88-15

Separation of 2-D Nonsingular MIMO Digital Filters Into a Cascade of 1-D Filters

SHAOHUA TAN AND JOOS VANDEWALLE

Abstract—**This correspondence provides a simple algebraic criterion to check if a given 2-D MIMO filter described by a nonsingular transfer matrix is in fact a cascade of two 1-D filters with a minimal number of inputs and outputs. The criterion offers the advantage that whenever a filter is found to be separable, it is almost trivial to construct the two 1-D filters.**

I. Introduction

The analysis and design of a 2-D digital filter can often be considerably simplified when the filter is separable in the denominator [1], [2], [7]. Further simplification can be achieved if the filter is actually a cascade of two 1-D filters since in this case, the analysis and design of the 2-D filter reduce to those of 1-D filters; thus, the well-established techniques for 1-D filters (e.g., stability analysis, realization, etc.) can be applied. Therefore, it is often desirable to know if a predefined or encountered filter structure is factorizable into a cascade of two (minimal) 1-D filters.

In this correspondence, a technique developed by Van Dorren *et al.* [3] for 1-D rational matrix factorization is generalized to 2-D multiinput multioutput (MIMO) filters. Based on the technique, we state a criterion which can be used to check whether or not a given 2-D nonsingular MIMO filter is minimally separable into a cascade of two 1-D filters.

The development proceeds by first defining, among other things, the concept of minimal cascade separation of 2-D square MIMO filters. Then we state and prove the criterion. It is shown that when a 2-D filter is indeed separable, the two 1-D filters can be readily constructed. A discussion on the general implication of the result is also given.

II. Preliminaries

Let a 2-D square MIMO filter be given by transfer matrix $G(z_1, z_2)$ of dimension $p \times p$. In this correspondence, we shall restrict our attention to the nonsingular filters, i.e., $\det G(z_1, z_2) \neq 0$. $G(z_1, z_2)$ is said to be separable if

$$G(z_1, z_2) = G_1(z_1)\, G_2(z_2). \tag{1}$$

Further, it is said to be *minimally* separable if, in addition to (1), the dimensions of $G_1(z_1)$, $G_2(z_2)$ are also $p \times p$. The physical interpretation of the minimal separation is that the 2-D filter is a cascade of two 1-D filters, and the dynamics of the filter propagates first along one direction, then the other direction. Moreover, the minimality implies that the number of inputs and outputs of the 1-D filters attains the lowest possible value, so that no additional complexity with respect to the input and output of the factors is introduced.

Manuscript received September 21, 1987; revised December 31, 1987.

The authors are with the E.S.A.T. Laboratory, Department of Electrical Engineering, Katholieke Universiteit Leuven, B-3000 Heverlee, Belgium.

IEEE Log Number 8821168.

For every 2-D transfer matrix $G(z_1, z_2)$ of dimension $p \times p$, there exists a so-called local state-space realization given by [4], [5]

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i, j)$$

$$y(i, j) = [C_1 \quad C_2] \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + Du(i, j) \tag{2}$$

where $x^h(i, j)$ is the n vector of horizontal states, $x^v(i, j)$ is the m vector of vertical states, $u(i, j)$ is the p vector of inputs, and $y(i, j)$ is the q vector of outputs; A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D are real matrices of compatible sizes. Clearly, D is a square matrix, and throughout we shall assume, without loss of generality, that D is nonsingular. Indeed, if this is not the case, a conformal mapping $z_1 \to 1/(z_1 - \alpha)$, $z_2 \to 1/(z_2 - \beta)$ can be applied to make it so where α, β are two constants and $G(\alpha, \beta)$ is nonsingular. It may be of interest to know that a simple realization algorithm is also available in [4] for actually obtaining (2) from a given $G(z_1, z_2)$.

The local state-space form (2) is said to realize $G(z_1, z_2)$ if the following equation holds:

$$G(z_1, z_2) = [C_1 \quad C_2] \begin{bmatrix} I_n z_1 - A_{11} & -A_{12} \\ -A_{21} & I_m z_2 - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D. \tag{3}$$

Finally, when a realization (2) of $G(z_1, z_2)$ is given, it can be alternatively represented in the system matrix form [6]

$$\begin{bmatrix} I_n z_1 - A_{11} & -A_{12} & B_1 \\ -A_{21} & I_m z_2 - A_{22} & B_2 \\ -C_1 & -C_2 & D \end{bmatrix}. \tag{4}$$

We shall see that the system matrix form (4) is a very convenient starting point for our development.

III. The Criterion

Consider a 2-D square transfer matrix $G(z_1, z_2)$ represented by (4). We can then state and prove the following separability criterion.

Criterion 1: The 2-D filter represented by (4) is minimally separable if and only if one of the following two sets of conditions holds:

a) $A_{21} = 0$ and

$$\operatorname{rank} \begin{bmatrix} -A_{12} & B_1 \\ -C_2 & D \end{bmatrix} = p$$

b) $A_{12} = 0$ and

$$\operatorname{rank} \begin{bmatrix} -A_{21} & B_2 \\ -C_1 & D \end{bmatrix} = p.$$

Proof: We only supply a proof for a). A proof for b) is completely analogous.

Suppose that $G(z_1, z_2)$ is minimally separable; then there exist two 1-D rational matrices $G_1(z_1)$, $G_2(z_2)$ of dimension $p \times p$ such that $G(z_1, z_2) = G_1(z_1)\, G_2(z_2)$. Let the system matrices for $G_1(z_1)$ and $G_2(z_2)$ be

$$\begin{bmatrix} I_n z_1 - \bar{A} & \bar{B} \\ -\bar{C} & \bar{D} \end{bmatrix}, \begin{bmatrix} I_m z_2 - \tilde{A} & \tilde{B} \\ -\tilde{C} & \tilde{D} \end{bmatrix},$$

respectively, where by our convention (see Section I), the matrices $\bar{D}$ and $\tilde{D}$ are nonsingular and of dimension $p \times p$. Then the system

0096-3518/88/0700-1090$01.00 © 1988 IEEE

matrix for $G(z_1, z_2)$ ($= G_1(z_1)\, G_2(z_2)$) is easily checked to be

$$\begin{bmatrix} I_n z_1 - \bar{A} & -\bar{B}\bar{C} & \bar{B}\bar{D} \\ 0 & I_m z_2 - \tilde{A} & \tilde{B} \\ -\bar{C} & -\bar{D}\tilde{C} & \bar{D}\tilde{D} \end{bmatrix}. \tag{5}$$

We can see from (5) that $A_{21} = 0$ and

$$\operatorname{rank} \begin{bmatrix} -A_{12} & B_1 \\ -C_2 & D \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -\bar{B}\bar{C} & \bar{B}\bar{D} \\ -\bar{D}\bar{C} & \bar{D}\bar{D} \end{bmatrix} = p.$$

Conversely, let (4) be the system matrix of $G(z_1, z_2)$. If the condition a) is true, the following constant matrix factorization is possible:

$$\begin{bmatrix} -A_{12} & B_1 \\ -C_2 & D \end{bmatrix} = \begin{bmatrix} B_1^* \\ D_1^* \end{bmatrix} [-C_2^* \quad D_2^*] \tag{6}$$

where the first matrix on the right-hand side of (6) has full column rank p and the second has full row rank p. Hence, (4) can be seen to be minimally separable, i.e., $G(z_1, z_2) = G_1(z_1)\, G_2(z_2)$ where $G_1(z_1)$ and $G_2(z_2)$ have the system matrices given by

$$\begin{bmatrix} I_n z_1 - A_{11} & B_1^* \\ -C_1 & D_1^* \end{bmatrix}, \begin{bmatrix} I_m z_2 - A_{22} & B_2 \\ -C_2^* & D_2^* \end{bmatrix}.$$

The criterion is thus proven.

The reader may note that the above proof is constructive, and can be used to obtain the 1-D filters whenever the separation is possible. For convenience of the application, the main steps of the separability test are summarized as follows.

1) Verify that $G(z_1, z_2)$ is square and nonsingular; otherwise, the criterion may not apply.

2) Check if $D = \lim_{z_1 \to \infty, z_2 \to \infty} G(z_1, z_2)$ is nonsingular. If not, find a point $(z_1, z_2) = (\alpha, \beta)$ such that $G(\alpha, \beta)$ is nonsingular, then apply the conformal mapping $z_1 \to 1/(z_1 - \alpha)$, $z_2 \to 1/(z_2 - \beta)$ so that $G(1/z_1 - \alpha, 1/z_2 - \beta)$ is in the desired form.

3) Realize $G(z_1, z_2)$ into a local state-space form (4).

4) Apply the criterion to test the minimal separability. If a) or b) holds, $G(z_1, z_2)$ is minimally separable, and the two 1-D filters can be constructed via the factorization (6); otherwise, it is not minimally separable.

To illustrate, consider a 2-D filter $G(z_1, z_2)$ which has been realized by

$$\begin{bmatrix} z_1 - 0.1 & -1 & 4 & 0.2 & 0.6 \\ 0 & z_2 - 0.4 & 0.2 & 0.7 & -2 \\ 0 & -1 & z_2 - 0.1 & -0.5 & 0.1 \\ 3 & -2 & 8 & 0.4 & 1.2 \\ -7 & 0 & -1 & 1 & 0 \end{bmatrix}. \tag{7}$$

Since $A_{21}^T = [0 \quad 0]$ and

$$\operatorname{rank} \begin{bmatrix} -A_{12} & B_1 \\ -C_2 & D \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -1 & 4 & 0.2 & 0.6 \\ -2 & 8 & 0.4 & 1.2 \\ 0 & -1 & 1 & 0 \end{bmatrix} = 2$$

it follows from the criterion that (7) is minimally separable. Furthermore, the desired 1-D filters are easily calculated to be

$$\begin{bmatrix} z_1 - 0.1 & -4.1711 & 0.0474 \\ 3 & -8.3421 & 0.0948 \\ -7 & 0.9232 & 1.0713 \end{bmatrix},$$

$$\begin{bmatrix} z_2 - 0.4 & 0.2 & 0.7 & -2 \\ -1 & z_2 - 0.1 & -0.5 & 0.1 \\ 0.2374 & -0.9602 & -0.037 & -0.1425 \\ -0.2046 & -0.106 & 0.9653 & 0.1228 \end{bmatrix}.$$

IV. Conclusions

A factorization technique for 1-D rational matrices has been generalized to check the minimal separability of 2-D square MIMO filters. The criterion developed is interesting since it provides a simple way to test if a given filter is minimally separable. Once the filter is minimally separable, the two 1-D filters can be obtained directly.

Although the development has been confined to nonsingular MIMO filters, it is readily seen that the separation extends to square singular or nonsquare cases as well. However, in these cases, the minimality of the separation can no longer apply and needs to be redefined. In fact, some interesting results along this line have been obtained recently [8]. Observe that the minimality requirement in this correspondence implies that the intermediate connections between the two factors should not be greater than the number of external inputs or outputs. If this constraint is removed, one can expect more general separability results. The readers are referred to [8] for further details, and to [9], [10] for a number of interesting consequences.

References

[1] T. S. Huang, *Picture Processing and Digital Filtering*. New York: Springer-Verlag, 1975.

[2] B. Lashgari, L. M. Silverman, and J.-F. Abramatic, "Approximation of 2-D separable in denominator filters," *IEEE Trans. Circuits Syst.*, vol. CAS-30, pp. 107–121, 1983.

[3] P. M. Van Dooren and P. Dewilde, "Minimal cascade factorization of real and complex rational transfer matrices," *IEEE Trans. Circuits Syst.*, vol. CAS-28, pp. 390–400, 1981.

[4] R. P. Roesser, "A discrete state-space modal for linear image processing," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 1–10, 1975.

[5] S. Kung, B. C. Levy, M. Morf, and T. Kailath, "New results in 2-D system theory, Part II: 2-D state space modals-realization and the notions of controllability, observability and minimality," *Proc. IEEE*, vol. 65, pp. 945–961, 1977.

[6] H. H. Rosenbrock, *State Space and Multivariable Theory*. London: Nelson, 1970.

[7] M. Kawamata and T. Higuchi, "Synthesis of 2-D separable denominator digital filters with minimum roundoff noise and no overflow oscillations," *IEEE Trans. Circuits Syst.*, vol. CAS-33, p. 365–372, 1986.

[8] T. Lin, M. Kawamata, and T. Higuchi, "Decomposition of 2-D separable denominator systems: Existence, uniqueness, and applications," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 292–296, 1987.

[9] ——, "Design of 2-D separable denominator digital filters based on the reduced dimensional decompositions," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 934–941, 1987.

[10] ——, "On controllability, observability, and minimality of 2-D separable denominator systems: A new approach based on the reduced dimensional decomposition," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 962–967, 1987.

0096-3518/88/0700-1091$01.00 © 1988 IEEE