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Brief paper

Analytical representation of the state-dependent coefficients in the SDRE/SDDRE scheme for multivariable systems☆

Li-Gang Lin [a,b,1], Joos Vandewalle [a], Yew-Wen Liang [b]

[a] ESAT-STADIUS, KU Leuven, Kasteelpark Arenberg 10, B-3001 Leuven-Heverlee, Belgium

[b] Institute of Electrical Control Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, 30010, Taiwan

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ABSTRACT

Recently, the easy-to-implement state-dependent Riccati equation (SDRE) strategy has been shown effective for numerous practical applications. Since being similar to SDRE, the newly emerged state-dependent differential Riccati equation (SDDRE) approach shares most of the benefits of SDRE, and exhibits interesting potential from both the analytical and practical viewpoints. By generalizing previous works to the general-order nonlinear time-variant systems, we try to resolve several problems related to the design flexibility (the infinitely many candidates of the state-dependent coefficient matrix), since they appear at the beginning of the implementation of both schemes. Finally, we demonstrate the proposed scheme via examples.

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1. Introduction

Among various nonlinear control techniques (Khalil, 1996), the state-dependent Riccati equation (SDRE) scheme has recently attracted considerable attention, and has become a promising and popular synthesis tool (e.g. Cloutier, D'Souza, and Mracek (1996), Erdem and Alleyne (2004), Mracek and Cloutier (1998) and Shamma and Cloutier (2003)). Inspired by SDRE, the state-dependent differential Riccati equation (SDDRE) scheme is progressing promisingly for nonlinear system control (e.g. Heydari and Balakrishnan (2013); Heydari, Landers, and Balakrishnan (2014); Pan, Kumar, Liu, and Furuta (2009)). Unlike aiming at infinite-time horizon nonlinear optimal control (ITHNOC) for the SDRE strategy, the SDDRE scheme focuses on the finite-time horizon nonlinear optimal control (FTHNOC). Associated with FTHNOC (resp., ITHNOC), the Hamilton–Jacobi–Bellman, HJB, (resp., Hamilton–Jacobi, HJ) equation is required to calculate the optimal control law, and is generally difficult to solve (Heydari & Balakrishnan, 2013) (resp., Çimen (2012)). Therefore, the goal of the SDDRE (resp., SDRE) control strategy is trying to deal with the FTHNOC (resp., ITHNOC) problem in a more implementable way.

In the literature (e.g. Çimen (2010, 2012); Liang and Lin (2013); Lin (2014)), the SDRE scheme is known to include many beneficial properties, and most of them are also possessed by the SDDRE scheme due to the similarity between both schemes. Besides, many practical applications successfully performed by the SDRE scheme have also been reported. However, it seems that the achievements in practical applications have outpaced that of theoretical fundamentals, which motivates this article to provide more theoretical support. In this article, we mainly consider the three problems on how to either relate the state-dependent coefficient (SDC) matrices with the corresponding solutions' properties, or easily construct feasible SDC matrices (Problems 1–3, which are encountered during the early-stage design of both schemes), and try to formulate implementable solutions to these problems. By extending our previous findings in SDC factorizations (Liang & Lin, 2011, 2013) to general-order nonlinear time-variant systems, we attempt to provide more theoretical results, and establish connections to existing contributions on both schemes.

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E-mail addresses: charles.lin@esat.kuleuven.be (L.-G. Lin), Joos.Vandewalle@esat.kuleuven.be (J. Vandewalle), ywliang@cn.nctu.edu.tw (Y.-W. Liang).

[1] Tel.: +32 16 321709; fax: +32 16 321970.

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The SDDRE scheme for nonlinear systems can be described as follows, which is performed *pointwise* in $\mathbf{x}$ and in the finite-time horizon $t_0 \le t \le t_f$. Consider a class of nonlinear time-variant systems and a quadratic-like performance index as (1)–(2):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u} \tag{1}$$

$$J_{\text{SDDRE}} = \frac{1}{2}\mathbf{x}^T(t_f)S(t)\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left\{\mathbf{x}^T Q(\mathbf{x}, t)\mathbf{x} + \mathbf{u}^T R(\mathbf{x}, t)\mathbf{u}\right\} dt, \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^p$ denote the system states and control inputs, respectively, $\mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^n$, $B(\mathbf{x}, t) \in \mathbb{R}^{n\times p}$, $S^T(t) = S(t) \ge 0$, $Q^T(\mathbf{x}, t) = Q(\mathbf{x}, t) \ge 0$, $R^T(\mathbf{x}, t) = R(\mathbf{x}, t) > 0$, and $\mathbf{f}(\mathbf{x}, t)$, $B(\mathbf{x}, t)$, $Q(\mathbf{x}, t)$, $R(\mathbf{x}, t) \in C^k$, $k \ge 1$. Conceptually, the procedure to obtain the SDDRE feedback control is briefed as the following three steps (Heydari & Balakrishnan, 2013; Heydari et al., 2014; Pan et al., 2009):

(i) Factorize $\mathbf{f}(\mathbf{x}, t)$ into the appropriate SDC matrix representation as $\mathbf{f}(\mathbf{x}, t) = A(\mathbf{x}, t)\mathbf{x}$, where $A(\mathbf{x}, t) \in \mathbb{R}^{n\times n}$.

(ii) If possible, solve pointwisely the following SDDRE for $P(\mathbf{x}, t)$:

$$A^T(\mathbf{x}, t)P(\mathbf{x}, t) + P(\mathbf{x}, t)A(\mathbf{x}, t) + Q(\mathbf{x}, t) - P(\mathbf{x}, t)B(\mathbf{x}, t)R^{-1}(\mathbf{x}, t)B^T(\mathbf{x}, t)P(\mathbf{x}, t) = -\dot{P}(\mathbf{x}, t) \quad \text{and} \quad P(\mathbf{x}, t_f) = S(t), \tag{3}$$

where $C(\mathbf{x}, t) \in \mathbb{R}^{q\times n}$ has full row rank and satisfies $Q(\mathbf{x}, t) = C(\mathbf{x}, t)^T C(\mathbf{x}, t)$, and $\dot{P}(\mathbf{x}, t)$ denotes the total time derivative of $P(\mathbf{x}, t)$ (Heydari & Balakrishnan, 2013).

(iii) The control law is thus given by

$$\mathbf{u}_{\text{SDDRE}} = -R^{-1}(\mathbf{x}, t)B^T(\mathbf{x}, t)P(\mathbf{x}, t)\mathbf{x}. \tag{4}$$

Similar to the SDDRE scheme, the SDRE strategy instead considers the infinite-time horizon counterpart ($t_f \to \infty$) with corresponding performance index as Çimen (2010) and Çimen (2012)

$$J_{\text{SDRE}} = \frac{1}{2}\int_{t_0}^{\infty}\left\{\mathbf{x}^T Q(\mathbf{x}, t)\mathbf{x} + \mathbf{u}^T R(\mathbf{x}, t)\mathbf{u}\right\} dt \tag{5}$$

and solves pointwisely the following SDRE for $P(\mathbf{x}, t)$:

$$A^T(\mathbf{x}, t)P(\mathbf{x}, t) + P(\mathbf{x}, t)A(\mathbf{x}, t) + Q(\mathbf{x}, t) - P(\mathbf{x}, t)B(\mathbf{x}, t)R^{-1}(\mathbf{x}, t)B^T(\mathbf{x}, t)P(\mathbf{x}, t) = 0; \tag{6}$$

while the control law is given by

$$\mathbf{u}_{\text{SDRE}} = -R^{-1}(\mathbf{x}, t)B^T(\mathbf{x}, t)P(\mathbf{x}, t)\mathbf{x}. \tag{7}$$

Obviously, both schemes highly rely on the feasible choices of the SDC matrices $A(\mathbf{x}, t)$, $B(\mathbf{x}, t)$, $Q(\mathbf{x}, t)$, and $R(\mathbf{x}, t)$ for successful implementations. In this article, the relations between the SDC matrices and the corresponding solutions' properties, namely existence, uniqueness, positive semi-definiteness, and positive definiteness, of SDRE (resp., SDDRE) are investigated. Specifically, the following problems are going to be addressed afterwards:

Problem 1. Given an SDC matrix, how to efficiently *determine* the property (i.e., existence, uniqueness, positive semi-definiteness, and positive definiteness) of solution of the corresponding SDRE (resp., SDDRE)?

Problem 2. When the SDRE (resp., SDDRE) solvability condition is satisfied, how to easily *construct* an appropriate SDC matrix with specific property (i.e., existence, uniqueness, positive semi-definiteness, and positive definiteness) of the corresponding solution?

Problem 3. In connection with Problem 2, how to *parameterize* all the feasible SDC matrices with specific property (i.e., existence, uniqueness, positive semi-definiteness, and positive definiteness) of the corresponding solution?

2. Mathematical representations

Unless otherwise mentioned in this paper, we denote for brevity $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$, $A = A(\mathbf{x}, t)$, $B = B(\mathbf{x}, t)$, $C = C(\mathbf{x}, t)$, $Q = Q(\mathbf{x}, t)$, $R = R(\mathbf{x}, t)$, $P = P(\mathbf{x}, t)$, and assume that, without loss of any generality, B (resp., C) has full column (resp., row) rank. Besides, in accordance with Liang and Lin (2013), if $W \in \mathbb{R}^{p\times n}$ with $p < n$ and $\text{rank}(W) = p$, then we define $W^\perp = N(W)$, null space of W, and $W_\perp \in \mathbb{R}^{n\times(n-p)}$ as a selected constant matrix having orthonormal columns and satisfying $WW_\perp = 0$. Similarly, if $W \in \mathbb{R}^{n\times q}$ and $\text{rank}(W) = q < n$, then we define $W^\perp = \{\mathbf{w}^T \mid \mathbf{w} \in N(W^T)\}$ and $W_\perp \in \mathbb{R}^{(n-q)\times n}$ as a selected constant matrix having orthonormal rows and satisfying $W_\perp W = 0$. Moreover, denote the SDC matrix sets $\mathcal{A}_{\mathbf{xf}}$, $\mathcal{A}^c$, $\mathcal{A}^s$, $\mathcal{A}^o$, $\mathcal{A}^d$, and $\mathcal{A}^i$ as the sets of A such that $A\mathbf{x} = \mathbf{f}$, (A, B) is controllable, (A, B) is stabilizable, (A, C) is observable, (A, C) is detectable, and (A, C) has no unobservable mode on the $j\omega$-axis, respectively. Note that, by choosing $A_p = \frac{1}{\|\mathbf{x}\|^2}\mathbf{f}\mathbf{x}^T$, we have

$$\mathcal{A}_{\mathbf{xf}} = \left\{A_p + K\mathbf{x}_\perp \mid K \in \mathbb{R}^{n\times(n-1)}\right\} \subset \mathbb{R}^{n\times n}. \tag{8}$$

In addition, denote $\mathcal{A}^{sl}_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}} \cap \mathcal{A}^s \cap \mathcal{A}^l$ and similarly to others, where $\mathcal{A}^l$ denotes the set of A such that (A, C) has no unobservable mode on the left-half plane (LHP) and $j\omega$-axis. Then, we summarize the following lemma to provide fundamental requirements/assumptions on the solvability of SDDRE/SDRE ((3)/(6)) (proof at Heydari et al. (2014); Liang and Lin (2013); Zhou, Doyle, and Glover (1996)).

Lemma 4. (i) $\mathcal{A}^{si}_{\mathbf{xf}} \ne \emptyset \Leftrightarrow$ "$C\mathbf{x} \ne \mathbf{0}$ *or* $\mathbf{f} \ne \mathbf{0}$" $\Leftrightarrow$ *"SDRE (6) admits pointwisely a unique, symmetric, and stabilizing* $P \ge 0$".

(ii) $\mathcal{A}^{sl}_{\mathbf{xf}} \ne \emptyset \Leftrightarrow$ "$C\mathbf{x} \ne \mathbf{0}$ *or* $\mathbf{f} \ne \mu\mathbf{x}$, *for some* $\mu \le 0$" $\Leftrightarrow$ *"SDRE (6) admits pointwisely a unique, symmetric, and stabilizing* $P > 0$".

(iii) $\mathcal{A}^{so}_{\mathbf{xf}} \ne \emptyset \Leftrightarrow$ "$C\mathbf{x} \ne \mathbf{0}$ *or* $\{\mathbf{x}, \mathbf{f}\}$ *are linearly independent (LI)" $\Rightarrow$ "SDRE (6) admits pointwisely a unique, symmetric, and stabilizing* $P \ge 0$" *(resp., "SDDRE (3) admits pointwisely a unique and symmetric* $P \ge 0$").

Given the results in Section 3.1 (Liang & Lin, 2013), with extension to the case of $\mathcal{A}^l$, Problem 1 can be easily solved by direct calculation, which alleviates the computational burden to determine the property of the solution of the corresponding SDDRE/SDRE ((3)/(6)). For brevity, we explain in detail the solutions only for Problems 2–3.

2.1. Solution to Problem 2

If the solvability conditions are satisfied as in Lemma 4, then via the following algorithm we could easily construct a real and diagonalizable SDC matrix in $\mathcal{A}^{s\alpha}_{\mathbf{xf}}$, $\alpha = d, l, i$, respectively. Note that, those findings are found to be ably generalized from our previous works on $\mathcal{A}^{sd}_{\mathbf{xf}}$ and $\mathcal{A}^{so}_{\mathbf{xf}}$ (Liang & Lin, 2011), so details of the proof are omitted.

Algorithm 1. Given pointwisely the $\mathbf{x}$, $\mathbf{f}$, and B,

(1) choose $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{R}^-$;

(2) if $\{\mathbf{x}, \mathbf{f}\}$ are linearly independent (LI), then

(a) choose $\lambda_n \in \mathbb{R}^-$ with $\lambda_n \ne \lambda_{n-1}$;

(b) choose $\{\mathbf{q}_1^T, \ldots, \mathbf{q}_{n-2}^T\}$ being a LI subset of $\{\mathbf{x}, \mathbf{f}\}^\perp$;

(c) choose $\{\mathbf{q}_{n-1}^T, \mathbf{q}_n^T\} \subseteq \text{span}\{\mathbf{x}, \mathbf{f}\}$ satisfying $\mathbf{q}_i^T(\lambda_i\mathbf{x} - \mathbf{f}) = 0$ for $i = n-1, n$;

else
(a) choose $\lambda_n = \lambda$ (resp. $\lambda_n = -\lambda$), where $\mathbf{f} = \lambda\mathbf{x}$;
(b) choose $\{\mathbf{q}_1^T, \ldots, \mathbf{q}_{n-1}^T\}$ being a LI subset of $\{\mathbf{x}\}^\perp$;
(c) choose $\mathbf{q}_n^T \notin B^\perp \cup \{\mathbf{x}\}^\perp$;

(3) $A = M^{-1}DM \in \mathcal{A}_{\mathbf{xf}}^{sd}$ (resp. $A \in \mathcal{A}_{\mathbf{xf}}^{s\alpha}$, $\alpha = i, l$), where $M = M(\mathbf{x}) = [\mathbf{q}_1, \ldots, \mathbf{q}_n]^T$ and $D = D(\mathbf{x}) = diag[\lambda_1, \ldots, \lambda_n]$.

Proof. Since $\mathcal{A}^l = (-\mathcal{A})^d := \{-A \mid A \in \mathcal{A}^d\}$, the case for $\mathcal{A}_{\mathbf{xf}}^{sl}$ can be similarly derived from Corollary 7 in Liang and Lin (2011) by treating the pair $(A, \mathbf{x}, \mathbf{f})$ as $(-A, \mathbf{x}, -\mathbf{f})$. On the other hand, since $\mathcal{A}^l \subset \mathcal{A}^i$, together with the fact that $\mathcal{A}^l = \emptyset \Leftrightarrow$ "$C\mathbf{x} = \mathbf{0}$ and $\mathbf{f} = \mu\mathbf{x}$, $\mu \le 0$", we only need to consider the following two cases for $\mathcal{A}_{\mathbf{xf}}^{si}$. If $\mu = 0$, then by (ii) of Lemma 4, $\mathcal{A}^i = \mathcal{A}_{\mathbf{xf}}^{si} = \emptyset$; If $\mu < 0$, then by (ii) of Lemma 4, $\mathcal{A}^i \neq \emptyset$, and by similar arguments of the proof of Corollary 7 in Liang and Lin (2011), the result thus follows. □

2.2. Solution to Problem 3

We consider the case of rank(B), rank$(C) \ge n-1$, and the remaining case follows similar derivations. When rank$(B) = n$ (resp., rank$(C) = n$), (A, B) (resp., (A, C)) is controllable (resp., observable). Thus, $\mathcal{A}_{\mathbf{xf}}^c = \mathcal{A}_{\mathbf{xf}}^s = \mathcal{A}_{\mathbf{xf}}$ (resp., $\mathcal{A}_{\mathbf{xf}}^o = \mathcal{A}_{\mathbf{xf}}^d = \mathcal{A}_{\mathbf{xf}}^l = \mathcal{A}_{\mathbf{xf}}^i = \mathcal{A}_{\mathbf{xf}}$), which is a $(n^2 - n)$-dimension linear variety. Next, if rank(B) =rank$(C) = n-1$, to describe the structure of $\mathcal{A}_{\mathbf{xf}}^c$, $\mathcal{A}_{\mathbf{xf}}^s$, $\mathcal{A}_{\mathbf{xf}}^o$, $\mathcal{A}_{\mathbf{xf}}^d$, $\mathcal{A}_{\mathbf{xf}}^l$, and $\mathcal{A}_{\mathbf{xf}}^i$, we need Lemmas 5–7, whose proofs are included in Lin (2014).

Lemma 5. *Let $\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $M \in \mathbb{R}^{n\times(n-1)}$ having full column rank. Then rank $(\mathbf{v}_\perp M) \ge n-2$. Moreover, $\mathbf{v}_\perp M$ is singular if and only if $\mathbf{v} \in span\{M\}$.*

Lemma 6. *Let $B \in \mathbb{R}^{n\times(n-1)}$ having full column rank, $M \in \mathbb{R}^{(n-1)\times(n-1)} \setminus \{0\}$ and $\boldsymbol{\xi} \in \mathbb{R}^{n-1}$. Define $\mathbf{m}_p \in \mathbb{R}^{n-1}$ to be such that $M^T\mathbf{m}_p = \boldsymbol{\xi}$. Then the set of $K \in \mathbb{R}^{n\times(n-1)}$ such that $(B_\perp K)M = \boldsymbol{\xi}^T$ can be parameterized as $K = \{B_\perp^T(\mathbf{m}_p^T + \kappa\boldsymbol{\eta}^T) + \tilde{B}K' \mid \kappa \in \mathbb{R}, \boldsymbol{\eta} \in \mathcal{N}(M^T)$ and $K' \in \mathbb{R}^{(n-1)\times(n-1)}\}$. In case M being nonsingular, we have $\mathbf{m}_p = \boldsymbol{\xi}M^{-T}$ and $\boldsymbol{\eta} = \mathbf{0}$.*

Lemma 7. *Let $C \in \mathbb{R}^{(n-1)\times n}$ having full row rank and $\boldsymbol{\nu}, \boldsymbol{\xi} \in \mathbb{R}^{n-1}$. Define $\mathbf{m}_p \in \mathbb{R}^n$ to be such that $C\mathbf{m}_p = \boldsymbol{\xi}$. Then the set of $K \in \mathbb{R}^{n\times(n-1)}$ such that $CK\boldsymbol{\nu} = \boldsymbol{\xi}$ can be parameterized as $K = \{\frac{1}{\|\boldsymbol{\nu}\|^2}(\mathbf{m}_p + \kappa C_\perp)\boldsymbol{\nu}^T + K'\boldsymbol{\nu}_\perp \mid K' \in \mathbb{R}^{n\times(n-2)}$ and $\kappa \in \mathbb{R}\}$.*

Therefore, the sets $\mathcal{A}_{\mathbf{xf}}^c$, $\mathcal{A}_{\mathbf{xf}}^s$, $\mathcal{A}_{\mathbf{xf}}^o$, $\mathcal{A}_{\mathbf{xf}}^d$, $\mathcal{A}_{\mathbf{xf}}^l$, and $\mathcal{A}_{\mathbf{xf}}^i$ can be easily obtained from Lemmas 5–7, as below:

Theorem 8. *Let $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\mathbf{f} \in \mathbb{R}^n$, and $B, C^T \in \mathbb{R}^{n\times(n-1)}$ of full column rank. Then*

(i) $\mathcal{A}_{\mathbf{xf}}^c = \begin{cases} \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{c}(1)} & \text{if } \mathbf{x} \notin span\{B\}; \\ \mathcal{A}_{\mathbf{xf}} & \text{if } \mathbf{x} \in span\{B\} \,\&\, \mathbf{f} \notin span\{B\}; \\ \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{c}(2)} & \text{if } \mathbf{x}, \mathbf{f} \in span\{B\}, \end{cases}$

where $\mathcal{A}_{\mathbf{xf}}^{\bar{c}(1)} := \{A_p^{\bar{c}(1)} + \tilde{B}K'\mathbf{x}_\perp \mid A_p^{\bar{c}(1)} = \frac{\mathbf{f}\mathbf{x}^T}{\|\mathbf{x}\|^2} - \frac{B_\perp\mathbf{f}}{\|\mathbf{x}\|^2}B_\perp^T(\mathbf{x}_\perp\tilde{B})^{-1}(\mathbf{x}^T\tilde{B})\mathbf{x}_\perp$ & $K' \in \mathbb{R}^{(n-1)\times(n-1)}\}$ is a $(n-1)^2$- dimension linear variety, $\mathcal{A}_{\mathbf{xf}}^{\bar{c}(2)} := \{A_p + (\kappa B_\perp^T\boldsymbol{\eta}^T + \tilde{B}K')\mathbf{x}_\perp \mid \tilde{B}^T\mathbf{x}_\perp^T\boldsymbol{\eta} = \mathbf{0}, \kappa \in \mathbb{R}$ & $K' \in \mathbb{R}^{(n-1)\times(n-1)}\}$ is a $(n^2 - 2n + 2)$- dimension linear variety, both $\mathcal{A}_{\mathbf{xf}}^{\bar{c}(1)}$ and $\mathcal{A}_{\mathbf{xf}}^{\bar{c}(2)}$ are in $\mathcal{A}_{\mathbf{xf}}$ and in which (A, B) is uncontrollable.

(ii) $\mathcal{A}_{\mathbf{xf}}^s = \begin{cases} \mathcal{A}_{\mathbf{xf}} & \text{if "}\mathbf{x} \notin span\{B\} \,\&\, \psi < 0\text{" or} \\ & \text{"}\mathbf{x} \in span\{B\} \,\&\, \mathbf{f} \notin span\{B\};\text{"} \\ \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{c}(1)} & \text{if } \mathbf{x} \notin span\{B\} \,\&\, \psi \ge 0; \\ \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{s}} & \text{if } \mathbf{x}, \mathbf{f} \in span\{B\}, \end{cases}$

where $\psi = (B_\perp\mathbf{f})\left[-(\mathbf{x}_\perp\tilde{B})^{-1}(\mathbf{x}^T\tilde{B})(\mathbf{x}_\perp B_\perp^T) + B_\perp\mathbf{x}\right]$ and $\mathcal{A}_{\mathbf{xf}}^{\bar{s}} := \left\{A \in \mathcal{A}_{\mathbf{xf}}^{\bar{c}(2)} \mid sign(\boldsymbol{\eta}^T\mathbf{x}_\perp B_\perp^T)\cdot\kappa \ge 0\right\}$ is half of $\mathcal{A}_{\mathbf{xf}}^{\bar{c}(2)}$ and in which (A, B) is unstabilizable.

(iii) $\mathcal{A}_{\mathbf{xf}}^o = \begin{cases} \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{o}} & \text{if } C\mathbf{x} \neq 0; \\ \mathcal{A}_{\mathbf{xf}} & \text{if } C\mathbf{x} = 0 \,\&\, \{\mathbf{x}, \mathbf{f}\} \text{ are LI}; \\ \emptyset & \text{if } C\mathbf{x} = 0 \,\&\, \{\mathbf{x}, \mathbf{f}\} \text{ are LD}, \end{cases}$

where $\mathcal{A}_{\mathbf{xf}}^{\bar{o}} := \left\{A_p^{\bar{o}} + \left[\kappa\frac{C_\perp(\mathbf{x}_\perp C_\perp)^T}{\|\mathbf{x}_\perp C_\perp\|^2} + K'(\mathbf{x}_\perp C_\perp)_\perp\right]\cdot\mathbf{x}_\perp \mid \kappa \in \mathbb{R}, K' \in \mathbb{R}^{n\times(n-2)}, A_p^{\bar{o}} = \frac{\mathbf{m}_p(\mathbf{x}_\perp C_\perp)^T}{\|\mathbf{x}_\perp C_\perp\|^2}\mathbf{x}_\perp + \frac{\mathbf{f}\mathbf{x}^T}{\|\mathbf{x}\|^2}$ & $\tilde{C}\mathbf{m}_p = -\frac{\mathbf{x}^T C_\perp}{\|\mathbf{x}\|^2}(\tilde{C}\mathbf{f})\right\}$ is a $(n-1)^2$-dimension linear variety in $\mathcal{A}_{\mathbf{xf}}$ and in which (A, C) is unobservable.

(iv) $\mathcal{A}_{\mathbf{xf}}^d = \begin{cases} \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{d}} & \text{if } C\mathbf{x} \neq 0; \\ \mathcal{A}_{\mathbf{xf}} & \text{if "}C\mathbf{x} = 0 \,\&\, \{\mathbf{x}, \mathbf{f}\} \text{ are LI" or} \\ & \text{"}C\mathbf{x} = 0 \,\&\, \mathbf{f} = \mu\mathbf{x},\ \mu < 0;\text{"} \\ \emptyset & \text{if } C\mathbf{x} = 0 \,\&\, \mathbf{f} = \mu\mathbf{x},\ \mu \ge 0, \end{cases}$

where $\mathcal{A}_{\mathbf{xf}}^{\bar{d}} := \left\{A \in \mathcal{A}_{\mathbf{xf}}^{\bar{o}} \mid \kappa \ge -\frac{(C_\perp^T\mathbf{f})(\mathbf{x}^T C_\perp)}{\|\mathbf{x}\|^2} - C_\perp^T\mathbf{m}_p\right\}$ is a half of $\mathcal{A}_{\mathbf{xf}}^{\bar{o}}$ and in which (A, C) is undetectable.

(v) $\mathcal{A}_{\mathbf{xf}}^l = \begin{cases} \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{l}} & \text{if } C\mathbf{x} \neq 0; \\ \mathcal{A}_{\mathbf{xf}} & \text{if "}C\mathbf{x} = 0 \,\&\, \{\mathbf{x}, \mathbf{f}\} \text{ are LI" or} \\ & \text{"}C\mathbf{x} = 0 \,\&\, \mathbf{f} = \mu\mathbf{x},\ \mu > 0;\text{"} \\ \emptyset & \text{if } C\mathbf{x} = 0 \,\&\, \mathbf{f} = \mu\mathbf{x},\ \mu \le 0, \end{cases}$

where $\mathcal{A}_{\mathbf{xf}}^{\bar{l}} := \left\{A \in \mathcal{A}_{\mathbf{xf}}^{\bar{o}} \mid \kappa \le -\frac{(C_\perp^T\mathbf{f})(\mathbf{x}^T C_\perp)}{\|\mathbf{x}\|^2} - C_\perp^T\mathbf{m}_p\right\}$ is a half of $\mathcal{A}_{\mathbf{xf}}^{\bar{o}}$ and in which (A, C) has an unobservable mode on the LHP and $j\omega$-axis.

(vi) $\mathcal{A}_{\mathbf{xf}}^i = \begin{cases} \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{i}} & \text{if } C\mathbf{x} \neq 0; \\ \mathcal{A}_{\mathbf{xf}} & \text{if "}C\mathbf{x} = 0 \,\&\, \mathbf{f} \neq \mathbf{0};\text{"} \\ \emptyset & \text{if } C\mathbf{x} = 0 \,\&\, \mathbf{f} = \mathbf{0}, \end{cases}$

where $\mathcal{A}_{\mathbf{xf}}^{\bar{i}} := \left\{A \in \mathcal{A}_{\mathbf{xf}}^{\bar{o}} \mid \kappa = -\frac{(C_\perp^T\mathbf{f})(\mathbf{x}^T C_\perp)}{\|\mathbf{x}\|^2} - C_\perp^T\mathbf{m}_p\right\}$ is a subspace of $(n^2 - 2n)$-dimension in $\mathcal{A}_{\mathbf{xf}}^{\bar{o}}$ and in which (A, C) has an unobservable mode on the $j\omega$-axis.

Proof. To derive the representation, we need to collect the results from Section 3.1 (Liang & Lin, 2013), together with Lemmas 5–7. For brevity, we only derive for the sets $\mathcal{A}_{\mathbf{xf}}^c$ and $\mathcal{A}_{\mathbf{xf}}^s$, while the others follow similar derivations.

Let $M_B = [B_\perp^T \vdots \tilde{B}]$, $\bar{A} = M_B^T A M_B$, and $\bar{B} = M_B^T B$, where the columns of $\tilde{B}$ form an orthonormal basis of the range space of B. Then $\bar{A}_{12} = \frac{B_\perp\mathbf{f}}{\|\mathbf{x}\|^2}(\mathbf{x}^T\tilde{B}) + (B_\perp K)(\mathbf{x}_\perp\tilde{B})$ and $\bar{A}_{11} = \frac{(B_\perp\mathbf{f})(B_\perp\mathbf{x})}{\|\mathbf{x}\|^2} + (B_\perp K)(\mathbf{x}_\perp B_\perp^T)$. From Section 3.1 (Liang & Lin, 2013), (A, B) is uncontrollable $\Leftrightarrow \bar{A}_{12} = 0$; and (A, B) is unstabilizable $\Leftrightarrow \bar{A}_{12} = 0$ and $\bar{A}_{11} \ge 0$. By Lemma 5, $\mathbf{x}_\perp\tilde{B}$ is singular $\Leftrightarrow \mathbf{x} \in$ span$\{B\}$. If $\mathbf{x} \notin$ span$\{B\}$ (i.e., $\mathbf{x}_\perp\tilde{B}$ is nonsingular), then those of K such that $\bar{A}_{12} = 0$ can be parameterized via Lemma 6 with $(B, M, \boldsymbol{\xi})$ being replaced by $\left(\tilde{B}, \mathbf{x}_\perp\tilde{B}, -\frac{(B_\perp\mathbf{f})}{\|\mathbf{x}\|^2}(\tilde{B}^T\mathbf{x})\right)$. Combining the parameterization of K with the expression of $\mathcal{A}_{\mathbf{xf}}$ gives the set $\mathcal{A}_{\mathbf{xf}}^{\bar{c}(1)}$. Consequently, $\mathcal{A}_{\mathbf{xf}}^c = \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{c}(1)}$. Besides, within $\mathcal{A}_{\mathbf{xf}}^{\bar{c}(1)}$ (i.e., $\bar{A}_{12} = 0$), K satisfies the relation $B_\perp K = -\frac{(B_\perp\mathbf{f})(\mathbf{x}^T\tilde{B})}{\|\mathbf{x}\|^2}(\mathbf{x}_\perp\tilde{B})^{-1}$. Inserting this relation into $\bar{A}_{11}$ yields $\bar{A}_{11} = \frac{\psi}{\|\mathbf{x}\|^2}$. Thus, $\bar{A}_{11} < 0 \Leftrightarrow \psi < 0$. Therefore, $\mathcal{A}_{\mathbf{xf}}^s = \mathcal{A}_{\mathbf{xf}}$ if $\mathbf{x} \notin$ span$\{B\}$ and $\psi < 0$; and $\mathcal{A}_{\mathbf{xf}}^s = \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{c}(1)}$ if $\mathbf{x} \notin$ span$\{B\}$ and $\psi \ge 0$. We now consider the case that $\mathbf{x}_\perp\tilde{B}$ is singular (i.e., $\mathbf{x} \in$ span$\{B\}$). If $\mathbf{f} \notin$ span$\{B\}$ (i.e., $B_\perp\mathbf{f} \neq 0$), since $\left[\frac{(B_\perp\mathbf{f})}{\|\mathbf{x}\|^2}\mathbf{x}^T + (B_\perp K)\mathbf{x}_\perp\right] \notin B_\perp$, we have $\bar{A}_{12} = \left[\frac{(B_\perp\mathbf{f})}{\|\mathbf{x}\|^2}\mathbf{x}^T + (B_\perp K)\mathbf{x}_\perp\right]\tilde{B} \neq 0$ and thus $\mathcal{A}_{\mathbf{xf}}^c = \mathcal{A}_{\mathbf{xf}}^s = \mathcal{A}_{\mathbf{xf}}$. When $\mathbf{f} \in$ span$\{B\}$ (i.e., $B_\perp\mathbf{f} = 0$), then those of K such that $\bar{A}_{12} = 0$ can be parameterized via Lemma 6 with $(B, M, \boldsymbol{\xi})$ being replaced by $\left(\tilde{B}, \mathbf{x}_\perp\tilde{B}, \mathbf{0}\right)$. Combining the parameterization of K with the expression of $\mathcal{A}_{\mathbf{xf}}$ gives the set $\mathcal{A}_{\mathbf{xf}}^{\bar{c}(2)}$. Consequently, $\mathcal{A}_{\mathbf{xf}}^c = \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{c}(2)}$. Besides, within $\mathcal{A}_{\mathbf{xf}}^{\bar{c}(2)}$ (i.e., $\bar{A}_{12} = 0$) and via the parameterization of K, we have $\bar{A}_{11} = \kappa\boldsymbol{\eta}^T\mathbf{x}_\perp B_\perp^T \ge 0 \Leftrightarrow$ sign$(\boldsymbol{\eta}^T\mathbf{x}_\perp B_\perp^T)\cdot\kappa \ge 0$ and thus $\mathcal{A}_{\mathbf{xf}}^s = \mathcal{A}_{\mathbf{xf}} \setminus \mathcal{A}_{\mathbf{xf}}^{\bar{s}}$. □

Remark 9. From Theorem 8, the sets $\mathcal{A}^c_{\mathbf{xf}}$ and $\mathcal{A}^s_{\mathbf{xf}}$ are always non-empty as long as rank$(B) \geq n-1$, and the results of Theorem 8 agree with Lemma 4. Moreover, after having the sets $\mathcal{A}_{\mathbf{xf}}$, $\mathcal{A}^s_{\mathbf{xf}}$, $\mathcal{A}^l_{\mathbf{xf}}$ and $\mathcal{A}^i_{\mathbf{xf}}$ presented above, the solutions $\mathcal{A}^{sl}_{\mathbf{xf}}$ and $\mathcal{A}^{si}_{\mathbf{xf}}$ can be easily obtained according to the ranks of B and C.

3. Impact and relevance

3.1. Estimating the domain of attraction

In Bracci, Innocenti, and Pollini (2006); Erdem and Alleyne (2002), the authors have described procedures for estimating the domain of attraction (DOA) via SDRE, but without guidelines on the construction of feasible SDC matrices. Therefore it is possible to enlarge the estimation by combining with the proposed results, specifically the overall DOA estimation is just the largest of all estimations by applying every SDC matrix in $\mathcal{A}^{si}_{\mathbf{xf}}$, $\mathcal{A}^{sl}_{\mathbf{xf}}$, or others. For example, the SDC matrix in Eq. (21) of Example 2 (Erdem & Alleyne, 2002) can be represented by $A_p + K\mathbf{x}_\perp \in \mathcal{A}^{so}_{\mathbf{xf}}$, where $K = \left(\frac{-2x_1^3 - x_2}{\|\mathbf{x}\|}, \frac{x_1}{\|\mathbf{x}\|}\right)$.

3.2. Relating to the optimality

Consider the SDDRE scheme in time-invariant case, noting that $\mathbf{f}, A, B, Q$ and R depend only on the state, while the derivation for the SDRE scheme is similar and thus omitted. Denote $J^* = J^*(\mathbf{x}, t) = \min_u J_{SDDRE}$ and assume that J^* is given, the optimal control for the FTHNOC problem can be obtained from the following:

$$u^* = -R^{-1}B^T\left(\frac{\partial J^*}{\partial \mathbf{x}}\right)^T, \tag{9}$$

where $J^*_{\mathbf{x}}$ satisfies the well-known HJB equation:

$$\frac{\partial J^*}{\partial t} + \left(\frac{\partial J^*}{\partial \mathbf{x}}\right)^T \mathbf{f} - \frac{1}{2}\frac{\partial J^*}{\partial \mathbf{x}} BR^{-1}B^T\left(\frac{\partial J^*}{\partial \mathbf{x}}\right)^T + \frac{1}{2}\mathbf{x}^T Q\mathbf{x} = 0. \tag{10}$$

Comparing (4) and (9), it is obvious that $u_{SDDRE} = u^* \Leftrightarrow \Delta \in B^\perp$, where $\Delta = \Delta(\mathbf{x}, t) = P\mathbf{x} - (\frac{\partial J^*}{\partial \mathbf{x}})^T$. Rewrite $\frac{\partial J^*}{\partial \mathbf{x}} = (P\mathbf{x} - \Delta)^T$ and by Lemma 2–22 of Berger and Berger (1968),

$$\begin{aligned} J^* &= \mathbf{x}^T \int_0^1 [P(\xi\mathbf{x}, t)\xi\mathbf{x} - \Delta(\xi\mathbf{x}, t)]\, d\xi \\ &= \mathbf{x}^T \left[\int_0^1 P(\xi\mathbf{x}, t)\xi\, d\xi\right]\mathbf{x} - \mathbf{x}^T \int_0^1 \Delta(\xi\mathbf{x}, t) d\xi \\ &= \frac{1}{2}\mathbf{x}^T P(\mathbf{x}, t)\mathbf{x} - \frac{1}{2}\mathbf{x}^T \left[\int_0^1 \frac{\partial P(\xi\mathbf{x}, t)}{\partial \xi}\xi^2 d\xi\right]\mathbf{x} \\ &\quad -\mathbf{x}^T \int_0^1 \Delta(\xi\mathbf{x}, t) d\xi, \end{aligned}$$

and thus $\frac{\partial J^*}{\partial t} = \frac{1}{2}\mathbf{x}^T\dot{P}(\mathbf{x}, t)\mathbf{x} + \Gamma$, where for brevity $\Gamma = \Gamma(\mathbf{x}, t)$ denotes the remaining terms, and some lengthy derivations are included in Lin (2014). Reformulate HJB (10) as

$$\begin{aligned} &\frac{1}{2}\mathbf{x}^T\dot{P}\mathbf{x} + \Gamma + (\mathbf{x}^T P - \Delta^T)A\mathbf{x} + \frac{1}{2}\mathbf{x}^T Q\mathbf{x} \\ &= \frac{1}{2}(\mathbf{x}^T P - \Delta^T)BR^{-1}B^T(P\mathbf{x} - \Delta) \\ &\Leftrightarrow \mathbf{x}^T(\dot{P} + A^T P + PA - PBR^{-1}B^T P + Q)\mathbf{x} = 2(\Delta^T\mathbf{f} - \Gamma), \end{aligned}$$

since $\mathbf{x}^T PA\mathbf{x} = \mathbf{x}^T A^T P\mathbf{x} \in \mathbb{R}$

then we summarize the following theorem and remark.

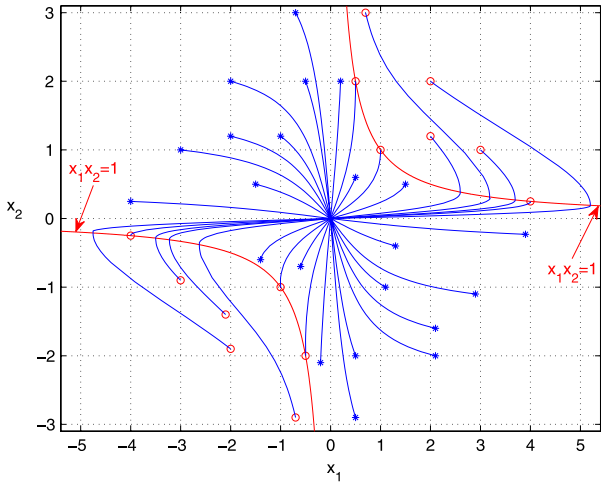

Fig. 1. Phase plot of various initial states.

Theorem 10. *If $\Delta \in B^\perp$ and $\Delta^T\mathbf{f} = \Gamma$ (resp., $\Delta \in [B, \mathbf{f}]^\perp$), then the SDDRE (3) (resp., SDRE (6)) is the corresponding HJB (resp., HJ) equation.*

Remark 11. The SDDRE (resp., SDRE) scheme recovers the optimal control if the corresponding solution of the SDDRE (3) (resp., SDRE (6)) satisfies the conditions in Theorem 10, which generalizes from Theorem 2 in Huang and Lu (1996). As a special case, the SDC matrix recovering the optimality in Example 7 of Huang and Lu (1996) can be represented by $A_p + K\mathbf{x}_\perp \in \mathcal{A}^{so}_{\mathbf{xf}}$, where $K = \left(\frac{-x_1^2 x_2 - x_2}{\|\mathbf{x}\|}, \frac{x_1}{\|\mathbf{x}\|} - x_1 \cdot \|\mathbf{x}\|\right)$. It is worth mentioning that Heydari and Balakrishnan (2013) also address the optimality issue, but relates the SDDRE (3) to the HJB (10) in a different way and outcome as above, with the main assumption made for the solvability of the SDDRE. In this article, we make it more convenient for the user by removing the assumption made for the solvability of the SDDRE/SDRE ((3)/(6)). Specifically, while the other contributions have a difficult-to-test assumption of stabilizability (resp., controllability) and observability made in Heydari et al. (2014) (resp., Pan et al. (2009)), the newly-summarized Lemma 4 has replaced it by a necessary and sufficient condition, and accordingly a feasible SDC matrix can be easily constructed by Algorithm 1.

4. Examples

Example 12. Consider the following system (Khalil, 1996)

$$\dot{x}_1 = x_1^2 x_2 - x_1 \quad \text{and} \quad \dot{x}_2 = u. \tag{11}$$

Clearly, this system is in the form of (1) with $\mathbf{x} = [x_1, x_2]^T$, $\mathbf{f}(\mathbf{x}) = [x_1^2 x_2 - x_1, 0]^T$ and $B(\mathbf{x}) = [0, 1]^T$. To demonstrate the SDRE design, we simply choose $Q(\mathbf{x}) = I_2$, $R(\mathbf{x}) = 1$ and an intuitive SDC matrix $A(\mathbf{x})$ with $a_{11}(\mathbf{x}) = x_1 x_2 - 1$ and the other entries being zero. Obviously, $(A(\mathbf{x}), B(\mathbf{x}))$ is stabilizable everywhere, except in the region $\{x_1 x_2 \geq 1\}$ where the SDRE solvability condition is violated; however, by Lemma 4, $\mathcal{A}^{s\gamma}_{\mathbf{xf}} \neq \emptyset$ for $\gamma = o, d, i$ at every nonzero state because $C(\mathbf{x})\mathbf{x} = \mathbf{x} \neq \mathbf{0}$.

From Section 2.1, we may easily construct feasible SDC matrices in the region $\{x_1 x_2 \geq 1\}$ as follows: (i) in the region $\{x_1 x_2 > 1\}$, where $\{\mathbf{x}, \mathbf{f}\}$ are LI, choose $\lambda_1 = -1$, $\lambda_2 = -2$, and $\mathbf{q}_i$ such that $\mathbf{q}_i^T(\lambda_i\mathbf{x} - \mathbf{f}) = 0$, $\forall i = 1, 2$; and (ii) on the hyperbola $\{x_1 x_2 = 1\}$, where $\{\mathbf{x}, \mathbf{f}\}$ are LD, choose $\lambda_1 = -1$, $\lambda_2 = 0$, $\mathbf{q}_1 = [x_2, x_1]^T$, and $\mathbf{q}_2 = B + \frac{\mathbf{x}}{\|\mathbf{x}\|}$. Note that in the region $\{x_1 x_2 < 1\}$, we just adopt the above-mentioned intuitive SDC matrix. The effectiveness of such multi-SDC SDRE scheme is illustrated in Fig. 1, where all

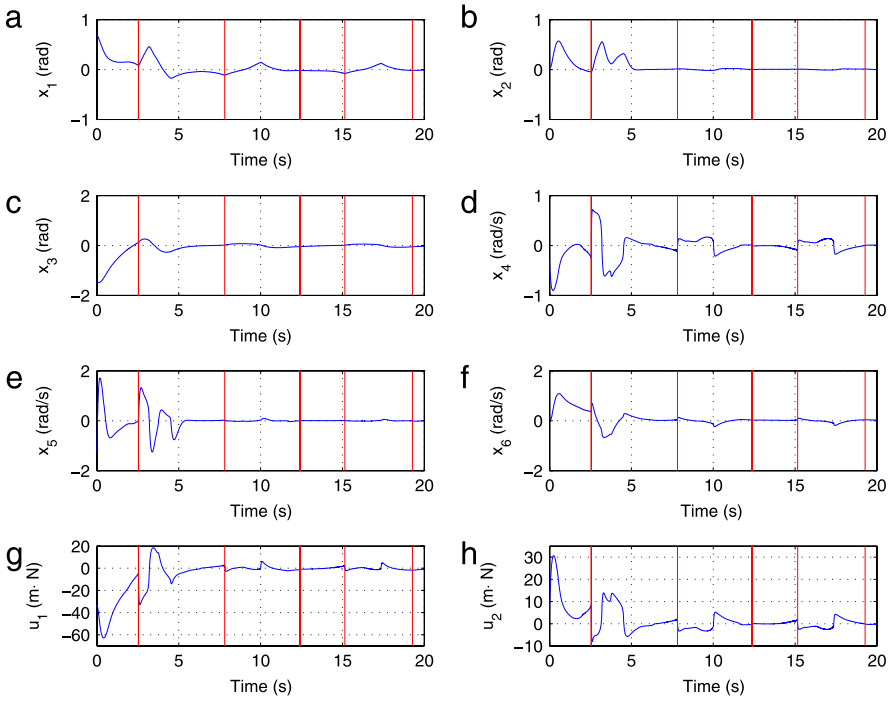

Fig. 2. Time history of the six system states and the two control inputs.

trajectories converge to the origin. For clarity, in Fig. 1 we mark the initial state residing in the region $\{x_1x_2 \geq 1\}$ with a red circle; while in the other region with a blue star.

Example 13. Consider an attitude model for a spacecraft along a circular orbit (Liang, Ting, & Lin, 2012). The three Euler's angles (ϕ, θ, ψ) and their derivatives are adopted as the six state variables. To demonstrate the proposed scheme, we assume that the thruster is the only applied control force, and only two actuators are available for attitude stabilization (since any three actuators would make the system pointwisely controllable, i.e. $\mathcal{A}_{\mathbf{xf}} = \mathcal{A}^c_{\mathbf{xf}}$). Given $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)^T = (\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi})^T$, $\mathbf{u} = (u_1, u_2)^T$, $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^6$, $B \in \mathbb{R}^{6\times 2}$, and the detailed descriptions could be found in Liang et al. (2012).

For the SDRE scheme, we choose $R(\mathbf{x}) = 0.1 \cdot I_2$ and $Q(\mathbf{x}) = diag(10^2, 10^2, 10^2, 10^2, 1, 10)$. To illustrate the proposed scheme, at first we follow the general guidelines summarized in Çimen (2010) to construct an SDC matrix, such as the following factorization:

$$2\cos^2(x_3)\sin(2x_1) = \begin{bmatrix} \dfrac{\sin(2x_1)}{x_1} + \cos^2(x_3)\dfrac{\sin(2x_1)}{x_1} & 0 & \dfrac{\cos^2(x_3)-1}{x_3}\sin(2x_1) \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x},$$

where every state component in the original dynamics contributes as a corresponding element in $A(\mathbf{x}, t)$, i.e. capture their state dependency in the proper entry of SDC matrix. When the SDRE solvability condition of such an SDC matrix is violated, we resort to the proposed scheme to construct alternatively feasible SDC matrices. Note that the existence of such feasible SDC matrix is guaranteed for all nonzero states by Lemma 4, since $Q(\mathbf{x})$ is chosen to be nonsingular and thus $C(\mathbf{x}) \cdot \mathbf{x} \neq 0$, $\forall \mathbf{x} \in \mathbb{R}^6 \setminus \{0\}$. From Fig. 2, such multi-SDC SDRE scheme stabilizes the spacecraft's attitude, with only two healthy actuators. The red lines correspond to the time instants when feasible SDC matrices are constructed to continue the SDRE scheme successfully.

5. Conclusion

In this article, we extend our previous findings on the study of SDC matrices to the general-order nonlinear time-variant systems, e.g. the easy construction and parameterization of feasible SDC matrices, and consequently endorses theoretically many successful practical applications to a certain level. Based on the proposed representation of feasible SDC matrices, we are now investigating analytically the effect/capability of switching between SDC matrices, especially how to relate the pointwise stability to the global consideration.

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Li-Gang Lin received the B.S. degree and M.S. degree in Electrical Control Engineering (ECE) from National Chiao Tung University (NCTU), Hsinchu, Taiwan, in 2008 and in 2010, respectively. In 2014, he received two Ph.D. degrees from KU Leuven (Department of Electrical Engineering, Arenberg Doctoral School) and NCTU (Institute of ECE, and the only recipient of Outstanding Graduate Award with appreciation speech by College of Electrical and Computer Engineering). His research interests include nonlinear control systems, state-dependent (differential/difference) Riccati equation, reliable and robust control.



Joos Vandewalle obtained the electrical engineering degree and doctorate in applied sciences from KU Leuven, Belgium in 1971 and 1976. Until October 2013 he was a full professor at the Department of Electrical Engineering (ESAT), KU Leuven, Belgium; head of the SCD division at ESAT, with more than 150 researchers. Since October 2013 he is a professor emeritus with assignments at KU Leuven. His present tasks include chairing the positioning test for engineering in Flanders, board member of the Flemish Academy in Brussels, chairing Ph.D. defenses.

He held visiting positions at University of California, Berkeley and I3S CNRS Sophia Antipolis, France. He taught courses in linear algebra, linear and nonlinear system and circuit theory, signal processing and neural networks. His research interests are in mathematical system theory and its applications in circuit theory, control, signal processing, cryptography and neural networks. He (co-)authored more than 300 international journal papers and obtained several best paper awards and research awards. He is a Fellow of IEEE, IET, and EURASIP and member of the Academia Europaea and of the Belgian Academy of Sciences. He is Chairman of the class of Technical sciences of the Belgian Academy KVAB. http://www.ae-info.org/ae/User/Vandewalle_Joseph



Yew-Wen Liang received the B.S. degree in Mathematics from Tung Hai University, Taichung, Taiwan, in 1982, the M.S. degree in Applied Mathematics in 1984 and the Ph.D. degree in Electrical and Control Engineering in 1998 from the National Chiao Tung University, Hsinchu, Taiwan. Since August 1986, he has been with the National Chiao Tung University, where he is currently a Professor of Electrical and Computer Engineering. His research interests include linear and nonlinear systems control, fault-tolerant control, and fault detection and diagnosis issues.