Polynomial/Rational Models from Model-compliant Data

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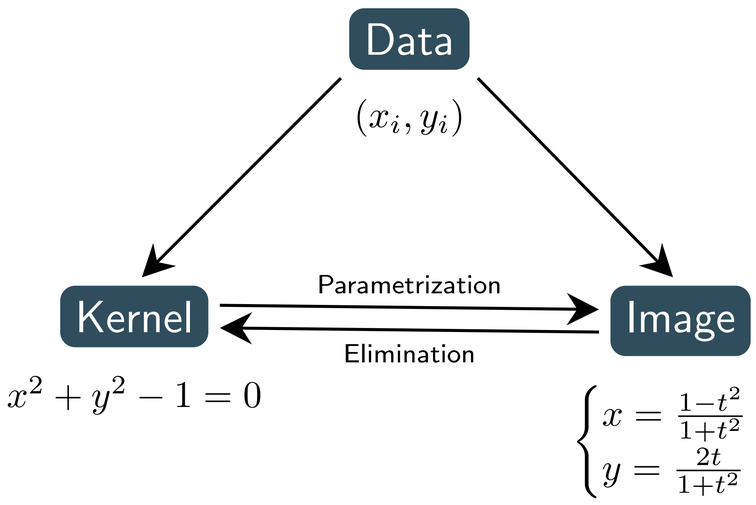
Problem statement

A static system can be described by different but equivalent representations:



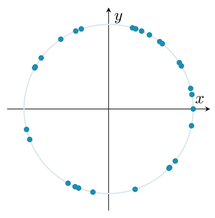
(How) can we convert between these representations?

In this poster, we illustrate these representations and conversions using the simple didactic example of the unit circle. The underlying questions and methods, however, extend to general static systems.

Data → kernel conversion

Objective: compute a kernel representation that is exactly satisfied by the data:

$$p(x_i, y_i) = 0, \qquad i = 1, 2, \ldots, N.$$



- Construct a Vandermonde matrix V from the data points:

$$V = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_N \\ y_1 & y_2 & \ldots & y_N \\ x_1^2 & x_2^2 & \ldots & x_N^2 \\ x_1y_1 & x_2y_2 & \ldots & x_Ny_N \\ y_1^2 & y_2^2 & \ldots & y_N^2 \end{pmatrix}$$

- The coefficient vector of the polynomial lies in the left null space of V:

$$\begin{array}{c} 1 \;\; x \;\; y \;\; x^2 \;\; xy \;\; y^2 \\ \boldsymbol{p}^T = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \\ \Rightarrow p(x, y) = -1 + x^2 + y^2 = 0 \end{array}$$

Parametrization

kernel → image

The conversion kernel → image raises several fundamental questions. Algebraic geometry provides some theoretical answers. However, from an engineering perspective, the concepts are often abstract and difficult to interpret.

1 Does a rational parametrization exist?

- Not every kernel representation admits a rational parametrization.
- For plane curves, a rational parametrization exists if and only if the genus is zero. [1]

2 Under what conditions does a polynomial parametrization exist?

- A curve admits a polynomial parametrization if and only if it admits a rational one and has exactly one place at infinity. [2]
- Points at infinity are determined by the highest-degree terms: $x^2 + y^2 = 0$ has 2 solutions $y = \pm ix$, so the circle is not parametrizable by a polynomial.

3 How many parameters are required?

- The number of parameters equals the dimension of the variety.

4 How can such a realization be computed algorithmically?

- Practical algorithms exist for some cases, but no general, user-friendly framework is known to us.

Elimination

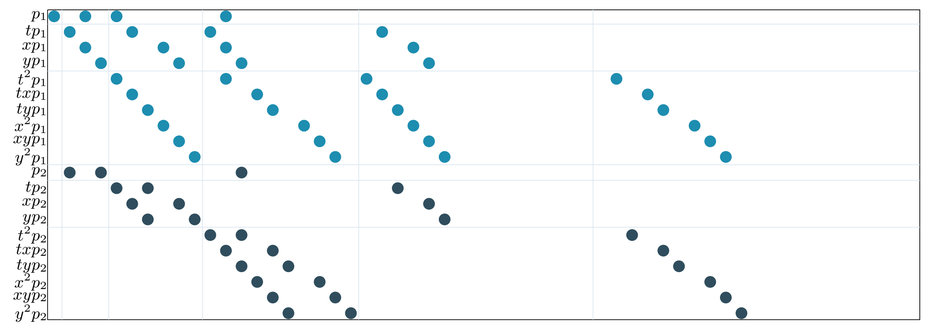
image → kernel

By eliminating the parameters, we can convert from an image to a kernel representation.

- Rewrite image representation as polynomials:

$$\begin{cases} p_1 = t^2x - t^2 + x + 1 & = 0 \\ p_2 = t^2y - 2t + y & = 0 \end{cases}$$

- Build the Macaulay matrix from polynomials and their shifts:



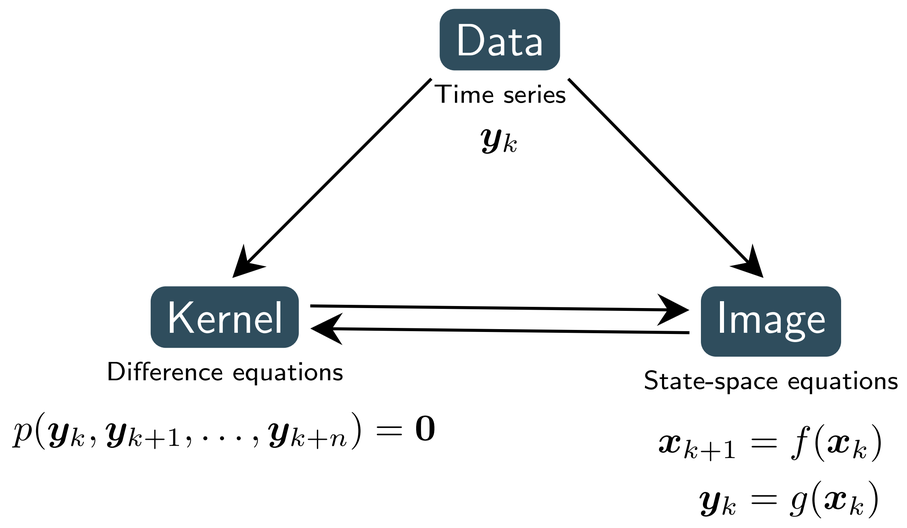
- To eliminate t, we search for vectors in the row space of the Macaulay matrix with a specified zero pattern [3]:

The blue zero entries indicate positions that must be zero.

$$\begin{array}{c} 1 \;\; t \;\; x \;\; y \;\; t^2 \;\; tx \;\; ty \;\; x^2 \;\; xy \;\; y^2 \;\; t^3 \;\; \ldots \\ p = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \ldots & 0 \end{pmatrix} \\ \Rightarrow p(x, y) = -1 + x^2 + y^2 = 0 \end{array}$$

Future research: dynamical systems

We aim to perform the same conversions for polynomial/rational **dynamical** systems. The representations are slightly different now, as they involve time-dependent dynamics:



Here, p is a polynomial and f and g are rational functions.
To tackle this, we plan to combine tools from numerical linear algebra, system theory, and algebraic geometry.

References

[1] J. Schicho, "Rational Parametrization of Surfaces," *Journal of Symbolic Computation*, vol. 26, no. 1, pp. 1–29, Jul. 1998.

[2] S. S. Abhyankar, *Algebraic Geometry for Scientists and Engineers*. American Mathematical Soc., 1990.

[3] K. Batselier, P. Dreesen, and B. De Moor, "The geometry of multivariate polynomial division and elimination," *SIAM Journal on Matrix Analysis and Applications*, vol. 34, no. 1, pp. 102–125, 2013.

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