Globally optimal least-squares misfit identification of multidimensional autonomous systems

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1 Introduction

For autonomous systems, previous work has shown that the globally optimal least-squares misfit identification can be translated to a multiparameter eigenvalue problem (MEVP). Such an MEVP can be solved using a linear algebra framework, allowing the characterization of the globally optimal solution and its system theoretic properties [1]. We will extend these results to the identification of two-dimensional, and by extension, general multidimensional systems.

2 Multidimensional autonomous systems

Multidimensional autonomous system models contain more than one independent variable, e.g. time and space. We consider two-dimensional, linear, shift-invariant, single output commutative state space models of the form:

$$\mathbf{x}_{k+1,l} = \mathbf{A}_1\mathbf{x}_{k,l},\ \mathbf{x}_{k,l+1} = \mathbf{A}_2\mathbf{x}_{k,l},\ y_{k,l} = \mathbf{C}\mathbf{x}_{k,l}(0)$$

$$\text{with}\quad \mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1.$$

As a first step towards solving this multidimensional modelling problem, starting from a given output sequence $y_{k,l}$, we confine ourselves to the identification of systems for which $\mathbf{A}_1$ and $\mathbf{A}_2$ are simultaneously diagonizable. More precisely, we assume that $\mathbf{A}_1 = \mathbf{V}\mathbf{\Lambda}_1\mathbf{V}^{-1}$ and $\mathbf{A}_2 = \mathbf{V}\mathbf{\Lambda}_2\mathbf{V}^{-1}$, where the $\mathbf{\Lambda}_i$-matrices are diagonal matrices containing the eigenvalues of the $\mathbf{A}_i$-matrices and the matrix $\mathbf{V}$ has the common eigenvectors as its columns. Note that this is a sufficient, though not a necessary condition for the matrices to commute. Under this assumption, an n-th order autonomous 2-D model can be parameterized by the $2n$ eigenvalues $\lambda_j^{(i)}$ of the $\mathbf{A}_i$-matrices, the n components ξ_j of the initial state $\mathbf{x}_{0,0}$ and the n components c_j of the row vector $\mathbf{C}$. This parameterization can of course be easily extended to even higher dimensional models.

3 Least-squares misfit identification

In the misfit identification framework, the given output sequence $y_{k,l}$ is altered by a misfit sequence $\tilde{y}_{k,l}$ to obtain the so-called 'exact' data $\hat{y}_{k,l} = y_{k,l} - \tilde{y}_{k,l}$. As the name implies, the 'exact' data $\hat{y}_{k,l}$ is required to satisfy the equations of the identified state space model exactly. For some predefined order n, misfit modelling then aims to find the nth order system that minimizes the 2-norm of the misfit sequence. More formally, this corresponds to solving the following nonlinear least-squares problem [2]:

$$\min_{\lambda_j^{(1)},\lambda_j^{(2)},\xi_j,c_j} ||\tilde{\mathbf{y}}||_2^2 = \sum_{k=0}^{N_1}\sum_{k=0}^{N_2}(y_{k,l} - \hat{y}_{k,l})^2$$

$$\text{s.t. } \hat{y}_{k,l} = \sum_{j=1}^{n} c_j\xi_j\left(\lambda_j^{(1)}\right)^k\left(\lambda_j^{(2)}\right)^l.$$

Obviously, the misfit modelling optimisation problem is non-linear, yet, upon closer inspection, its nature is multivariate polynomial: the objective function is a sum of squares and all unknowns also appear as multivariate polynomials in the constraints. Upon invoking Lagrange multipliers, the first order necessary conditions for optimality are also multivariate polynomials, the common roots of which characterise all stationary points, including the global optimal solutions. Rooting such problems is equivalent to solving an MEVP. Furthermore, this characterization allows us to describe the system theoretic properties of the globally optimal model and its misfits. In particular, we aim to extend Walsh's theorem to the multidimensional setting [1, 3].

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