Canonical Polyadic Decomposition via QZ Factorization

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***Abstract*—The canonical polyadic decomposition (CPD) is a fundamental tensor decomposition which expresses a tensor as a sum of rank one tensors. In stark contrast to the matrix case, with light assumptions, the CPD of a low rank tensor is (essentially) unique. The essential uniqueness of CPD makes this decomposition a powerful tool in many applications as it allows for extraction of component information from a signal of interest.**

One popular algorithm for algebraic computation of a CPD is the generalized eigenvalue decomposition (GEVD) which selects a matrix subpencil of a tensor, then computes the generalized eigenvectors of the pencil. In this article, we present a simplification of GEVD which improves the accuracy of the algorithm. Surprisingly, the generalized eigenvector computation in GEVD is in fact unnecessary and can be replaced by a QZ decomposition which factors a pair of matrices as a product of unitary and upper triangular matrices. Computing a QZ decomposition is a standard first step when computing generalized eigenvectors, so our algorithm can been seen as a direct simplification of GEVD.

***Index Terms*—Tensors, CPD, QZ, GEVD, Multilinear Algebra**

I. Introduction

Tensors, or multiindexed numerical arrays, are higher order generalizations of matrices and are natural structures for expressing data and signals which have inherent higher order structure. In this article we study the canonical polyadic decomposition (CPD) which expresses a tensor as a sum of rank one compoents. The CPD plays an important role in many applications due to the fact that, with mild assumptions, a low rank tensor has a unique CPD [1]–[3]. As such, one can recover underlying component information by computing a CPD of a low rank signal tensor [4]–[6]. The essential uniqueness of CPD has helped make tensors and tensor methods common place in machine learning and signal processing [4], [5].

A standard approach for computing a CPD of a low rank tensor is to first algebraically approximate the decomposition, then to refine the approximation with optimization routines. These algebraic approximations play an important role in optimization routines, as they are relatively inexpensive to compute and the strong initializations they provide can both improve final accuracy and reduce total computation time. Notably, computing a best low rank approximation of a noisy low rank signal tensor is nonconvex and NP-hard [7], so good algebraic initializations greatly aid in getting reliable solutions.

A common approach for algebraic CPD computation is to first compute the generalized eigenvectors of a matrix subpencil of a tensor [8]–[11]. One of the tensors factors can then be obtained by computing the inverse transpose of the matrix of generalized eigenvectors for the pencil. This generalized eigenvalue decomposition (GEVD) algorithm dates back to 1970 when it was communicated by Harshman who credits the algorithm to Jennrich [12]. Since then GEVD has been rediscovered by many authors [10], [11], [13].

One very popular algorithm for algebraic computation of a CPD is the generalized eigenvalue decomposition (GEVD) [8]–[11]. GEVD exploits the fact that one factor matrix of a tensor can be revealed by computing the generalized eigenvectors of a subpencil of the tensor. The GEVD algorithm dates back to 1970 when it was communicated by Harshman who credits the algorithm to Jennrich [12]. Since then GEVD has been rediscovered by many authors [10], [11], [13].

We present a simplification of the GEVD algorithm which is more accurate than the original algorithm. As it turns out, the generalized eigenvector computation in GEVD is not needed. Instead, one need only compute a $\mathbf{QZ}$ decomposition of a subpencil of the tensor. For a generic low rank tensor, the computed $\mathbf{Q}$ and $\mathbf{Z}$ will simultaneously upper triangularize all frontal slices of the tensor. Our key observation is that, once one has upper triangularized all frontal slices of the tensor, one can simply read off one of the tensors factor matrices from the diagonal entries of the upper triangular slices. Borrowing the perspective of [14], upper triangularizing the frontal slices of a low rank tensor reveals its "joint generalized eigenvalues."

Computing a $\mathbf{QZ}$ decomposition is a standard first step in computing generalized eigenvectors, so our algorithm can be seen as a direct simplification of GEVD. Intuitively, this simplification is obtained by basing CPD computation on (joint) generalized eigenvalue computation rather than (joint) generalized eigenvector computation. The increase in accuracy of in our algorithm is due to the fact that computing the $\mathbf{QZ}$ decomposition only relies on unitary matrices as opposed to the general invertible matrices needed for generalized eigenvectors. Furthermore, the $\mathbf{QZ}$ method eliminates an inverse computation needed by GEVD.

It is worth noting that ours is not the first algorithm for CPD computation based on simultaneous upper triangularization of the frontal slices of a tensor. For example, in [15] frontal slices are jointly upper triangularized by minimizing the

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Frobenius norm of the lower triangular portion of the slices. However, the method used in [15] to obtain factor matrices after simultaneous upper triangularization is more involved.

The article is organized as follows. Our **QZ** based CPD algorithm is presented in Section II. In Section III we prove that our algorithm successfully computes the CPD of a generic low rank tensor. The article ends with Section IV where we illustrate the performance of our algorithm on direction-of-arrival retrieval and in a fluorescence data experiment using the amino acid data set from [16].

Notation and terminology. Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. We denote scalars, vectors, matrices, and tensors by lower case (a), bold lower case $(\mathbf{a})$, bold upper case $(\mathbf{A})$, and calligraphic script $(\mathcal{A})$, respectively. For a matrix $\mathbf{A} \in \mathbb{K}^{I\times J}$, we let $\mathbf{A}^{\mathrm{T}}$ denote the transpose of $\mathbf{A}$ while $\mathbf{A}^{\mathrm{H}}$ denotes the conjugate transpose of $\mathbf{A}$. If $\mathbf{A}$ is invertible, then we let $\mathbf{A}^{-\mathrm{T}}$ denote the inverse of $\mathbf{A}^{\mathrm{T}}$. We use $D_r(\mathbf{A})$ denote the diagonal matrix whose diagonal entries are given by the rth row of $\mathbf{A}$. In addition, we say a matrix is generic to mean that it lies in a full measure set. Roughly speaking, a matrix is generic with probability equal to one.

A **tensor** is a multiindexed array with entries in $\mathbb{K}$. The **order** of a tensor is the number of indices. Given a collection nonzero vectors $\mathbf{u}^{(1)} \in \mathbb{K}^{I_1}, \dots, \mathbf{u}^{(N)} \in \mathbb{K}^{I_N}$, let

$$\mathbf{u}^{(1)} \otimes \cdots \otimes \mathbf{u}^{(N)} \in \mathbb{K}^{I_1\times\cdots\times I_N}$$

denote the $I_1 \times \cdots \times I_N$ tensor with $i_1, i_2, \dots, i_N$ entry equal to $u^{(1)}_{i_1} u^{(2)}_{i_2} \cdots u^{(N)}_{i_N}$. A tensor of this form is called a **rank one tensor**. The minimal integer R such that

$$\mathcal{T} = \sum_{r=1}^{R} \mathbf{u}^{(1)}_r \otimes \cdots \otimes \mathbf{u}^{(N)}_r$$

where each $\mathbf{u}^{(n)}_r$ has entries in $\mathbb{K}$ is called the $\mathbb{K}$-rank of the tensor $\mathcal{T}$, and a decomposition of this form is called a **canonical polyadic decomposition** (CPD) of $\mathcal{T}$. Compactly we write $\mathcal{T} = [\![\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \dots, \mathbf{U}^{(N)}]\!]$. Here the matrix $\mathbf{U}^{(n)} \in \mathbb{K}^{I_n\times R}$ has $\mathbf{u}^{(n)}_r$ as its rth column. The matrix $\mathbf{U}^{(n)}$ is called a **factor matrix** of $\mathcal{T}$.

A **mode-ℓ fiber** of a tensor is a vector obtained by fixing all indices but the ℓth. Defined for a third order tensor, the **mode-ℓ unfolding** $\mathbf{T}_{[\ell;j,i]}$ of $\mathcal{T}$ is the matrix obtained by stacking all mode-ℓ fibers of $\mathcal{T}$ as columns of a matrix, where the mode i indices increment faster than the mode-j indices. The **ℓ-mode** product $\mathcal{T} \cdot_\ell \mathbf{A}$ between a matrix $\mathbf{A}$ and a tensor $\mathcal{T}$ is the tensor with mode-ℓ unfolding equal to $\mathbf{A}\mathbf{T}_{[\ell;j,i]}$. We also make use of more general unfoldings for tensors of order greater than three. E.g., if $\mathcal{T} \in \mathbb{K}^{I_1\times I_2\times I_3\times I_4}$, then $\mathbf{T}_{[4,3;2,1]}$ is an $I_1I_2 \times I_3I_4$ matrix with $I_1(i_2-1)+i_1, I_3(i_4-1)+i_3$ entry equal to t_{i_1,i_2,i_3,i_4}.

We often consider order three subtensors of a tensor. Given a tensor $\mathcal{T} \in \mathbb{K}^{I_1\times\cdots\times I_N}$ and an integer $3 \le n \le N$ define

$$\mathcal{T}[n] := \mathcal{T}(:,:,1,\dots,1,:,1,\dots,1)$$

where the third : occurs in the nth mode of $\mathcal{T}$. That is, $\mathcal{T}[n]$ is an order three subtensor of $\mathcal{T}$ formed by fixing all but the first, second, and nth. In the case that $\mathcal{T}$ has order 3, we call the matrices $\{\mathbf{t}(:,:,k)\}_{k=1}^{I_3}$ the **frontal slices** of $\mathcal{T}$.

II. CPD BY QZ

We now present the CPDQZ algorithm. In our discussion we let $\mathcal{M} \in \mathbb{K}^{I_1\times\cdots\times I_N}$ denote a measured tensor of interest, and we assume that $\mathcal{M}$ has the form $\mathcal{M} = \mathcal{T} + \mathcal{N}$ where $\mathcal{T}$ is the signal portion of $\mathcal{M}$ and where $\mathcal{N}$ is noise. A standard assumption for generalized eigenvalue based algorithms such as GEVD and GESD is that that the tensor $\mathcal{T}$ has at least two factor matrices with full column rank. In particular, letting R denote the rank $\mathcal{T}$, one has $R \le \min\{I_1, I_2\}$ up to a permutation of indices. For this reason, by computing a (truncated) orthogonal compression of $\mathcal{M}$ such as a multilinear singular value decomposition, see e.g. [17], we can restrict to the case where $\mathcal{M}$ has rank R and size $R\times R\times R_3\times\cdots\times R_N$ with $R \ge R_n$ for all n.[1]

We also note that, when compared to GEVD, our algorithm requires one additional mild assumption. Namely, we assume that the matrix $\mathbf{U}^{(N)}\odot\cdots\odot\mathbf{U}^{(3)}$ has full column rank. Here $\odot$ denotes the Khatri-Rao between matrices. In the case of the upcoming CPDQZS variation of our algorithm, one instead must make the stronger assumption that there is an index $3 \le n \le N$ such that $R_n = R$ and $\mathbf{U}^{(n)}$ is invertible. These deterministic assumptions are satisfied for generic factor matrices provided the matrices in question all have at least as many rows as columns.

The key observation behind the CPDQZ algorithm is that one may compute all but two factor matrices of a tensor using a single **QZ** decomposition together with n-mode products. In particular, given a rank R tensor $\mathcal{T} \in \mathbb{K}^{R\times R\times R_3\times\cdots\times R_N}$ which satisfies our assumptions, to obtain factor matrices $\mathbf{U}^{(n)}$ for $n \ge 3$, one need only compute unitary matrices $\mathbf{Q}$ and $\mathbf{Z}$ such that

$$\mathbf{Q}\mathbf{T}[3](:,:,1)\mathbf{Z} \qquad \mathbf{Q}\mathbf{T}[3](:,:,2)\mathbf{Z}$$

is a **QZ** decomposition of the matrix pencil

$$(\mathbf{T}[3](:,:,1), \mathbf{T}([3](:,:,2)),$$

i.e., such that both of these matrices are upper triangular.[2]

Define $\mathcal{T}_{qz} := \mathcal{T} \cdot_1 \mathbf{Q} \cdot_2 \mathbf{Z}^{\mathrm{T}}$. It is then a matter of technical formula manipulation to show that

$$\mathbf{U}^{(n)}(r,:) = \mathbf{t}_{qz}[n](r,r,:)$$

for all $n = 3, \dots, N$ and all $r = 1, \dots, R$.

It remains to compute $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$. To do this we first compute $\mathbf{U}^{(2)} \odot \mathbf{U}^{(1)}$ by solving the overdetermined system

$$\mathbf{T}_{[N,N-1,\dots,3;2,1]} = (\mathbf{U}^{(N)} \odot \cdots \odot \mathbf{U}^{(3)})(\mathbf{U}^{(2)} \odot \mathbf{U}^{(1)})^{\mathrm{T}}.$$

The rth column of the matrix $\mathbf{U}^{(2)} \odot \mathbf{U}^{(1)}$ is the Kronecker product of the rth column of $\mathbf{U}^{(2)}$ with that of $\mathbf{U}^{(1)}$ and can therefore be reshaped into a rank-1 matrix. It follows that the columns of these factors can be obtained by computing a rank-1 approximation of each reshaped column of $\mathbf{U}^{(2)} \odot \mathbf{U}^{(1)}$.

As an alternative, suppose $\mathbf{U}^{(n)} \in \mathbb{K}^{R\times R}$ has full rank for some $n \in \{3, \dots, N\}$. For ease of exposition we take $n = N$.

[1] We assume $R \ge R_n$ since the rank of a tensor upper bounds its multilinear ranks.

[2] We emphasize that one may take linear combinations of matrices of the form $\mathbf{T}(:,:,i_3,\dots,i_N)$ to form the matrix pencil used for **QZ** computation. A fixed pencil is used here for the sake of exposition.

Then the remaining factor matrices can be computed by first solving for $(\mathbf{U}^{(N-1)} \odot \cdots \odot \mathbf{U}^{(1)})$ in

$$\mathbf{T}_{[N-1,\ldots,1;N]} = (\mathbf{U}^{(N-1)} \odot \cdots \odot \mathbf{U}^{(1)})(\mathbf{U}^{(N)})^{\mathrm{T}},$$

then computing rank-1 approximations of the appropriately reshaped columns of this matrix. We call the first approach CPDQZ and the second approach CPDQZS(ingle). For clarity we emphasize that there is no difference between CPDQZ and CPDQZS for tensors of order three.

For tensors or order greater than three, the key difference between these methods is that the CPDQZS method uses rank-1 tensor approximations to compute most of its factors, while the CPDQZ relies more heavily on the initial $\mathbf{QZ}$ decomposition to obtain factors. Since a best rank-1 tensor approximation can often be accurately computed and since there is less opportunity for error accumulation before rank-1 tensor approximations are employed in CPDQZS, the CPDQZS method is expected to be more accurate than CPDQZ. This expectation is supported by our numerical experiments; however, CPDQZS is more computationally expensive than CPDQZ.

III. Algorithm derivation

We now present a mathematical derivation of the CPDQZ algorithm. For the sake of exposition, we first consider the case where $\mathcal{T} \in \mathbb{K}^{R\times R\times R}$ has order three. The first observation used in the derivation of the CPDQZ algorithm is that in the special case where $\mathcal{T} = \llbracket \mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \mathbf{U}^{(3)} \rrbracket$ has upper triangular frontal slices and meets our assumptions, then in an appropriate column ordering, the matrix $\mathbf{U}^{(1)}$ is upper triangular while the matrix $\mathbf{U}^{(2)}$ is lower triangular.

This fact follows quickly from the main observation behind the GEVD algorithm itself. Namely, if $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ are $R \times R$ invertible matrices, then the columns of the matrix $(\mathbf{U}^{(2)})^{-\mathrm{T}}$ are equal to the generalized eigenvectors of the matrix pencil

$$(\mathbf{T}(:,:,1), \mathbf{T}(:,:,2)).$$

In the case that $\mathbf{T}(:,:,1)$ and $\mathbf{T}(:,:,2)$ are upper triangular, a routine argument shows that the pencil's generalized eigenvectors can be ordered so that $(\mathbf{U}^{(2)})^{-\mathrm{T}}$ is upper triangular, hence $\mathbf{U}^{(2)}$ is lower triangular. Having shown that $\mathbf{U}^{(2)}$ is lower triangular, one may use the assumption that $\mathcal{T}$ has upper triangular frontal slices together with the formula

$$\mathbf{T}(:,:,r) = \mathbf{U}^{(1)} D_r(\mathbf{U}^{(3)})(\mathbf{U}^{(2)})^{\mathrm{T}} \qquad \text{for } r = 1,\ldots,R \quad (1)$$

to conclude not only that $\mathbf{U}^{(1)}$ is upper triangular, but also that, up to scaling, one has

$$\mathbf{U}^{(3)}(:,r) = \mathrm{diag}(\mathbf{T}(:,:,r)) \qquad \text{for } r = 1,\ldots,R.$$

From this point we need only show that if $\mathcal{T} = \llbracket \mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \mathbf{U}^{(3)} \rrbracket \in \mathbb{K}^{R\times R\times R}$ is an arbitrary tensor which meets our assumptions and if $\mathbf{Q}$ and $\mathbf{Z}$ are matrices which give a $\mathbf{QZ}$ decomposition of the matrix pencil

$$(\mathbf{QT}(:,:,1)\mathbf{Z}, \mathbf{QT}(:,:,2)\mathbf{Z}),$$

then the tensor $\mathcal{T} \cdot_1 \mathbf{Q} \cdot_2 \mathbf{Z}^{\mathrm{T}} = \llbracket \mathbf{QU}^{(1)}, \mathbf{Z}^{\mathrm{T}}\mathbf{U}^{(2)}, \mathbf{U}^{(3)} \rrbracket$ has upper triangular frontal slices. As we shall explain, this fact follows from repeating preceding argument.

Since $\mathbf{QT}(:,:,1)\mathbf{Z}$ and $\mathbf{QT}(:,:,2)\mathbf{Z}$ are upper triangular by definition of the $\mathbf{QZ}$ factorization, the matrix of generalized eigenvectors of this pencil, i.e, the inverse transpose of the second factor matrix of the tensor $\mathbf{T} \cdot_1 \mathbf{Q} \cdot_2 \mathbf{Z}^{\mathrm{T}}$, can be taken to be upper triangular, hence $\mathbf{Z}^{\mathrm{T}}\mathbf{U}^{(2)}$ can be taken to be lower triangular. Applying equation (1) to $\mathbf{T} \cdot_1 \mathbf{Q} \cdot_2 \mathbf{Z}^{\mathrm{T}}$ then shows that $\mathbf{QU}^{(1)}$ is upper triangular and moreover that all frontal slices of this tensor are upper triangular. Since $\mathcal{T}$ and $\mathcal{T} \cdot_1 \mathbf{Q} \cdot_2 \mathbf{Z}^{\mathrm{T}}$ have the same third factor matrix it then easily follows that the CPDQZ algorithm successfully recovers the CPD of a tensor which meets the assumptions stated at the beginning of Section II.

The extension of the CPDQZ to tensors of order greater than three follows a routine argument using the fact that an order N tensor with CPD $\llbracket \mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \ldots, \mathbf{U}^{(N)} \rrbracket$ can be reshaped to an order three tensor with CPD $\llbracket \mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \mathbf{U}^{(N)} \odot \cdots \odot \mathbf{U}^{(3)} \rrbracket$. A similar argument can be used to show that CPDQZS successfully computes the CPD of any tensor which satisfies our assumptions.

IV. Experiments

The proposed QZ methods make different trade-offs with respect to accuracy and speed. In this section, we compare both methods with the classical GEVD algorithm by Jennrich [18] and the more recent GESD algorithm [19]. We use a machine with an AMD Ryzen 5 5600H CPU at 3,30GHz and 16GB of RAM using MATLAB R2021b and Tensorlab 3.0 [20].

In a first experiment, we generate fourth-order low-rank tensors $\mathcal{T}$ by sampling the entries of factor matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of dimensions $\mathbb{R}80 \times R$ from the uniform distribution on $[0, 1]$ and we normalize all columns to unit length. The rank R is varied in the range $[4, 16]$ and Gaussian noise is added such that the SNR is $40\,\mathrm{dB}$. For the estimated factor matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$, we show maximal relative factor matrix errors compared to the true factors, defined as

$$\max\left(\frac{\|\mathbf{A}-\hat{\mathbf{A}}\|_{\mathrm{F}}}{\|\mathbf{A}\|_{\mathrm{F}}}, \frac{\|\mathbf{B}-\hat{\mathbf{B}}\|_{\mathrm{F}}}{\|\mathbf{B}\|_{\mathrm{F}}}, \frac{\|\mathbf{C}-\hat{\mathbf{C}}\|_{\mathrm{F}}}{\|\mathbf{C}\|_{\mathrm{F}}}, \frac{\|\mathbf{D}-\hat{\mathbf{D}}\|_{\mathrm{F}}}{\|\mathbf{D}\|_{\mathrm{F}}}\right),$$

where columns of $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ and have been optimally permuted and scaled to match the columns of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, respectively. The results are shown in the top plots of Figure 1. CPDQZ is the fastest method, but is less accurate compared to CPDQZS and GEVD. CPDQZS is as fast as GEVD, but is more accurate over the whole range of ranks. In the bottom plots of Figure 1, the experiment is repeated, but now the SNR is varied between 10 and $60\,\mathrm{dB}$, with the rank $R = 10$ fixed. The same relative performance can be seen for the four methods.

In a more applied experiment, the QZ approach is compared to GEVD in a direction-of-arrival (DOA) retrieval experiment for line-of-sight signals impinging on a uniform rectangular array (URA). The CPD can be applied to find the DOAs in this case [21]–[23]. The URA has $M \times M$ sensors, where $M = 20$, and collects $K = 20$ samples from $R = 8$ far-field sources. These have azimuths $[15\ 20\ 25\ 30\ 35\ 40\ 50\ 55]$ and elevations $[5\ 10\ 20\ 25\ 30\ 35\ 40\ 45]$ degrees, respectively. We consider omnidirectional sensors, evenly-spaced with inter-sensor spacing

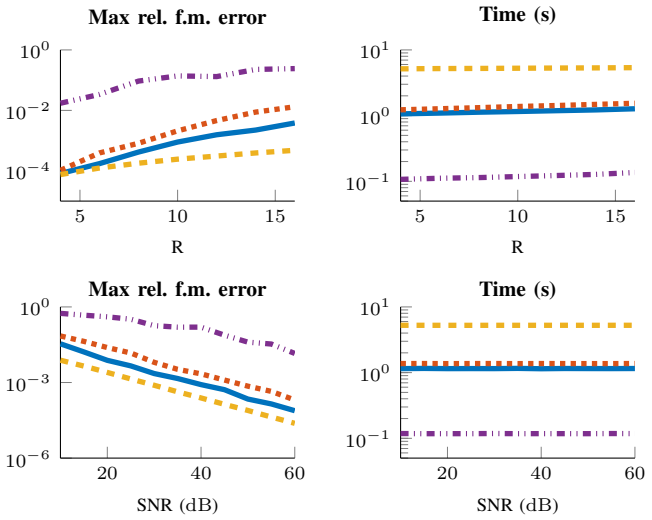


Fig. 1. The QZ methods make different trade-offs with respect to time and accuracy. Over 50 trials, CPDQZ ( ) is the fastest, but least accurate, while GESD ( ) is the most accurate and the slowest. CPDQZS ( ) and GEVD ( ) are equally fast, but the former is more accurate. The relative performance of the methods for the fourth-order tensors is consistent over varying ranks (with SNR 40 dB) and SNRs (with rank $R = 10$).

Δ. The azimuths $\mathbf{Z} \in \mathbb{C}^{M\times R}$ and elevations $\mathbf{L} \in \mathbb{C}^{M\times R}$ of the R sources that we collect at each sensor yield an observed tensor $\mathcal{T} \in \mathbb{C}^{M\times M\times K}$, where the kth frontal slice has a low rank decomposition $\mathbf{T}_k = \mathbf{A}^{(k)}\mathrm{diag}(\mathbf{s}^{(k)})(\mathbf{E}^{(k)})^{\mathrm{T}}$. The matrix $\mathbf{A}^{(k)} \in \mathbb{C}^{M\times R}$ has entries $a^{(k)}_{mr} = \exp((m-1)2\pi/\lambda \sin(z_{rk}\pi/180)\Delta i)$, while $\mathbf{E}^{(k)} \in \mathbb{C}^{M\times R}$ has entries $e^{(k)}_{mr} = \exp((m-1)2\pi/\lambda \sin(l_{rk}\pi/180)\Delta i)$. The vector $\mathbf{s}^{(k)} \in \mathbb{C}^{R}$ holds the sources and λ is the signal wavelength. By computing a rank-R CPD of a few frontal slices of $\mathcal{T}$, the azimuths and elevations of the sources can be recovered from the first two CPD factors. The tensor $\mathcal{T}$ is perturbed with Gaussian distributed noise in the range $[-20, 40]$dB and we compare the accuracy and speed of the CPDs obtained with GEVD and CPDQZS. In Figure 2, one can find the median error (left) and computation time (right) over 500 trials for the azimuth and elevation estimations of the eight sources. QZ is more accurate than GEVD, especially for the estimation of the source elevations, and is faster as well.

For a final experiment, we use the amino acid dataset from [16]. This dataset holds the measured emission and excitation spectra of five amino acid mixtures in a tensor $\mathcal{T}$ of dimensions $5 \times 201 \times 61$. The first mode corresponds to the five mixtures and the second and third modes refer to the emission and excitation spectra, respectively. In the left plot of Figure 3, the theoretical emission spectra are shown for the three amino acids. Following [16], $\mathcal{T}$ admits a rank-3 CPD with each component corresponding to one amino acid. We add Gaussian noise with SNRs between -20 and 20 dB and use GEVD, GESD and CPDQZS to compute a CPD of the perturbed $\mathcal{T}$. We compare the accuracy of the estimated concentrations of the amino acids in each mixture, provided by the mode-1 factor matrix of the CPD. The relative mode-1 factor matrix errors are shown in the center plot of Figure 3, while the computation times are shown in the right plot. CPDQZS is about as accurate

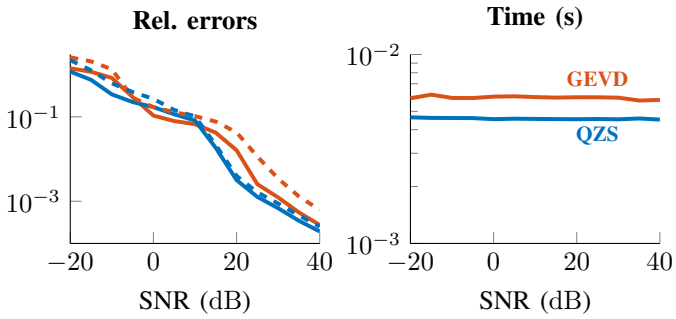


Fig. 2. CPDQZS estimates the elevations of the sources more accurately than GEVD outside of a small SNR window. In addition, CPDQZS has a lower computational cost. Left: mean relative errors over the eight sources of the azimuths ( ) and elevations ( ) that are estimated for CPDQZS and the azimuths ( ) and elevations ( ) that are estimated for GEVD. Right: computation time for both methods. Medians over 500 trials are shown.

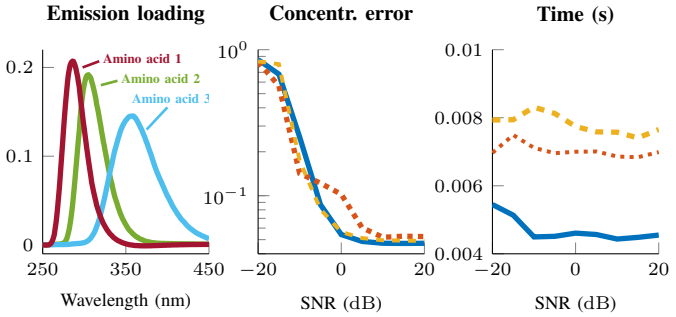


Fig. 3. The accuracy of CPDQZS ( ) for the estimation of the amino acid concentrations is as least as good as GEVD ( ) and GESD ( ), while requiring a lower computation time. The theoretical emission loading for the three amino acids in function of the wavelength is shown in the left plot. In the center plot, the mode-1 factor matrix error is shown, which corresponds to the estimated concentrations of the amino acids in the five mixtures. The right plot shows the computation times. All results are medians over 100 trials.

as GEVD and GESD over the range of SNRs and is also notably faster.

V. Conclusion

We presented the novel CPDQZ and CPDQZS algorithms for algebraic CPD computation which can be viewed as direct simplifications of the popular GEVD algorithm. These algorithms replace the generalized eigenvector computation of GEVD with a simple **QZ** decomposition. This **QZ** decomposition is used to simultaneously upper triangularize the frontal slices of a tensor at which point one factor matrix is obtained by reading the diagonal entries of the upper triangular slices. We showed in experiments that, in the case of CPDQZS, this simplification results in an increase in accuracy when compared to GEVD. For tensors of order four or more[3], CPDQZ is observed to be less accurate but faster than GEVD.

In future work we will investigate a deflation style algorithm in the spirit of GESD where the deflation step is based on a **QZ** computation rather than generalized eigenspace computations.

[3]Recall that CPDQZ and CPDQZS are identical for tensors of order three.

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