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# On a variational formulation of the QSVD and the RSVD☆

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**Abstract**

Recently, M.T. Chu, R.F. Funderlic and G.H. Golub [SIAM J. Matrix Anal. Appl., 18 (1997) 1082–1092] presented a variational formulation for the quotient singular value decomposition (QSVD) of two matrices $A \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times m}$ which is a generalization to two matrices of the ordinary singular value decomposition (SVD) and characterizes the role of

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two orthogonal matrices in QSVD. In this paper, we give an alternative derivation of this variational formulation and extend it to establish an analogous variational formulation for the Restricted Singular Value Decomposition (RSVD) of a matrix triplet $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{p\times m}$, which provides a new understanding of the orthogonal matrices appearing in this decomposition. © 2000 Elsevier Science Inc. All rights reserved.

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## 1. Introduction

The ordinary singular value decomposition (OSVD) of a given matrix $A \in \mathbb{R}^{n\times m}$ is

$$U^{\mathrm{T}} A V = \begin{matrix} \\ r_a \\ n-r_a \end{matrix} \begin{matrix} \begin{matrix} r_a & m-r_a \end{matrix} \\ \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \end{matrix}, \tag{1}$$

with $V \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{r_a\times r_a}$ and

$$\begin{matrix} \\ U = n \end{matrix} \begin{matrix} \begin{matrix} r_a & n-r_a \end{matrix} \\ \begin{bmatrix} U_1 & U_2 \end{bmatrix} \end{matrix}, \quad \begin{matrix} \\ V = m \end{matrix} \begin{matrix} \begin{matrix} r_a & m-r_a \end{matrix} \\ \begin{bmatrix} V_1 & V_2 \end{bmatrix} \end{matrix},$$
$$\Sigma = \operatorname{diag}\{\sigma_1, \ldots, \sigma_{r_a}\}, \quad \sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{r_a} > 0, \quad r_a = \operatorname{rank}(A),$$

where $U, V$ are orthogonal matrices. The $\sigma_1, \ldots, \sigma_{r_a}$ are the non-trivial singular values of $A$, and the columns of $U_1$ and $V_1$ are, respectively, the non-trivial left and right singular vectors of $A$. In this paper, $\|\cdot\|$ denotes the two-norm of a vector. The following theorem is well known [4].

**Theorem 1.** *Given $A \in \mathbb{R}^{n\times m}$ with OSVD* (1).
(a) *Consider the optimization problem*

$$\max_{\substack{x\in\mathbb{R}^m, y\in\mathbb{R}^n \\ y=Ax,\ y\neq 0}} \frac{\|y\|}{\|x\|}. \tag{2}$$

*Then the non-trivial singular values $\sigma_1, \ldots, \sigma_{r_a}$ of $A$ are precisely the stationary values, i.e., the functional evaluations at the stationary points, of* (2). *Let the stationary points in* (2) *corresponding to the stationary values $\sigma_1, \ldots, \sigma_{r_a}$ be $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \ldots, \begin{bmatrix} x_{r_a} \\ y_{r_a} \end{bmatrix}$. Then*

$$V_1 = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \cdots & \frac{x_{r_a}}{\|x_{r_a}\|} \end{bmatrix}.$$

*Moreover, if $n = m = r_a$, then*

$$U_1 = \begin{bmatrix} \frac{y_1}{\|y_1\|} & \cdots & \frac{y_{r_a}}{\|y_{r_a}\|} \end{bmatrix}.$$

(b) *Consider the dual optimization problem*

$$\max_{\substack{x\in\mathbb{R}^n, y\in\mathbb{R}^m \\ y^{\mathrm{T}}=x^{\mathrm{T}}A,\ y\neq 0}} \frac{\|y\|}{\|x\|}. \tag{3}$$

*Then the non-trivial singular values* $\sigma_1, \ldots, \sigma_{r_a}$ *of A are precisely the stationary values of* (3). *And, let the stationary points in* (3) *corresponding to the stationary values* $\sigma_1, \ldots, \sigma_{r_a}$ *be* $\begin{bmatrix} cx_1 \\ y_1 \end{bmatrix}, \ldots, \begin{bmatrix} x_{r_a} \\ y_{r_a} \end{bmatrix}$. *Then*

$$U_1 = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \cdots & \frac{x_{r_a}}{\|x_{r_a}\|} \end{bmatrix}.$$

*Moreover, if* $n = m = r_a$, *then*

$$V_1 = \begin{bmatrix} \frac{y_1}{\|y_1\|} & \cdots & \frac{y_{r_a}}{\|y_{r_a}\|} \end{bmatrix}.$$

Recently, in [1] Theorem 1 has been generalized to the quotient singular value decomposition (QSVD) [3,5–11,13,14] of two matrices $A \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times m}$. Naturally, it is very interesting to generalize the result in [1] to the restricted singular value decomposition (RSVD) [8–11,15,16] of a matrix triplet. After a study, we found that it is not trivial to get an analogous variational formulation for the RSVD via the approach in [1], this motivates us to re-derive the result in [1] via a different approach.

The purpose of this paper is twofold. Firstly, we present an alternative derivation of the variational formulation in [1] directly based on the QSVD of two matrices $A, C$. Then we extend this result to the RSVD [8–11,15,16] of a matrix triplet and obtain an analogous variational formulation that provides new understanding of the orthogonal matrices appearing in this decomposition.

Our approach is quite different from that one in [1]. We will show in Section 2 that an orthogonal reduction can be applied to the matrices $A \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times m}$ to get lower dimensional non-singular matrices $A_{22}$ and $C_{22}$ such that the non-trivial quotient singular values of the pair $(A, C)$ are just the standard singular values of $A_{22}C_{22}^{-1}$ and there is a close relationship between the two orthogonal matrices in the QSVD of $(A, C)$ and the (left and right) singular vectors of $A_{22}C_{22}^{-1}$ (see Lemma 2 and Theorem 3). Thus, the non-trivial quotient singular values of $(A, C)$ can be characterized by Theorem 1 with matrix $A_{22}C_{22}^{-1}$. Hence, Theorem 1 can be used as a bridge to re-derive the variational formulation in [1] for QSVD of $(A, C)$. Moreover, same idea also works for the RSVD (see Lemma 5 and Theorem 6), so it offers a springboard to leap forward to the variational formulation of the RSVD.

In order to prove our main results, we will establish two condensed forms based on orthogonal matrix transformations. The QSVD of two matrices and the RSVD of matrix triplets can be obtained and the variational formulation for QSVD and RSVD can be proved directly based on these two condensed forms.

In this paper, we use the following notation:

- $\mathscr{S}_\infty(M)$ denotes a matrix with orthogonal columns spanning the right nullspace of a matrix $M$.
- $\mathscr{T}_\infty(M)$ denotes a matrix with orthogonal columns spanning the right nullspace of a matrix $M^{\mathrm{T}}$.
- $M^\perp$ denotes the orthogonal complement of the space spanned by the columns of $M$.
- For a matrix $M$, $\mathscr{T}_\infty^{\mathrm{T}}(M)$ and $\mathscr{T}_\infty^\perp(M)$ are defined by $(\mathscr{T}_\infty(M))^{\mathrm{T}}$ and $(\mathscr{T}_\infty(M))^\perp$, respectively.
- Unless noted, we do not distinguish between a matrix with orthogonal columns and the space spanned by its columns.

We also use the following notation for any given matrices $A, B, C$ with compatible sizes: denote

$$r_a = \operatorname{rank}(A), \quad r_b = \operatorname{rank}(B), \quad r_c = \operatorname{rank}(C),$$

$$r_{ab} = \operatorname{rank}\begin{bmatrix} A & B \end{bmatrix}, \quad \bar{r}_{ac} = \operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix}, \quad r_{abc} = \operatorname{rank}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix},$$

$$k_1 = r_{abc} - r_b - r_c, \quad k_2 = r_{ab} + r_c - r_{abc},$$

$$k_3 = \bar{r}_{ac} + r_b - r_{abc}, \quad k_4 = r_a + r_{abc} - r_{ab} - \bar{r}_{ac}.$$

## 2. A variational formulation for QSVD

Several generalizations of the OSVD have been proposed and analyzed. One that is well known is the generalized SVD as introduced by Paige and Saunders in [5], which was proposed by De Moor and Golub [11] to rename as the QSVD. Another one is the RSVD, introduced in its explicit form by Zha [16] and further developed and discussed by De Moor and Golub [8].

In this section, we will give an alternative proof of the variational formulation for the QSVD of [1] based directly on QSVD itself. Firstly, we present a condensed form to derive the QSVD of two matrices.

**Lemma 2.** *Given matrices $A \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times m}$. Then there exist three orthogonal matrices $U_a \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{m\times m}$, $V_c \in \mathbb{R}^{p\times p}$ such that*

$$U_a^{\mathrm{T}} A W = \begin{array}{c} \\ \bar{r}_{ac} - r_c \\ r_a + r_c - \bar{r}_{ac} \\ n - r_a \end{array} \begin{array}{c} \begin{array}{cccc} \bar{r}_{ac} - r_c & r_a + r_c - \bar{r}_{ac} & \bar{r}_{ac} - r_a & m - \bar{r}_{ac} \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{4}$$

$$V_c^{\mathrm{T}} C W = \begin{array}{c} \\ p - r_c \\ r_a + r_c - \bar{r}_{ac} \\ \bar{r}_{ac} - r_a \end{array} \begin{array}{c} \begin{array}{cccc} \bar{r}_{ac} - r_c & r_a + r_c - \bar{r}_{ac} & \bar{r}_{ac} - r_a & m - \bar{r}_{ac} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{22} & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 \end{bmatrix} \end{array},$$

*where $A_{11}$, $A_{22}$, $C_{22}$ and $C_{33}$ are non-singular.*

**Proof.** See Appendix A. □

Let the SVD of $A_{22}C_{22}^{-1}$ be

$$U_{22}^{\mathrm{T}} A_{22} C_{22}^{-1} V_{22} = \mathrm{diag}\{\sigma_1, \ldots, \sigma_s\} =: S_{AC}, \quad s = r_a + r_c - \bar{r}_{ac}, \tag{5}$$

where $U_{22}$, $V_{22}$ are orthogonal matrices, $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_s > 0$. Define

$$U := U_a \,\mathrm{diag}\{I_{\bar{r}_{ac}-r_c}, U_{22}, I_{n-r_a}\}, \tag{6}$$

$$V := V_c \,\mathrm{diag}\{I_{p-r_c}, V_{22}, I_{\bar{r}_{ac}-r_a}\}. \tag{7}$$

$$X = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -C_{33}^{-1}C_{31} & -C_{33}^{-1}C_{32} & C_{33}^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \times \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}C_{22}^{-1}V_{22} & 0 & 0 \\ 0 & C_{22}^{-1}V_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{8}$$

Then, as a direct consequence of the condensed form (4), we have the following well-known QSVD theorem.

**Theorem 3** (QSVD theorem). *Let $A \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times m}$. There exist orthogonal matrices $U \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{p\times p}$ and a non-singular matrix $X$ such that*

$$U^{\mathrm{T}} A X = \begin{array}{c} \\ \bar{r}_{ac} - r_c \\ r_a + r_c - \bar{r}_{ac} \\ n - r_a \end{array} \begin{array}{c} \begin{array}{cccc} \bar{r}_{ac} - r_c & r_a + r_c - \bar{r}_{ac} & \bar{r}_{ac} - r_a & m - \bar{r}_{ac} \end{array} \\ \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & S_{AC} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$V^{\mathrm{T}} C X = \begin{array}{c} \\ p - r_c \\ r_a + r_c - \bar{r}_{ac} \\ \bar{r}_{ac} - r_a \end{array} \begin{array}{c} \begin{array}{cccc} \bar{r}_{ac} - r_c & r_a + r_c - \bar{r}_{ac} & \bar{r}_{ac} - r_a & m - \bar{r}_{ac} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \end{array}, \tag{9}$$

*where $S_{AC}$ is of the form* (5)*, and $U, V$ and $X$ can be chosen to be given by* (6)–(8)*, respectively. $\sigma_i, i = 1, \ldots, s$, are defined to be the non-trivial quotient singular values of the matrix pair $A, C$.*

According to the uniqueness theorem in [16], we only need to characterize the matrices $U, V$ given by (6) and (7) in order to characterize the role of the orthogonal matrices in the QSVD. Let $U, V$ be given by (6) and (7) and partition these two orthogonal matrices as

$$U = \begin{array}{c} \bar{r}_{ac} - r_c \quad r_a + r_c - \bar{r}_{ac} \quad n - r_a \\ \left[ \begin{array}{ccc} U_1 & U_2 & U_3 \end{array} \right] \end{array}, \tag{10}$$

$$V = \begin{array}{c} p - r_c \quad r_a + r_c - \bar{r}_{ac} \quad \bar{r}_{ac} - r_a \\ \left[ \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \right] \end{array}. \tag{11}$$

Then, from Lemma 2 we have

$$U_3 = \mathscr{T}_\infty(A), \quad U_1 = \mathscr{T}_\infty^\perp(A\mathscr{S}_\infty(C)), \tag{12}$$

$$V_1 = \mathscr{T}_\infty(C), \quad V_3 = \mathscr{T}_\infty^\perp(C\mathscr{S}_\infty(A)). \tag{13}$$

Hence, in order to characterize the role of the orthogonal matrices $U, V$ in the QSVD, it should only characterize the role of $U_2, V_2$ in QSVD.

The following variational formulation has been established in [1] to characterize $U_2$ and $V_2$.

**Theorem 4.** *Given $A \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times m}$. Consider the optimization problem*

$$\max_{\substack{x\in\mathbb{R}^n, y\in\mathbb{R}^p,\ x\neq 0 \\ \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ \mathscr{T}_\infty^{\mathrm{T}}(A) & 0 \\ 0 & \mathscr{T}_\infty^{\mathrm{T}}(C) \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} = 0}} \frac{\|y\|}{\|x\|}. \tag{14}$$

*Then the non-trivial quotient singular values $\sigma_1, \ldots, \sigma_s$ of the matrix pair $A, C$ are precisely the stationary values for the problem* (14)*. Furthermore, let $\begin{bmatrix} x_1 \\ -y_1 \end{bmatrix}, \ldots, \begin{bmatrix} x_s \\ -y_s \end{bmatrix}$ be stationary points of the problem* (14) *with corresponding stationary values $\sigma_1, \ldots, \sigma_s$, then*

$$U_2 = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \cdots & \frac{x_s}{\|x_s\|} \end{bmatrix}, \qquad V_2 = \begin{bmatrix} \frac{y_1}{\|y_1\|} & \cdots & \frac{y_s}{\|y_s\|} \end{bmatrix}.$$

**Proof.** We prove Theorem 4 by the following three arguments.

- *Argument* 1. Firstly, we characterize the orthogonal matrices $U_{22}, V_{22}$ in (5). Consider the optimization problem

$$\max_{\substack{x_2,y_2\in\mathbb{R}^{r_a+r_c-\bar{r}_{ac}}\\ x_2^{\mathrm{T}}A_{22}=y_2^{\mathrm{T}}C_{22},x_2\neq 0}} \frac{\|y_2\|}{\|x_2\|}. \tag{15}$$

Since $A_{22}$, $C_{22}$ are both non-singular, by Theorem 1 the $\sigma_1,\ldots,\sigma_s$, i.e., the singular values of the matrix $A_{22}C_{22}^{-1}$ are precisely the stationary values of the problem (15), and, if $\begin{bmatrix} x_2^1\\ -y_2^1\end{bmatrix},\ldots,\begin{bmatrix} x_2^s\\ -y_2^s\end{bmatrix}$ are the stationary points of the problem (15) with corresponding stationary values $\sigma_1,\ldots,\sigma_s$, then

$$U_{22}=\begin{bmatrix} \frac{x_2^1}{\|x_2^1\|} & \cdots & \frac{x_2^s}{\|x_2^s\|}\end{bmatrix},\quad V_{22}=\begin{bmatrix} \frac{y_2^1}{\|y_2^1\|} & \cdots & \frac{y_2^s}{\|y_2^s\|}\end{bmatrix}.$$

- *Argument* 2. Secondly, define

$$\mathscr{F}=\left\{\begin{bmatrix} x\\ -y\end{bmatrix}|x\in\mathbb{R}^n,\ y\in\mathbb{R}^p,\ U_a^{\mathrm{T}}x=\begin{matrix}\bar{r}_{ac}-r_c\\ r_a+r_c-\bar{r}_{ac}\\ n-r_a\end{matrix}\begin{bmatrix}0\\ x_2\\ 0\end{bmatrix},\right.$$

$$\left. V_c^{\mathrm{T}}y=\begin{matrix}p-r_c\\ r_a+r_c-\bar{r}_{ac}\\ \bar{r}_{ac}-r_a\end{matrix}\begin{bmatrix}0\\ y_2\\ 0\end{bmatrix},\ x^{\mathrm{T}}A=y^{\mathrm{T}}C, x\neq 0\right\}.$$

Consider the optimization problem

$$\max_{\begin{bmatrix} x\\ -y\end{bmatrix}\in\mathscr{F}} \frac{\|y\|}{\|x\|}. \tag{16}$$

Obviously, we have that $\begin{bmatrix} x\\ -y\end{bmatrix}$ is a stationary point of the problem (16) with stationary value $\sigma$ if and only if $\begin{bmatrix} x_2\\ -y_2\end{bmatrix}$ is a stationary point of the problem (15) with the same stationary point $\sigma$ and furthermore

$$U_a^{\mathrm{T}}x=\begin{matrix}\bar{r}_{ac}-r_c\\ r_a+r_c-\bar{r}_{ac}\\ n-r_a\end{matrix}\begin{bmatrix}0\\ x_2\\ \\ 0\end{bmatrix},\qquad V_c^{\mathrm{T}}y=\begin{matrix}p-r_c\\ r_a+r_c-\bar{r}_{ac}\\ \bar{r}_{ac}-r_a\end{matrix}\begin{bmatrix}0\\ y_2\\ 0\end{bmatrix}.$$

- *Argument* 3. Finally, for any $x\in\mathbb{R}^n$, $y\in\mathbb{R}^p$, partition

$$U_a^{\mathrm{T}}x=\begin{matrix}\bar{r}_{ac}-r_c\\ r_a+r_c-\bar{r}_{ac}\\ n-r_a\end{matrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},\qquad V_c^{\mathrm{T}}y=\begin{matrix}p-r_c\\ r_a+r_c-\bar{r}_{ac}\\ \bar{r}_{ac}-r_a\end{matrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}.$$

Since

$$U_a^{\mathrm{T}}\mathscr{T}_\infty(A) = \begin{matrix} \bar{r}_{ac}-r_c \\ r_a+r_c-\bar{r}_{ac} \\ n-r_a \end{matrix}\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix},$$

$$V_c^{\mathrm{T}}\mathscr{T}_\infty(C) = \begin{matrix} p-r_c \\ r_a+r_c-\bar{r}_{ac} \\ \bar{r}_{ac}-r_a \end{matrix}\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix},$$

it is easy to know that $\left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right] \in \mathscr{F}$ if and only if

$$\begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ \mathscr{T}_\infty^{\mathrm{T}}(A) & 0 \\ 0 & \mathscr{T}_\infty^{\mathrm{T}}(C) \end{bmatrix}\begin{bmatrix} x \\ -y \end{bmatrix} = 0, \quad x \neq 0.$$

Note that

$$U_a^{\mathrm{T}}U_2 = \begin{matrix} \bar{r}_{ac}-r_c \\ r_a+r_c-\bar{r}_{ac} \\ n-r_a \end{matrix}\begin{bmatrix} 0 \\ U_{22} \\ 0 \end{bmatrix}, \qquad V_c^{\mathrm{T}}V_2 = \begin{matrix} p-r_c \\ r_a+r_c-\bar{r}_{ac} \\ \bar{r}_{ac}-r_a \end{matrix}\begin{bmatrix} 0 \\ V_{22} \\ 0 \end{bmatrix},$$

thus, Theorem 4 follows directly from the above Arguments 1–3. □

## 3. A variational formulation for RSVD

In Section 2, we have derived the QSVD of two matrices $A, C$ based on the condensed form (4). Now we will establish the RSVD of a matrix triplet $(A, B, C)$ via an analogous condensed form.

**Lemma 5.** *Given $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{p\times m}$. Then there exist orthogonal matrices $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times m}$, $U_b \in \mathbb{R}^{l\times l}$, $V_c \in \mathbb{R}^{p\times p}$ such that*

$$PAQ = \begin{matrix} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \\ r_{ab}-r_a \\ n-r_{ab} \end{matrix}\begin{matrix} \begin{matrix} k_1 & k_2 & k_3 & k_4 & \bar{r}_{ac}-r_a & m-\bar{r}_{ac} \end{matrix} \\ \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & 0 & 0 \\ A_{41} & A_{42} & 0 & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix},$$

$$PBU_b = \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \\ r_{ab}-r_a \\ n-r_{ab} \end{array} \begin{array}{c} \begin{array}{cccc} l-r_b & k_3 & k_4 & r_{ab}-r_a \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & B_{14} \\ 0 & 0 & 0 & B_{24} \\ 0 & B_{32} & B_{33} & B_{34} \\ 0 & 0 & B_{43} & B_{44} \\ 0 & 0 & 0 & B_{54} \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{17}$$

$$V_c^{\mathrm{T}} C Q = \begin{array}{c} \\ p-r_c \\ k_2 \\ k_4 \\ \bar{r}_{ac}-r_a \end{array} \begin{array}{c} \begin{array}{cccccc} k_1 & k_2 & k_3 & k_4 & \bar{r}_{ac}-r_a & m-\bar{r}_{ac} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{22} & 0 & 0 & 0 & 0 \\ C_{31} & C_{32} & 0 & C_{34} & 0 & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & 0 \end{bmatrix} \end{array},$$

*where* $A_{11}, A_{22}, A_{33}, A_{44}, B_{32}, B_{43}, B_{54}, C_{22}, C_{34}$ *and* $C_{45}$ *are non-singular.*

**Proof.** See Appendix B. □

Let the SVD of $B_{43}^{-1} A_{44} C_{34}^{-1}$ be

$$U_{44}^{\mathrm{T}} B_{43}^{-1} A_{44} C_{34}^{-1} V_{44} = \mathrm{diag}\{\sigma_1, \ldots, \sigma_{k_4}\} =: S_{ABC}, \tag{18}$$

where $U_{44}$, $V_{44}$ are orthogonal matrices, $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{k_4} > 0$. Define

$$U := U_b \begin{bmatrix} I_{l-r_b} & & & \\ & I_{k_3} & & \\ & & U_{44} & \\ & & & I_{r_{ab}-r_a} \end{bmatrix}, \tag{19}$$

$$V := V_c \begin{bmatrix} I_{p-r_c} & & & \\ & I_{k_2} & & \\ & & V_{44} & \\ & & & I_{\bar{r}_{ac}-r_a} \end{bmatrix}. \tag{20}$$

Similarly to Theorem 3, from Lemma 5 directly, we have the following theorem.

**Theorem 6** (RSVD theorem). *Given* $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{p\times m}$. *Then there exist non-singular matrices* $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{m\times m}$ *and orthogonal matrices* $U \in \mathbb{R}^{l\times l}$, $V \in \mathbb{R}^{p\times p}$ *such that*

$$X^{\mathrm{T}}AY = \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \\ r_{ab}-r_a \\ n-r_{ab} \end{array} \begin{array}{c} \begin{array}{cccccc} k_1 & k_2 & k_3 & k_4 & \bar{r}_{ac}-r_a & m-\bar{r}_{ac} \end{array} \\ \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{ABC} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$X^{\mathrm{T}}BU = \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \\ r_{ab}-r_a \\ n-r_{ab} \end{array} \begin{array}{c} \begin{array}{cccc} l-r_b & k_3 & k_4 & r_{ab}-r_a \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{21}$$

$$V^{\mathrm{T}}CY = \begin{array}{c} \\ p-r_c \\ k_2 \\ k_4 \\ \bar{r}_{ac}-r_a \end{array} \begin{array}{c} \begin{array}{cccccc} k_1 & k_2 & k_3 & k_4 & \bar{r}_{ac}-r_a & m-\bar{r}_{ac} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix} \end{array},$$

*where $S_{ABC}$ is of the form* (18)*, and $U$, $V$ can be chosen to be given by* (19) *and* (20)*, respectively, $\sigma_1, \ldots, \sigma_{k_4}$ are defined to be the non-trivial restricted singular values of the matrix triplet $A$, $B$, $C$.*

From the uniqueness theorem in [16], we only need to consider the matrices $U$, $V$ given by (19) and (20) in order to characterize the role of the orthogonal matrices in the RSVD. Let $U$, $V$ be defined by (19) and (20), respectively, and partition

$$U = \begin{array}{c} \begin{array}{cccc} l-r_b & k_3 & k_4 & r_{ab}-r_a \end{array} \\ \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \end{bmatrix} \end{array},$$

$$V = \begin{array}{c} \begin{array}{cccc} p-r_c & k_2 & k_4 & \bar{r}_{ac}-r_a \end{array} \\ \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \end{bmatrix} \end{array}.$$

We have

$$U_1 = \mathscr{S}_\infty(B), \quad U_4 = \mathscr{S}_\infty^{\perp}(\mathscr{T}_\infty^{\mathrm{T}}(A)B),$$
$$V_1 = \mathscr{T}_\infty(C), \quad V_4 = \mathscr{T}_\infty^{\perp}(C\mathscr{S}_\infty(A)).$$

Furthermore, if we define

$$\Psi_1 := C\mathscr{S}_\infty(\mathscr{T}_\infty^{\mathrm{T}}(B)A),$$

$$\Psi_2 := A\mathscr{S}_\infty(\mathscr{T}_\infty^{\mathrm{T}}(B)A),$$

$$\Psi_3 := (\mathscr{T}_\infty(\Psi_2\mathscr{S}_\infty(\Psi_1)))^{\mathrm{T}}B,$$

then,

$$\Psi_1 = V_c \times \begin{matrix} \\ p-r_c \\ k_2 \\ k_4 \\ \bar{r}_{ac}-r_a \end{matrix} \begin{matrix} \begin{matrix} k_3 & k_4 & \bar{r}_{ac}-r_a & m-\bar{r}_{ac} \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & C_{34} & 0 & 0 \\ C_{43} & C_{44} & C_{45} & 0 \end{bmatrix} \end{matrix},$$

$$\Psi_2 = P^{\mathrm{T}} \times \begin{matrix} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \\ r_{ab}-r_a \\ n-r_{ab} \end{matrix} \begin{matrix} \begin{matrix} k_3 & k_4 & \bar{r}_{ac}-r_a & m-\bar{r}_{ac} \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{33} & A_{34} & 0 & 0 \\ 0 & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix},$$

$$\Psi_3 = \begin{matrix} k_1 \\ k_2 \\ k_4 \\ r_{ab}-r_a \\ n-r_{ab} \end{matrix} \begin{matrix} \begin{matrix} l-r_b & k_3 & k_4 & r_{ab}-r_a \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & B_{14} \\ 0 & 0 & 0 & B_{24} \\ 0 & 0 & B_{43} & B_{44} \\ 0 & 0 & 0 & B_{54} \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix} U_b^{\mathrm{T}}. \tag{22}$$

Hence, from (22) we have

$$\begin{bmatrix} U_1 & U_2 \end{bmatrix} = \mathscr{S}_\infty(\Psi_3).$$

Thus, in order to characterize the role of orthogonal matrices $U, V$ in the RSVD we only need to characterize the role of $U_3, V_3$ of $U, V$ in the RSVD. This can be done by the following variational formulation.

**Theorem 7.** *Given matrices $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{p\times m}$. Consider the optimization problem*

$$\max_{x\in\mathbb{R}^m,\ y\in\mathbb{R}^l,\ x\neq 0} \frac{\|y\|}{\|Cx\|}. \tag{23}$$

$$\begin{matrix} \begin{matrix} A & B \end{matrix} \\ \begin{bmatrix} \mathscr{S}_\infty^{\mathrm{T}}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) & 0 \\ \mathscr{S}_\infty^{\mathrm{T}}(A)C^{\mathrm{T}}C & 0 \\ 0 & \mathscr{S}_\infty^{\mathrm{T}}(B) \\ 0 & \mathscr{T}_\infty^{\mathrm{T}}(B\mathscr{S}_\infty^{\perp}(\Psi_3))B \end{bmatrix} \end{matrix} \begin{bmatrix} x \\ -y \end{bmatrix} = 0$$

*Then the stationary values for the problem* (23) *are precisely the non-trivial restricted singular values* $\sigma_1, \ldots, \sigma_{k_4}$ *of the matrix triplet* $A, B, C$. *Moreover, if* $\begin{bmatrix} x_1 \\ -y_1 \end{bmatrix}, \ldots, \begin{bmatrix} x_{k_4} \\ -y_{k_4} \end{bmatrix}$ *are the stationary point of the problem* (23) *corresponding to the stationary values* $\sigma_1, \ldots, \sigma_{k_4}$, *respectively, then*

$$U_3 = \begin{bmatrix} \frac{y_1}{\|y_1\|} & \cdots & \frac{y_{k_4}}{\|y_{k_4}\|} \end{bmatrix}.$$

**Proof.** Same as the proof of Theorem 4, we prove part (a) by the following three arguments.

- *Argument* 1. Firstly, we characterize $U_{44}$ in (18). Consider the optimization problem

$$\max_{\substack{x_4, y_3 \in \mathbb{R}^{k_4}, y_3 \neq 0 \\ A_{44}x_4 = B_{43}y_3}} \frac{\|y_3\|}{\|C_{34}x_4\|}. \tag{24}$$

  Since $A_{44}, B_{43}, C_{34}$ are non-singular, so by Theorem 1, the stationary values of the problem (24) are precisely $\sigma_1, \ldots, \sigma_{k_4}$, i.e., the singular values of the matrix $B_{43}^{-1}A_{44}C_{34}^{-1}$, and, if let the corresponding stationary points be $\begin{bmatrix} x_4^1 \\ -y_3^1 \end{bmatrix}, \ldots, \begin{bmatrix} x_4^{k_4} \\ -y_3^{k_4} \end{bmatrix}$, then

$$U_{44} = \begin{bmatrix} \frac{y_3^1}{\|y_3^1\|} & \cdots & \frac{y_3^{k_4}}{\|y_3^{k_4}\|} \end{bmatrix}.$$

- *Argument* 2. Secondly, define

$$\mathscr{F} := \left\{ \begin{bmatrix} x \\ -y \end{bmatrix} \middle| x \in \mathbb{R}^m,\ y \in \mathbb{R}^l,\ U_b^T y = \begin{matrix} l - r_b \\ k_3 \\ k_4 \\ r_{ab} - r_a \end{matrix} \begin{bmatrix} 0 \\ 0 \\ y_3 \\ 0 \end{bmatrix},\ Q^T x = \begin{matrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ \bar{r}_{ac} - r_a \\ m - \bar{r}_{ac} \end{matrix} \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \\ x_5 \\ 0 \end{bmatrix}, \right.$$

$$\left. C_{43}x_3 + C_{44}x_4 + C_{45}x_5 = 0,\ Ax = By,\ y \neq 0 \right\}.$$

  Consider the optimization problem

$$\max_{\begin{bmatrix} x \\ -y \end{bmatrix} \in \mathscr{F}} \frac{\|y\|}{\|Cx\|}. \tag{25}$$

  Since $A_{33}, A_{44}, B_{43}$ and $C_{45}$ are non-singular, so a simple calculation yields that the problem (25) are equivalent to the problem (24) in the sense that the stationary

values of the problem (25) are precisely the stationary values of the problem (24), i.e., $\sigma_1, \ldots, \sigma_{k_4}$, and, $\left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right]$ is the stationary point of the problem (25) if and only if $\left[\begin{smallmatrix} x_4 \\ -y_3 \end{smallmatrix}\right]$ is the stationary point of the problem (24) with same stationary value.

- *Argument* 3. Thirdly, for any $x \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, denote

$$U_b^{\mathrm{T}} y = \begin{matrix} l - r_b \\ k_3 \\ k_4 \\ r_{ab} - r_a \end{matrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad Q^{\mathrm{T}} x = \begin{matrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ \bar{r}_{ac} - r_a \\ m - \bar{r}_{ac} \end{matrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}.$$

Since

$$\mathscr{S}_\infty^{\mathrm{T}} \left( \begin{bmatrix} A \\ C \end{bmatrix} \right) x = 0 \Longleftrightarrow x_6 = 0,$$

$$Ax = By \Longrightarrow x_1 = 0,\ x_2 = 0,\ y_4 = 0,$$

$$\mathscr{S}_\infty^{\mathrm{T}}(A) C^{\mathrm{T}} C x = 0 \Longleftrightarrow C_{41} x_1 + C_{42} x_2 + C_{43} x_3 + C_{44} x_4 + C_{45} x_5 = 0,$$

$$\mathscr{S}_\infty^{\mathrm{T}}(B) y = 0 \Longleftrightarrow y_1 = 0.$$

From (22), we also know

$$\mathscr{T}_\infty^{\mathrm{T}}(B \mathscr{S}_\infty^{\perp}(\Psi)) B y = 0 \Longleftrightarrow y_2 = 0.$$

Therefore, we have that $\left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right] \in \mathscr{F}$ if and only if

$$\begin{bmatrix} A & B \\ \mathscr{S}_\infty^{\mathrm{T}} \left( \begin{bmatrix} A \\ C \end{bmatrix} \right) & 0 \\ \mathscr{S}_\infty^{\mathrm{T}}(A) C^{\mathrm{T}} C & 0 \\ 0 & \mathscr{S}_\infty^{\mathrm{T}}(B) \\ 0 & \mathscr{T}_\infty^{\mathrm{T}}(B \mathscr{S}_\infty^{\perp}(\Psi_3)) B \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} = 0, \quad y \neq 0.$$

Note that

$$U_b^{\mathrm{T}} U_3 = \begin{matrix} l - r_b \\ k_3 \\ k_4 \\ r_{ab} - r_a \end{matrix} \begin{bmatrix} 0 \\ 0 \\ U_{44} \\ 0 \end{bmatrix},$$

so, Theorem 7 follows directly from the above Arguments 1–3. □

Similarily, we also have the dual result of Theorem 7 which characterizes the non-trivial generalized singular values $\sigma_1, \ldots, \sigma_{k_4}$ and the matrix $V_3$ in (20). For the sake of simplicity, we omit it here.

## 4. Conclusion

In this paper, we have studied generalized SVDs. We have given an alternative proof of the variational formulation for the QSVD in [1] and established an analogous variational formulation for the RSVD which provides a new understanding of the orthogonal matrices appearing in this decomposition.

The proofs of our main results (i.e., Theorems 4 and 7) provide deflation procedures for the numerical computation of two orthogonal matrices in the QSVD and RSVD. For example, we can compute the non-trivial restricted singular values and orthogonal matrix $U$ in the RSVD (21) as follows:

- Compute the condensed form (17) by QR factorizations.
- Apply existing deflation process [4] to the optimization problem (24) to get the non-trivial restricted singular values $\sigma_1, \ldots, \sigma_{k_4}$ in (21) and the corresponding orthogonal matrix $U_{44}$. The computational complexity of the optimization problem (24) is much lower than the variational formulation (23). This is one advantage of our deflation procedure.
- Compute the orthogonal matrix $U$ by (19).

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## Appendix A

In this appendix, we prove Lemma 2 constructively.

**Proof.** We prove Lemma 2 by four steps as follows:

*Step* 1: Perform simulaneous row and column compression:

$$U_1^{\mathrm{T}} A W_1 = \begin{matrix} r_a \\ n - r_a \end{matrix} \begin{bmatrix} \begin{matrix} r_a \\ A_{11}^{(1)} \\ 0 \end{matrix} & \begin{matrix} \bar{r}_{ac} - r_a \\ 0 \\ 0 \end{matrix} & \begin{matrix} m - \bar{r}_{ac} \\ 0 \\ 0 \end{matrix} \end{bmatrix},$$

$$V_1^{\mathrm{T}} C W_1 = \begin{array}{c} \\ p - r_c \\ r_a + r_c - \bar{r}_{ac} \\ \bar{r}_{ac} - r_a \end{array} \begin{array}{c} \begin{array}{ccc} r_a & \bar{r}_{ac} - r_a & m - \bar{r}_{ac} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ C_{21}^{(1)} & 0 & 0 \\ C_{31}^{(1)} & C_{33} & 0 \end{bmatrix} \end{array}$$

with $A_{11}^{(1)}$, $C_{33}$ non-singular and $C_{21}^{(1)}$ full row rank.

*Step* 2: Perform a column compression:

$$C_{21}^{(1)} W_2 = \begin{array}{c} \\ \end{array} \begin{array}{c} \begin{array}{cc} \bar{r}_{ac} - r_c & r_a + r_c - \bar{r}_{ac} \end{array} \\ \begin{bmatrix} 0 & C_{22} \end{bmatrix} \end{array}$$

with $C_{22}$ non-singular. Set

$$A_{11}^{(1)} W_2 = \begin{array}{c} \\ \end{array} \begin{array}{c} \begin{array}{cc} \bar{r}_{ac} - r_c & r_a + r_c - \bar{r}_{ac} \end{array} \\ \begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} \end{bmatrix} \end{array},$$

$$C_{31}^{(1)} W_2 = \begin{array}{c} \\ \end{array} \begin{array}{c} \begin{array}{cc} \bar{r}_{ac} - r_c & r_a + r_c - \bar{r}_{ac} \end{array} \\ \begin{bmatrix} C_{31} & C_{32} \end{bmatrix} \end{array}.$$

*Step* 3: Perform a row compression:

$$U_3^{\mathrm{T}} A_{11}^{(2)} = \begin{array}{c} \bar{r}_{ac} - r_c \\ r_a + r_c - \bar{r}_{ac} \end{array} \begin{bmatrix} A_{11} \\ 0 \end{bmatrix}$$

with $A_{11}$ non-singular. Set

$$U_3^{\mathrm{T}} A_{12}^{(2)} = \begin{array}{c} \bar{r}_{ac} - r_c \\ r_a + r_c - \bar{r}_{ac} \end{array} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}.$$

*Step* 4: Set

$$U_a = U_1 \begin{bmatrix} U_3 & \\ & I \end{bmatrix}, \qquad W = W_1 \begin{bmatrix} W_2 & \\ & I \end{bmatrix}, \quad V_c = V_1.$$

Then orthogonal matrices $U_a$, $V_c$ and $W$ satisfy (4). □

## Appendix B

Now we prove Lemma 5 constructively.

**Proof.** We prove Lemma 5 by five steps as follows:

*Step* 1: Compute orthogonal matrices $P_1$, $Q_1$ and $U_1$ by applying Lemma 2 to $(A^{\mathrm{T}}, B^{\mathrm{T}})$ such that

$$P_1 A Q_1 = \begin{array}{c} \\ r_{ab}-r_b \\ r_a+r_b-r_{ab} \\ r_{ab}-r_a \\ n-r_{ab} \end{array} \begin{array}{c} \begin{array}{ccc} r_{ab}-r_b & r_a+r_b-r_{ab} & m-r_a \end{array} \\ \begin{bmatrix} \Theta_{11} & 0 & 0 \\ \Theta_{21} & \Theta_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$P_1 B U_1 = \begin{array}{c} \\ r_{ab}-r_b \\ r_a+r_b-r_{ab} \\ r_{ab}-r_a \\ n-r_{ab} \end{array} \begin{array}{c} \begin{array}{ccc} l-r_b & r_a+r_b-r_{ab} & r_{ab}-r_a \end{array} \\ \begin{bmatrix} 0 & 0 & \Phi_{14} \\ 0 & \Phi_{32} & \Phi_{34} \\ 0 & 0 & B_{54} \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

with $\Theta_{11}$, $\Theta_{22}$, $\Phi_{32}$ and $B_{54}$ non-singular. Set

$$C Q_1 = \begin{array}{c} \begin{array}{ccc} r_{ab}-r_b & r_a+r_b-r_{ab} & m-r_a \end{array} \\ \begin{bmatrix} C_1^{(1)} & C_2^{(1)} & C_3^{(1)} \end{bmatrix} \end{array}.$$

*Step* 2: Compute orthogonal matrices $P_2$, $Q_2$ and $V_2$ based on Lemma 2 such that

$$P_2 \begin{bmatrix} \Theta_{22} & 0 \end{bmatrix} Q_2 = \begin{array}{c} \\ k_3 \\ k_4 \end{array} \begin{array}{c} \begin{array}{cccc} k_3 & k_4 & \bar{r}_{ac}-r_a & m-\bar{r}_{ac} \end{array} \\ \begin{bmatrix} A_{33} & A_{34} & 0 & 0 \\ 0 & A_{44} & 0 & 0 \end{bmatrix} \end{array},$$

$$V_2 \begin{bmatrix} C_2^{(1)} & C_3^{(1)} \end{bmatrix} Q_2 = \begin{array}{c} \\ p-k_4-\bar{r}_{ac}-r_a \\ k_4 \\ \bar{r}_{ac}-r_a \end{array} \begin{array}{c} \begin{array}{cccc} k_3 & k_4 & \bar{r}_{ac}-r_a & m-\bar{r}_{ac} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{34} & 0 & 0 \\ C_{43} & C_{44} & C_{45} & 0 \end{bmatrix} \end{array}$$

with $A_{33}$, $A_{44}$ and $C_{34}$, $C_{45}$ non-singular. Set

$$V_2 C_1^{(1)} = \begin{array}{c} \\ p-k_4-\bar{r}_{ac}-r_a \\ k_4 \\ \bar{r}_{ac}-r_a \end{array} \begin{bmatrix} C_{11}^{(2)} \\ C_{31}^{(2)} \\ C_{41}^{(2)} \end{bmatrix}.$$

*Step* 3: Perform a simultaneous row and column compression:

$$V_3 C_{11}^{(2)} Q_3 = \begin{array}{c} \\ p-r_c \\ k_2 \end{array} \begin{array}{c} \begin{array}{cc} k_1 & k_2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & C_{22} \end{bmatrix} \end{array}$$

with $C_{22}$ non-singular. Set

$$\begin{bmatrix} C_{31}^{(2)} \\ C_{41}^{(2)} \end{bmatrix} Q_3 = \begin{array}{c} \\ k_4 \\ \bar{r}_{ac}-r_a \end{array} \begin{array}{c} \begin{array}{cc} k_1 & k_2 \end{array} \\ \begin{bmatrix} C_{31} & C_{32} \\ C_{41} & C_{42} \end{bmatrix} \end{array}.$$

*Step* 4: Perform a row compression and a column compression:

$$P_4\Theta_{11}Q_3 = \begin{matrix} \\ k_1 \\ k_2 \end{matrix}\begin{matrix} \begin{matrix} k_1 & k_2 \end{matrix} \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \end{matrix}, \qquad P_2\Phi_{32}U_4 = \begin{matrix} \\ k_3 \\ k_4 \end{matrix}\begin{matrix} \begin{matrix} k_3 & k_4 \end{matrix} \\ \begin{bmatrix} B_{32} & B_{33} \\ 0 & B_{43} \end{bmatrix} \end{matrix},$$

where $A_{11}$, $A_{22}$ and $B_{32}$, $B_{43}$ are non-singular. Set

$$P_2\Theta_{21}Q_4 = \begin{matrix} \\ k_3 \\ k_4 \end{matrix}\begin{matrix} \begin{matrix} k_1 & k_2 \end{matrix} \\ \begin{bmatrix} A_{31} & A_{32} \\ A_{41} & A_{42} \end{bmatrix} \end{matrix}, \qquad \begin{bmatrix} P_4\Phi_{14} \\ P_2\Phi_{34} \end{bmatrix} = \begin{matrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{matrix}\begin{bmatrix} B_{14} \\ B_{24} \\ B_{34} \\ B_{44} \end{bmatrix}.$$

*Step* 5: Set

$$P = \begin{bmatrix} P_4 & \\ & I \end{bmatrix}\begin{bmatrix} I_{r_{ab}-r_b} & & \\ & P_2 & \\ & & I_{n-r_a} \end{bmatrix} P_1,$$

$$Q = Q_1\begin{bmatrix} I & \\ & Q_2 \end{bmatrix}\begin{bmatrix} Q_3 & \\ & I \end{bmatrix},$$

$$U_b = U_1\begin{bmatrix} I_{l-r_b} & & \\ & U_4 & \\ & & I_{r_{ab}-r_a} \end{bmatrix}, \quad V_c^{\mathrm{T}} = \begin{bmatrix} V_3 & \\ & I \end{bmatrix} V_2.$$

Then, the orthogonal matrices $P$, $Q$, $U_b$ and $V_c$ satisfy the condensed form (17). □

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