Exploiting the Macaulay matrix structure for the globally optimal identification of autonomous systems

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Abstract

Stationary points, including the globally optimal parameter values, of the least-squares misfit identification of autonomous systems can be found as the solutions (eigentuples) of a multiparameter eigenvalue problem (MEP). Eigentuples of MEP's can be found using the block-Macaulay method which relies on constructing a block-Macaulay matrix. From the null space of this matrix, the eigentuples of the MEP are distilled. This work proposes a novel, recursive approach to compute this null space without explicitly building the block-Macaulay matrix.

The block-Macaulay method for MEP's

- **Multiparameter eigenvalue problem (MEP)**: Find $\boldsymbol{x}$ such that matrix polynomial $\boldsymbol{A}(\boldsymbol{x})$ is rank-deficient. E.g., first-degree 2-parameter eigenvalue problem:

$$\boldsymbol{A}(x,y)\boldsymbol{v} = (\boldsymbol{A}_1 + x\boldsymbol{A}_2 + y\boldsymbol{A}_3)\boldsymbol{v} = \boldsymbol{0}.$$

- **Algorithm:** Arrange **shifted versions of MEP** in rows of block-Macaulay matrix. **Null space** is spanned by Vandermonde-like vectors associated with all solutions (x, y):

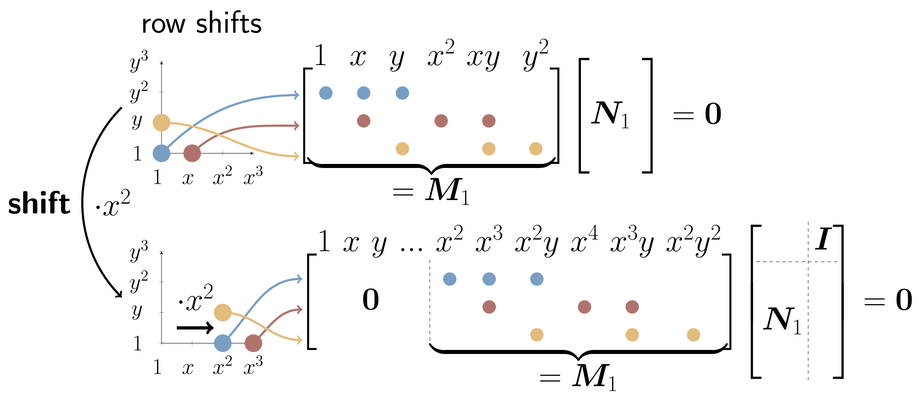
$$\begin{array}{r} \\ \boldsymbol{A}(x,y) \\ x\cdot\boldsymbol{A}(x,y) \\ y\cdot\boldsymbol{A}(x,y) \\ \vdots \end{array} \underbrace{\begin{array}{c} \begin{array}{ccccccc} 1 & x & y & x^2 & xy & y^2 & \dots \end{array} \\ \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & & & & \\ & \boldsymbol{A}_1 & & \boldsymbol{A}_2 & \boldsymbol{A}_3 & & \\ & & \boldsymbol{A}_1 & & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \\ & & & & & & \ddots \end{bmatrix} \end{array}}_{\text{Block Macaulay matrix } \boldsymbol{M}} \begin{bmatrix} \boldsymbol{v} \\ x\boldsymbol{v} \\ y\boldsymbol{v} \\ x^2\boldsymbol{v} \\ xy\boldsymbol{v} \\ y^2\boldsymbol{v} \\ \vdots \end{bmatrix} = \boldsymbol{0}.$$

  Adding all shifts of $\boldsymbol{A}(x,y)$ up to high enough degree as block-rows, **all solutions** (x,y) can be extracted from null space through several **eigenvalue decompositions**.

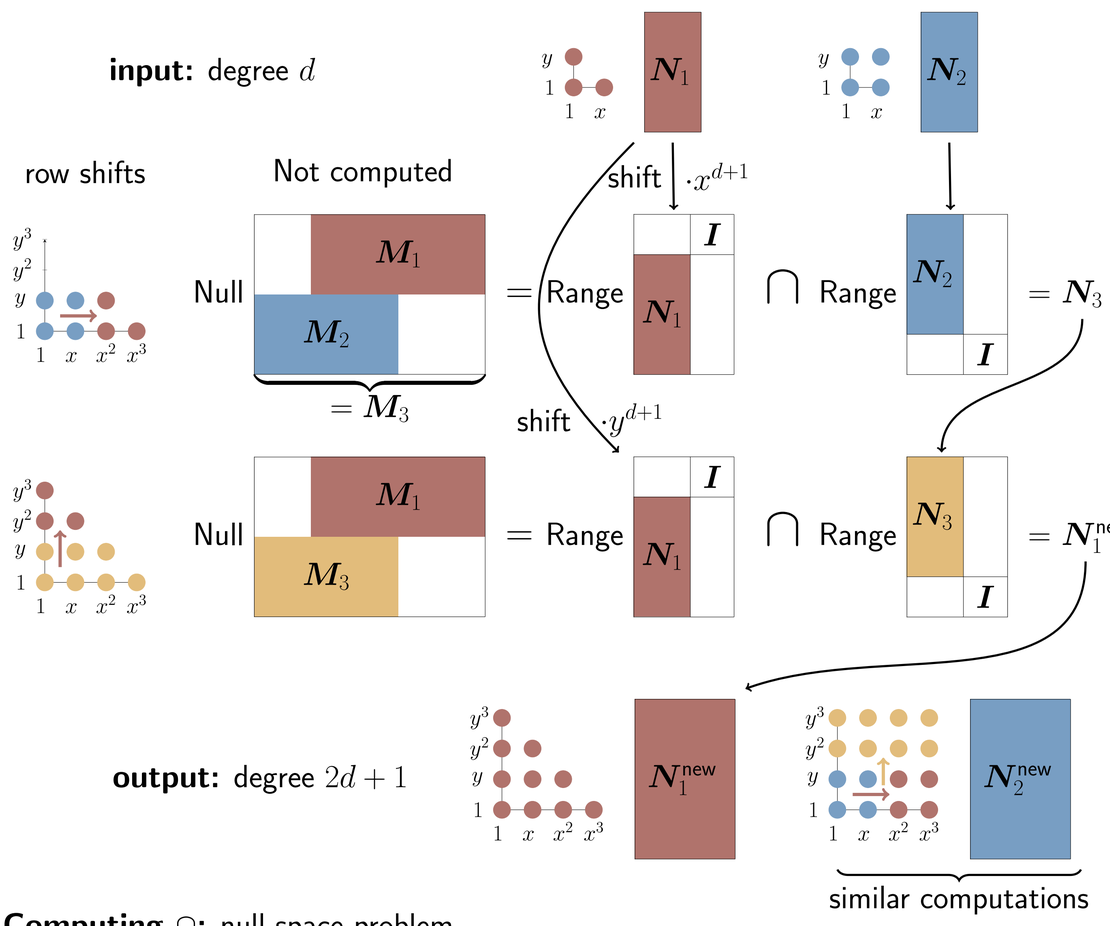
- **Bottleneck:** computation of null space of large block-Macaulay matrix.

Exploiting the Macaulay matrix structure

- Each block row has **same coefficients**, only positions differ → **set of row shifts fully determine the matrix**. This set can be 'shifted' without affecting nonzero part of matrix:



- Decompose Macaulay matrix rows into block rows by **decomposing set of row shifts**. Null space can be computed from null spaces of these block rows, i.e., smaller Macaulay matrices. E.g., for 2-parameter MEP (can be generalized to arbitrary dimensions):



- **Computing** $\cap$**:** null space problem.

$$\boldsymbol{p} \in (\text{Range } \boldsymbol{B} \cap \text{Range } \boldsymbol{C}) \leftrightarrow \exists \boldsymbol{x}, \boldsymbol{y} : \boldsymbol{p} = \boldsymbol{B}\boldsymbol{x} = \boldsymbol{C}\boldsymbol{y} \rightarrow \begin{bmatrix} \boldsymbol{B} & -\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{0}.$$

Identity rows can be ignored:

$$\underbrace{\begin{bmatrix} \boldsymbol{N}_{11} \\ \boldsymbol{N}_{12} \\ & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{B}} \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} & \boldsymbol{I} \\ \boldsymbol{N}_{22} & \\ \boldsymbol{N}_{21} & \end{bmatrix}}_{\boldsymbol{C}} \begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{bmatrix} \rightarrow \begin{cases} \begin{bmatrix} \boldsymbol{N}_{12} & -\boldsymbol{N}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{Y}_1 \end{bmatrix} = \boldsymbol{0} \\ \boldsymbol{X}_2 = \boldsymbol{N}_{21}\boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 = \boldsymbol{N}_{11}\boldsymbol{X}_1 \end{cases}$$

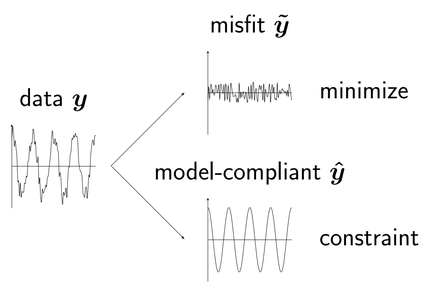
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Application: globally optimal misfit identification

Given N datapoints $\boldsymbol{y} \in \mathbb{R}^N$, determine globally optimal order n autonomous, LTI model w.r.t. least-squares, misfit-based cost criterium, i.e., find best signal approximation $\hat{\boldsymbol{y}}$ satisfying autonomous difference equation [1]: $\sum_{i=0}^{n} a_i \hat{y}_{k+i} = 0,\ \forall k = 0, ..., N-n-1.$

$$\min_{\boldsymbol{a}, \hat{\boldsymbol{y}}} ||\tilde{\boldsymbol{y}}||_2^2 \quad \text{s.t.} \begin{cases} \boldsymbol{y} = \tilde{\boldsymbol{y}} + \hat{\boldsymbol{y}} \\ \boldsymbol{T}_{N-n}^{\boldsymbol{a}} \hat{\boldsymbol{y}} = \boldsymbol{0}, \end{cases}$$

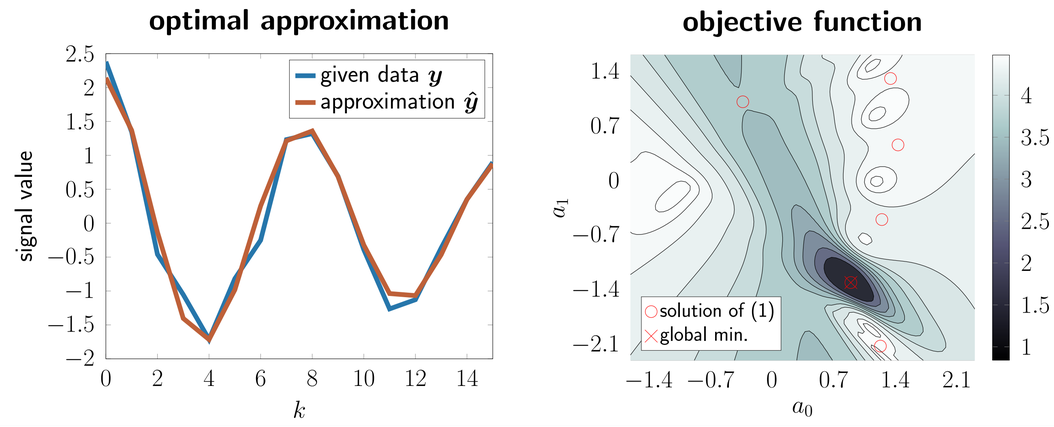
$$\boldsymbol{T}_{N-n}^{\boldsymbol{a}} = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} & 1 & & \\ & \ddots & \ddots & & \ddots & \ddots & \\ & & a_0 & a_1 & \cdots & a_{n-1} & 1 \end{bmatrix} \in \mathbb{R}^{N-n \times N}.$$



All minima and some saddle points are solutions of MEP (1) [2]. Use cost to filter out global minimum.

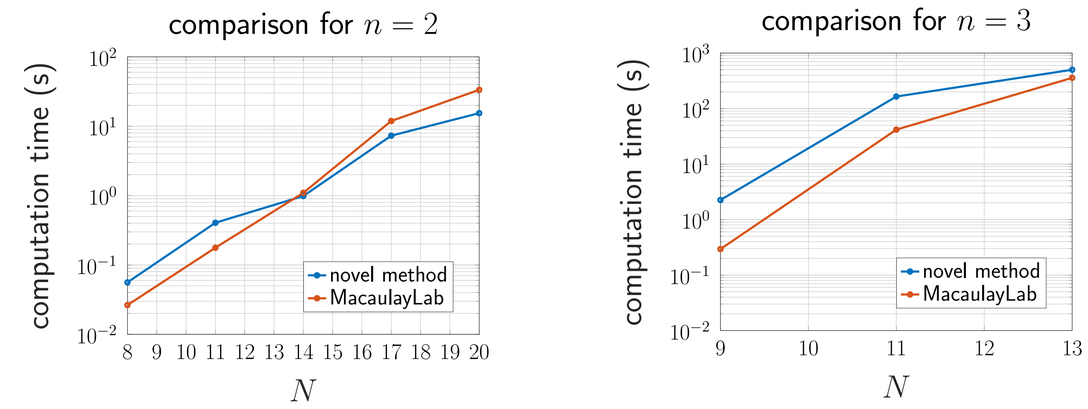
$$\boldsymbol{0} = \begin{bmatrix} \boldsymbol{T}_{N-n}^{\boldsymbol{a}}\boldsymbol{y} & \boldsymbol{T}_{N-n}^{\boldsymbol{a}}(\boldsymbol{T}_{N-n}^{\boldsymbol{a}})^{\mathsf{T}}(\boldsymbol{T}_{N-2n}^{\boldsymbol{a}})^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} -1 \\ \boldsymbol{g} \end{bmatrix}. \tag{1}$$

Example: Given 16 noisy data points generated by underlying second-order system with poles $0.6718 \pm 0.6718i$. Objective function and globally optimal model-compliant approximation visualized below. Optimal poles are $0.6537 \pm 0.6857i$.



Numerical experiments

Comparison: proposed algorithm vs. the **MacaulayLab implementation** [4]. For $n = 2$ and $n = 3$ and varying N, null space for which a *gap* appears is computed for MEP (1). This indicates the null space degree is large enough so that MEP can be solved [3]. The computation times visualized below in semi-log scale.



Seems to suggest scaling advantage in N, larger experiments needed to confirm.

Further research

- Attempt to tackle larger-scale misfit problems using a supercomputer.
- Carry out an asymptotic complexity analysis.
- Investigate the numerical stability of the proposed method.

Other applications:

- Solving general problems that can be formulated as an MEP.

References

[1] B. De Moor. "Least squares optimal realisation of autonomous LTI systems is an eigenvalue problem". In: *Communications in Information and Systems* 20.2 (2020), pp. 163–207.

[2] S. Lagauw, L. Vanpoucke, and B. De Moor. "Exact Characterization of the Global Optima of Least Squares Realization of Autonomous LTI Models as a Multiparameter Eigenvalue Problem". In: *2024 European Control Conference*. 2024, pp. 3446–3451.

[3] C. Vermeersch. "The (Block) Macaulay Matrix: Solving Systems of Multivariate Polynomial Equations and Multiparameter Eigenvalue Problems". eng. PhD thesis. Leuven, Belgium: KU Leuven, 2023.

[4] C. Vermeersch and B. De Moor. "Two complementary block Macaulay matrix algorithms to solve multiparameter eigenvalue problems". In: *Linear Algebra and its Applications* 654 (2022), pp. 177–209.