress, Milano (Italy) August 28 - September 2, 2011, pp. 12745–12750.
[18] J.C. Willems, Realization of systems with internal passivity and symmetry constraints, J. Franklin Inst. 301 (6) (1976) 605–621.
[19] A.J. van der Schaft, On the relation between port-Hamiltonian and gradient systems, in: 18th IFAC World Congress, Milano (Italy, 2011, pp. 3321–3326.
[20] A. van der S.chaft, D. Jeltsema, Port-Hamiltonian systems theory, An introductory overview, Found. Trends Syst. Control. 1 (2–3) (2014) 173–378.
[21] A.J. van der Schaft, Positive feedback interconnection of Hamiltonian systems, in: 50th IEEE Conf. on Decision and Control and European Control Conf, (CDC-ECC) Orlando, FL, USA, December (6510), 2011, pp. 12–15–6515.
[22] I.R. Petersen, A. Lanzon, Feedback control of negative-imaginary systems, IEEE Control Syst. Mag. 30 (5) (2010) 54–72.
[23] A. Lanzon, I.R. Petersen, Stability robustness of a feedback interconnection of systems with negative imaginary frequency response, IEEE Trans. Autom. Control 53 (4) (2008) 1042–1046.
[24] D. Angeli, Systems with counterclockwise input–output dynamics, IEEE Trans. Autom. Control 51 (7) (2006) 1130–1143.
[25] A.J. van der Schaft, Duality for linear systems: External and state space characterization of the adjoint system, in: B. Bonnard, B. Bride, J.P. Gauthier, I. Kupka (Eds.), 393–403 in Analysis of Controlled Dynamical Systems, Birkhaüser, Boston, 1991.
[26] A.J. van der Schaft, J.C. Willems, A new procedure for stochastic realization of spectral density matrices, SIAM J. Control. Optim. vol. 22 (1984) 845–855.
[27] K.R. Meyer, Hamiltonian systems with a discrete symmetry, J. Differential Equations 41 (1981) 228–238.
[28] J.C. Willems, Time-reversibility in deterministic and stochastic systems, in: R.R. Mohler, A. Ruberti (eds.) (Eds.), Recent Developments in Variable Structure Systems, Economics and Biology, in: Lect. Notes in Economics and Math. Systems, Springer, Berlin-New York, 1978, pp. 318–326.
[29] A.J. van der Schaft, Time-reversible Hamiltonian systems, Systems Control Lett. 1 (5) (1982) 295–300.
[30] A. Weinstein, Symplectic manifolds and their Lagrangian submanifolds, Adv. Math. 6 (1971) 329–346.
[31] A.M. Bloch, P.E. Crouch, Representation of Dirac structures on vector spaces and nonlinear lcv-circuits, in: H. Hermes, G. Ferraya, R. Gardner, H. Sussman (Eds.), Proc. of Symposia in Pure Mathematics, in: Differential Geometry and Control Theory, vol. 64, 1999, pp. 103–117.