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An alternative proof for the minimal factorization theorem for rational matrices

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Abstract: In this note, an alternative derivation of the minimal first degree factorization theorem for rational matrices is presented. It is shown that when a minimal state-space realization for $G(s)$ is put into the Jordan form, a minimal first degree factorization of $G(s)$ can be read out directly without computation.

Keywords: Rational matrices; minimal first degree factorization; Jordan form.

1. Introduction

In a recent paper [1], a factorization theorem is presented which states that any rational matrix $G(s)$ with McMillan degree n always admits the following factorization:

$$G(s) = G_1(s) \cdots G_n(s) \tag{1}$$

where the McMillan degree of each $G_i(s)$ $(i = 1, \ldots, n)$ is 1. A factorization of this kind is called a *minimal first degree factorization*, and has found wide applications in a number of fields (see for example [2]).

While the proof provided in [1] only uses simple basic principles, it is primarily inspired by the structure of the global state-space realization assembled from realizations of two cascade factors. This approach is dominant at present in the literature on the minimal factorization problem. In this note, we present yet another proof of (1) by following an entirely different approach. It will seen that when a minimal state-space realization for $G(s)$ is brought to the Jordan form, the minimal first degree factorization of $G(s)$ can be obtained almost trivially.

2. The derivation

Our task here is to show that any $G(s)$ with McMillan degree n and coefficients over the complex field can be factorized into (1). To start with, let a minimal state-space realization of $G(s)$ be given in the following system matrix form:

$$P(s) = \begin{bmatrix} I_n s - A & B \\ -C & D \end{bmatrix}. \tag{2}$$

It is easy to see that $P(s)$ can be equivalently put into the form

$$\tilde{P}(s) = \begin{bmatrix} I_n s - A & 0 & B \\ 0 & I_m & \tilde{B} \\ -C & -\tilde{C} & 0 \end{bmatrix} \tag{3}$$

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where $D = \tilde{C}\tilde{B}$, $m = \text{rank } D$. Note that while any decomposition of the form $D = \overline{C}\,\overline{B}$ can serve our purpose, the special choice we have made here will lead to factors with smaller dimensions. This will become clear later.

Applying a similarity transformation to bring A into its Jordan form A_J (the matrices B, C will also be transformed, and we shall denote their transformed versions to be B_J, C_J respectively) and after simple computation we can establish that

$$\begin{bmatrix} I_n s - A_J & 0 \\ 0 & I_m \end{bmatrix}^{-1} = \text{block diag}\{N_1(s), \ldots, N_r(s),\ I_m\} \tag{4}$$

in which

$$N_i(s) = \begin{bmatrix} 1/(s-\alpha_i) & 1/(s-\alpha_i)^2 & \ldots & 1/(s-\alpha_i)^{f_i} \\ 0 & 1/(s-\alpha_i) & \ldots & 1/(s-\alpha_i)^{f_i-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1/(s-\alpha_i) \end{bmatrix} \quad (i = 1, \ldots, r) \tag{5}$$

and f_i (≥ 1) is the size for the i-th Jordan block; $n = \sum_{i=1}^{r} f_i$.

By extracting the common scalar factor $1/(s-\alpha_i)$ from the first row of $N_1(s)$, we have

$$\begin{bmatrix} I_n s - A_J & 0 \\ 0 & I_m \end{bmatrix}^{-1} = \begin{bmatrix} 1/(s-\alpha_1) & 0 \\ 0 & I \end{bmatrix} \left(\text{block diag}\{\overline{N}_1(s), \ldots, N_r(s),\ I_m\}\right) \tag{6}$$

where I is a unit matrix of appropriate size; $\overline{N}_1(s)$ is given by

$$\overline{N}_1(s) = \begin{bmatrix} 1 & 1/(s-\alpha_1) & \ldots & 1/(s-\alpha_1)^{f_1-1} \\ 0 & 1/(s-\alpha_1) & \ldots & 1/(s-\alpha_1)^{f_1-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1/(s-\alpha_1) \end{bmatrix}. \tag{7}$$

Adding to the first row the second row times (-1) in (7), $\overline{N}_1(s)$ reduces to

$$E\overline{N}_1(s) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1/(s-\alpha_1) & \ldots & 1/(s-\alpha_1)^{f_1-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1/(s-\alpha_1) \end{bmatrix}. \tag{8}$$

In the above E is an elementary constant matrix describing the operation. Taking (8) into account, (6) can be factorized into

$$\begin{bmatrix} I_n s - A_J & 0 \\ 0 & I_m \end{bmatrix}^{-1} = \left(\begin{bmatrix} 1/(s-\alpha_1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}^{-1}\right) \left(\text{block diag}\{1,\ N_1^*(s), \ldots, N_r(s),\ I_m\}\right) \tag{9}$$

where $N_1^*(s)$ has the same structure as $N_1(s)$ and its dimension is $f_1^* = f_1 - 1$. Obviously, the degree of the first factor on the right hand side of (9) is 1, and the degree of the second factor is $n-1$.

The same procedure can be applied to $N_1^*(s)$, and to $N_2(s)$ till $N_r(s)$. It is apparent that after exactly $n-r$ times of the above extractions, the block diagonal matrix (4) reduces to the unit matrix I_{n+m} times a sequence of n first degree factors, i.e.,

$$\begin{bmatrix} I_n s - A_J & 0 \\ 0 & I_m \end{bmatrix}^{-1} = R_1(s) G_2(s) \cdots G_{n-1}(s) R_n(s) \tag{10}$$

where each of the factors on the right hand side is of degree 1.

To obtain (1), we simply note

$$G(s) = \begin{bmatrix} C_J & \tilde{C} \end{bmatrix} \begin{bmatrix} I_n s - A_J & 0 \\ 0 & I_m \end{bmatrix}^{-1} \begin{bmatrix} B_J \\ \tilde{B} \end{bmatrix}$$

$$= \begin{bmatrix} C_J & \tilde{C} \end{bmatrix} R_1(s) G_2(s) \cdots G_{n-1}(s) R_n(s) \begin{bmatrix} B_J \\ \tilde{B} \end{bmatrix}. \tag{11}$$

The minimal first degree factorization is achieved by defining

$$G_1(s) = \begin{bmatrix} C_J & \tilde{C} \end{bmatrix} R_1(s), \qquad G_n(s) = R_n(s) \begin{bmatrix} B_J \\ \tilde{B} \end{bmatrix}.$$

Note that both of the above factors have degree 1. Our assertion is thus proven.

The factors produced in the preceding scheme appear to have two special features. First, except the first and the last factors, every factor is an $n + m$ square matrix. Even the first (the last) factor is simply such a square matrix pre- (post-) multiplied by a constant matrix. Secondly, because of the properties of the elementary matrices, each of the factors in (10) can be obtained by very simple and regular constructions. As a result, if a minimal state-space realization for $G(s)$ is put into the Jordan form, it is almost trivial to write down its factorization. The following example is illustrative.

Example. Consider the following system:

$$P(s) = \begin{bmatrix} s+1 & -1 & 0 & 0 & 0 \\ 0 & s+1 & 0 & 1 & 0 \\ 0 & 0 & s+1 & -1 & 1 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 \end{bmatrix}.$$

Since it is already in the Jordan form, we can immediately write down its minimal first degree factorization as

$$G(s) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1/(s+1) & 1/(s+1) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/(s+1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/(s+1) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

3. Conclusions

An alternative proof to establish (1) is presented in this note. The fundamental idea of the proof is to explore the structure of Jordan form and to perform the combined extraction and reduction. The obtained factors have highly regular constructs and uniform dimensions, as compared to those given in [1].

It should be noted that the minimal factorization (1) is only aimed at splitting the poles into its factors, and the concept of zero is not involved. In fact, the locations of the finite zeros in the factors is not under control although we do know that each of the factors in (10) has a unique zero at infinity. It should also be realized that our construction of the first degree minimal factors is largely based on theoretical considerations. Therefore, its practical as well as numerical aspects, especially when it is used for the realization of digital and analog filters, remain to be explored.

References

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