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## RESEARCH ARTICLE

# On efficient computation of low-complexity controlled invariant sets for uncertain linear systems

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A method is presented for determining invariant low-complexity polytopic sets and associated linear feedback laws for linear systems with polytopic uncertainty. Conditions based on the relationship between 2- and $\infty$-norms are used to define an initial invariant low-complexity polytope as the solution of a semidefinite program. The problem of computing a maximal controlled invariant low-complexity polytopic set is then formulated as a bilinearly constrained problem, and a relaxation of this problem is derived as an iterative sequence of convex programs. The proposed method scales linearly with the state dimension, which allows the possibility of determining low-complexity robust controlled invariant sets for high-order systems.

*Key Words:* polytopic invariant sets; optimization; constrained control

## 1 Introduction

The notion of invariant sets arises in many problems concerning the analysis and control of dynamical systems. A controlled invariant set defines a region of state space in which there necessarily exists a control law ensuring that the state remains in the set at all times. Because of this property invariant sets are typically used as target sets in an MPC framework in order to ensure stability and recursive feasibility of the controller. An overview of set invariance can be found in Blanchini (1999).

In the interests of computational tractability, two types of convex sets have been proposed as candidate invariant sets, namely ellipsoidal and polytopic sets. In this paper the main focus is on low-complexity polytopes as defined in Cannon et al. (2003). Low-complexity polytopes have significant advantages compared with ellipsoidal and more general polytopic sets Gilbert and Tan (1991), Pluymers et al. (2005a). As pointed out in Pluymers et al. (2005a), use of ellipsoidal target sets for linear MPC leads to the formulation of the online MPC optimization as a Semi-Definite Program (SDP), which typically has a much higher computational cost than the Quadratic Program (QP) required for polytopic target sets. On the other hand maximal robust invariant polytopic sets are described by large numbers of inequality constraints which can lead to significant increase in the computational complexity of the QP problem, especially for high order systems. Moreover, existing methods for determining general invariant polytopic

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sets Gilbert and Tan (1991), Pluymers et al. (2005a,b) do not allow the feedback law to be optimized, while use of a fixed, predetermined feedback gain has a negative influence on the size of the resulting invariant set. On the other hand, in Blanchini (1991) and Hennet and Beziat (1991) methods are described that start from a fixed set and then determine a feedback law $K$ that makes this set invariant. It is straightforward to see that these approaches also lead to invariant sets which are conservative w.r.t. the volume. However, the existing algorithms for determining low-complexity polytopes as described in Cannon et al. (2003) and in this paper allow the feedback $K$ and the volume of the invariant set to be optimized simultaneously.

In Cannon et al. (2003) an algorithm is described for determining the maximal feasible invariant low-complexity polytope for nonlinear systems. The current paper proposes to extend these results to the case of uncertain linear systems with polytopic uncertainty. The first contribution of the paper consists of a method that enables the calculation of an initial feasible invariant low-complexity polytope using Linear Matrix Inequalities (LMI). Using a general relationship between 2- and $\infty$-norms, the proposed algorithm determines a feedback gain $K$ and an invariant low-complexity polytope satisfying linear state and input constraints. This initial set is used as starting point for an efficient optimization procedure that increases the volume of the initial set by solving a sequence of Convex Programs. In contrast with the method described in Cannon et al. (2003), the number of optimization variables and constraints of the proposed method scales linearly with the number of the primary vertices of the low-complexity polytope instead of exponentially. Therefore, this method makes it possible to calculate invariant low-complexity polytopes for high dimensional systems as it significantly reduces the required calculation time and memory usage. The effectiveness of the proposed method is illustrated with examples.

Notation: $A \geqslant 0$ is used to denote a matrix $A$ with non-negative elements, whereas $A \succeq 0$ denotes a positive definite matrix $A$.

## 2 Problem Description

In this work uncertain discrete-time systems are considered of the following form:

$$x_{k+1} = \widetilde{A}x_k + \widetilde{B}u_k. \tag{1}$$

with $\widetilde{A}$ and $\widetilde{B}$ uncertain, possibly time-varying belonging to the polyhedral uncertainty class

$$\Omega = \left\{ (\widetilde{A}, \widetilde{B}) = \sum_{i=1}^{n_p} \gamma_i (A_i, B_i) | \gamma_i \geq 0, \sum_{i=1}^{n_p} \gamma_i = 1 \right\}. \tag{2}$$

Note that for $n_p = 1$, (2) represents a linear time-invariant (LTI) system, so the results in this paper also apply to the LTI case. It is assumed that the system is subject to polytopic input and state constraints:

$$x_k \in \{x | \, \|Lx\|_\infty \leq 1\} \tag{3}$$

$$u_k \in \{u | \, \|u\|_\infty \leq \bar{u}\} \tag{4}$$

with $x \in \mathbb{R}^{n_x \times 1}, u \in \mathbb{R}^{n_u \times 1}$ and $L \in \mathbb{R}^{n_c \times n_x}$. The main focus of this work is to determine low-complexity controlled invariant sets with maximal volume for the uncertain system (1) subject to the input and state constraint (3) and (4). To ensure that this maximization problem has a well-defined solution, we make the simplifying assumption that the maximal controlled invariant

set for (1) subject to (3)-(4) is finite. Note that the paper's approach is applicable even if this assumption is not satisfied, but the results may not be optimal in this case.

The invariant sets are assumed to have the following form:

$$\varphi = \{x| \left\|Vx\right\|_\infty \leq \gamma\}, \gamma \geq 0 \tag{5}$$

with $V$ a square matrix (not necessarily symmetric) and of full-rank. Invariance of the uncertain system (1) under a linear state feedback $K$ requires $\forall x \in \varphi$ and $\forall(\widetilde{A}, \widetilde{B}) \in \Omega$:

$$\left\|V\left(\widetilde{A} + \widetilde{B}K\right)x\right\|_\infty \leq \left\|Vx\right\|_\infty . \tag{6}$$

## 3 Invariant Feasible Polytopes

### 3.1 *Initial Feasible Set*

In this subsection an iterative procedure is outlined in order to determine an invariant set with maximum volume. In order that the procedure guarantees a feasible solution at each iteration an initial feasible invariant set has to be determined that can be used as a feasible starting point for the procedure. In the sequel a method will be discussed to find such a set.

The following relation between 2- and $\infty$-norms holds:

$$\forall x \in \mathbb{R}^n : \left\|Vx\right\|_\infty \leq \left\|Vx\right\|_2 \leq \sqrt{n}\left\|Vx\right\|_\infty . \tag{7}$$

The following theorem uses this property in order to determine an initial invariant set for the system (1).

**Theorem 3.1 :** *The set $\varphi = \{x| \left\|Vx\right\|_\infty \leq \gamma\}$ is invariant for system (1) w.r.t. the state feedback gain $K$ if the following condition is satisfied $\forall(\widetilde{A}, \widetilde{B}) \in \Omega$:*

$$\left\|V\widetilde{\phi}x_k\right\|_2 \leq (\frac{1}{\sqrt{n}})\left\|Vx_k\right\|_2 \tag{8}$$

*with $\widetilde{\phi} = (\widetilde{A} + \widetilde{B}K)$.*

**Proof.** From (7) the following relationships follow

$$\left\|V\widetilde{\phi}x\right\|_\infty \leq \left\|V\widetilde{\phi}x\right\|_2 \leq \frac{1}{\sqrt{n}}\left\|Vx\right\|_2 \leq \left\|Vx\right\|_\infty \tag{9}$$

which proves the theorem. □

**Remark 1 :** In order to impose condition (8) for all $(\widetilde{A}, \widetilde{B}) \in \Omega$ it is sufficient to impose the

condition on the vertices of the polytopic uncertainty set $\Omega$:

$$\|V(A_i + B_iK)x\|_2 \leq (\frac{1}{\sqrt{n}}) \|Vx\|_2 , i = 1, \ldots, n_p. \tag{10}$$

In the case of $(\widetilde{A}, \widetilde{B})$ belonging to the polytopic uncertainty class (2), the condition (8) can be re-written as the following set of LMIs

$$\phi_i^T P \phi_i \preceq \alpha^2 P, \quad i = 1, \ldots, n_p \tag{11}$$

with $\alpha = 1/\sqrt{n}$, $P \succeq 0$ and $\phi_i = A_i + B_iK$. A matrix $P$ satisfying these LMIs leads to a set $\varphi = \left\{x | \ \|P^{\frac{1}{2}}x\|_\infty \leq 1\right\}$ which is invariant for the uncertain system under the stabilizing feedback law $K$. A suitable $P$ and $K$ satisfying conditions (11) and constraints (3)-(4) can be found by solving the following optimization scheme (Boyd et al. (1994), Kothare et al. (1996), Lee et al. (2005)):

**Algorithm 1:**

$$\min_{Q,X,Y} - \log \det(Q) \tag{12}$$

subject to

$$Q \succeq 0 \tag{13}$$

$$L(i,:)QL(i,:)^T \leq 1, \ i = 1, \ldots, n_c \tag{14}$$

$$\begin{bmatrix} X & Y \\ Y^T & Q \end{bmatrix} \succeq 0, \, X(j,j) \leq \bar{u}^2 , \ j = 1, \ldots, n_u \tag{15}$$

and

$$\begin{bmatrix} \alpha^2 Q & (A_iQ + B_iY)^T \\ A_iQ + B_iY & Q \end{bmatrix} \succeq 0, \ j = 1, \ldots, n_p \tag{16}$$

By taking $P = \frac{1}{\alpha^2}Q^{-1}$ and $K = YQ^{-1}$, a feedback gain is obtained stabilizing the uncertain system (1) in such a way that the corresponding $\infty$-norm set $\varphi$ is invariant and all the input and state constraints are satisfied.

**Remark 2:** In the case that the system is linear time-invariant (LTI) it can be shown that a necessary and sufficient condition for the optimization algorithm of Theorem 3.1 to be feasible is controllability of the LTI system. If the system is linear time-varying (LTV) then a sufficient condition is that the LTV system is quadratically controllable.

**Remark 3:** In Lee and Kouvaritakis (2000) a method is described for determining an initial invariant set for LTV systems. This procedure has some specific drawbacks when compared to the algorithm of Theorem 3.1:

- For the LTV case the procedure is difficult to use as the proposed invariance condition in Lee and Kouvaritakis (2000) is nonlinear. The method of Theorem 3.1 allows to find an initial

feasible invariant set by solving a convex optimization problem.

- In the LTI case an invariant set can be obtained by first determining a stabilizing feedback matrix $K$ with pole placement and by performing an eigenvalue decomposition of the closed-loop system. The drawback of this method is that no optimization is performed. Therefore, the resulting set is not optimal w.r.t. the volume of the set. In Theorem 3.1 the initial set is obtained as result from an optimization procedure were both the feedback matrix $K$ and $V$ are optimization variables and the volume of the set is maximized. Using this set as starting point in the volume maximization algorithms of subsection 3.2 typically leads to faster convergence and larger invariant sets.

### 3.2 *Volume Maximization*

In Cannon et al. (2003) a nonlinear program is described to determine the maximal volume low-complexity polytope. However, the complexity of the proposed optimization scales exponentially with the state dimension of the problem because the number of constraints as well as the number of unknown variables scale exponentially with the state dimension. Also the proposed iterative scheme solving successive LPs suffers from the same problem. Therefore, in this section a new nonlinear program is proposed that scales much better with the state dimension. This new optimization scheme is mostly based on Farkas' Lemma (see e.g. Blanchini (1999)).

**Lemma 3.2:** *Given two polyhedra* $\varphi_1 = \{x|F_1x \leq g_1\}$ *and* $\varphi_2 = \{x|F_2x \leq g_2\}$ *then* $\varphi_1 \subseteq \varphi_2$ *if and only if there exists a non-negative matrix* $H$ *such that* $HF_1 = F_2$ *and* $Hg_1 \leq g_2$.

In the case of low-complexity polytopes (5) these conditions can be reduced as indicated in the following theorem:

**Lemma 3.3:** *Given two polyhedra* $\varphi_1 = \{x| \left\|V_1x\right\|_\infty \leq 1\}$ *and* $\varphi_2 = \{x| \left\|V_2x\right\|_\infty \leq 1\}$ *then* $\varphi_1 \subseteq \varphi_2$ *if and only if there exist non-negative matrices* $H_1$ *and* $H_2$ *such that the following conditions are satisfied:*

$$(H_1 - H_2)V_1 = V_2 \tag{17}$$

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \leqslant \mathbf{1} \tag{18}$$

*with* $\mathbf{1}$ *a* $n$*-dimensional vector whose elements are all* 1.

**Proof.** The two sets can be rewritten as

$$\varphi_1 = \left\{ x \middle| \begin{bmatrix} V_1 \\ -V_1 \end{bmatrix} x \leqslant \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \right\} \tag{19}$$

$$\varphi_2 = \left\{ x \middle| \begin{bmatrix} V_2 \\ -V_2 \end{bmatrix} x \leqslant \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \right\}. \tag{20}$$

By Farkas' Lemma $\varphi_1 \subseteq \varphi_2$ if and only if the following conditions are satisfied:

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} V_1 \\ -V_1 \end{bmatrix} = \begin{bmatrix} V_2 \\ -V_2 \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \tag{22}$$

with $H_1, H_2, H_3, H_4 \geq 0$. Here the conditions on $H_1, H_2$ are identical to those on $H_3, H_4$ and (21) and (22) are therefore equivalent to (17) and (18).□

In Benzaouia and Burgat (1988), Bitsoris (1988) and Blanchini (1999) invariance conditions are derived for general polytopic sets by use of Farkas' Lemma. Applying these conditions to the low-complexity set (5) combined with Lemma 3.3 leads to the following invariance conditions for low-complexity polytopes:

**Lemma 3.4:** *The set $\varphi = \{x \,|\, \|Vx\|_\infty \leq 1\}$ is invariant for the system $x_{k+1} = A_j x_k + B_j u_k$ under linear state feedback $u_k = Kx_k$ if and only if there exist non-negative matrices $H_{1j}$ and $H_{2j}$ such that the following conditions are satisfied for $j = 1, \ldots, n_p$:*

$$(H_{1j} - H_{2j})V = V(A_j - B_j K) \tag{23}$$

$$\begin{bmatrix} H_{1j} & H_{2j} \end{bmatrix} \begin{bmatrix} \boldsymbol{1} \\ \boldsymbol{1} \end{bmatrix} \leqslant \boldsymbol{1} \tag{24}$$

$$H_{1j}, H_{2j} \geqslant 0 \tag{25}$$

**Proof.** Invariance of $\varphi = \{x \,|\, \|Vx\|_\infty \leq 1\}$ is equivalent to $\varphi \subseteq \left\{x | (\widetilde{A} + \widetilde{B}K)x \in \varphi, \forall (\widetilde{A}, \widetilde{B}) \in \Omega\right\}$, which is equivalent to $\varphi \subseteq \left\{x \,|\, \|V(A_j + B_j K)x\|_\infty \leq 1\right\}$ for $j = 1, \ldots, n_p$. By Lemma 3.3 this condition is satisfied if and only if (23)-(25) hold. □

The following theorem uses these results to determine the maximum volume feasible invariant low-complexity polytope:

**Theorem 3.5 :** *The maximum volume feasible invariant low-complexity polytope is the solution of the nonlinear program, for $j = 1, \ldots, n_p$:*

$$\min_{V, K, H_{1j}, H_{2j}, H_3, \ldots, H_6} \log(|\det(V)|) \tag{26}$$

*subject to the following constraints:*

$$(H_{1j} - H_{2j})V = V(A_j - B_j K) \tag{27}$$

$$\begin{bmatrix} H_{1j} & H_{2j} \end{bmatrix} \begin{bmatrix} \boldsymbol{1} \\ \boldsymbol{1} \end{bmatrix} \leqslant \boldsymbol{1} \tag{28}$$

$$(H_3 - H_4)V = K \tag{29}$$

$$\begin{bmatrix} H_3 & H_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{1} \\ \boldsymbol{1} \end{bmatrix} \leqslant \bar{u}\boldsymbol{1} \tag{30}$$

$$(H_5 - H_6)V = L \tag{31}$$

$$\begin{bmatrix} H_5 & H_6 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \leqslant \mathbf{1} \tag{32}$$

$$H_{1j}, H_{2j}, H_3, H_4, H_5, H_6 \geqslant 0 \tag{33}$$

**Proof.** Consider the following sets:

$$\varphi_1 = \{x | \, \|Vx\|_\infty \leq 1\} \tag{34}$$

$$\varphi_2 = \{x | \, \|Kx\|_\infty \leq \bar{u}\} \tag{35}$$

$$\varphi_3 = \{x | \, \|Lx\|_\infty \leq 1\} \tag{36}$$

$$\varphi_{4j} = \left\{x | \, \|V(A_j - B_j K)x\|_\infty \leq 1\right\} \tag{37}$$

The set $\varphi_1$ is invariant if and only if $\varphi_1 \subseteq \varphi_{4j}$ (Lemma 3.4). This is expressed by the conditions (27) and (28). In order for the feedback $K$ to satisfy the input constraints (4) the set $\varphi_1$ must satisfy $\varphi_1 \subseteq \varphi_2$ which is expressed by the conditions (29) and (30). In order for the set $\varphi_1$ to satisfy the state constraints the set $\varphi_1$ must satisfy $\varphi_1 \subseteq \varphi_3$ which is expressed by the conditions (31) and (32).□

Constraints (29) and (31) of Theorem 3.5 are bilinear constraints. By introducing the transformations $P = V^{-1}$ and $Q = KP$ these constraints can be made linear as given in the following theorem:

**Theorem 3.6 :** *The optimization of (26) subject to (27)-(33) is equivalent to:*

$$\min_{P,Q,H_{1j},H_{2j},H_{3,\ldots,6},j=1,\ldots,n_p} -\log(|\det(P)|) \tag{38}$$

*subject to the following equations for $j = 1, \ldots, n_p$:*

$$P(H_{1j} - H_{2j}) = A_j P + B_j Q \tag{39}$$

$$H_3 - H_4 = Q \tag{40}$$

$$H_5 - H_6 = LP \tag{41}$$

$$H_{1j}, H_{2j}, H_{3,\ldots,6} \geqslant 0 \tag{42}$$

$$\begin{bmatrix} H_{1j} & H_{2j} \\ H_3 & H_4 \\ H_5 & H_6 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{1} \\ \bar{u}\mathbf{1} \\ \mathbf{1} \end{bmatrix} \tag{43}$$

*with*

$$P = V^{-1} \tag{44}$$

*and*

$$Q = KP. \tag{45}$$

*The corresponding stabilizing feedback $K$ can be obtained as*

$$K = QP^{-1} \tag{46}$$

**Proof.** In (Cannon et al. (2003)) it was shown that in order to optimize the volume of the set (5) the objective $\left|\det(\left[\begin{array}{ccc} v_1 & \dots & v_n \end{array}\right])\right|$ needs to be maximized, with $v_{1,\dots,n}$ the primary vertices of the set (5). This objective can be written as $\left|\det(V^{-1}S)\right| = |\det(P)|\,|\det(S)| \sim |\det(P)|$ (where $S$ is a constant non-singular matrix with elements equal to $\pm 1$), which leads to the objective (38). The invariance constraint (27) can easily be re-written as $(H_{1j} - H_{2j})P^{-1} = P^{-1}(A_j + B_j K)$ which can easily be re-written as (39) taking into account that $Q = KP$. A similar approach applied to (29) and (31) leads to conditions (40) and (41).□

**Remark 4 :** Note that the state constraints (41) and input constraints (40) are linear in Theorem 3.6. In Theorem 3.5 these constraints are bilinear. This clearly shows the benefits of Theorem 3.6. However, the invariance conditions (39) are still bilinear and are therefore the reason that the optimization procedure in Theorem 3.6 is nonlinear.

In comparison with the method described in Cannon et al. (2003) this method leads to a significant reduction of the number of variables and constraints. In Cannon et al. (2003) for a state space with dimension $n$ the increase in variables and constraints is of order $n2^n$ while in the method proposed here the increase is of order $n^2$ which means that the optimization scheme described in Theorem 3.6 is scalable in high dimensions. Also note that this optimization scheme is nonlinear due to the cost function and the $n$ bilinear invariance constraints (39). This optimization scheme can be simplified by means of the concept of inverse-positive matrices Berman and Plemmons (1994).

**Definition 3.7:** A matrix is inverse-positive if all the entries of its inverse consists of positive elements.

A matrix $A$ satisfying $A = \alpha I - D$ with $\alpha \geq 0$, $D$ a nonnegative matrix and $\|D\|_\infty \leq \alpha$ is inverse-positive. Note that this is a sufficient condition, not necessary. The following theorem uses this concept in order to generate a sequence of convex programs leading to a sequence of feasible invariant sets with increasing volume.

**Theorem 3.8 :**

*The following iteration for $k = 1, \dots$:*

$$\min_{X,Q,R_1,R_2,H_3,\dots,H_6,a,D} \left(-\log\det\left(P_k X\right)\right) \tag{47}$$

*subject to the following convex constraints for $j = 1, \dots, n_p$:*

$$P_k(R_{1j} - R_{2j}) = A_j P_k X + B_j Q \tag{48}$$

$$H_3 - H_4 = Q \tag{49}$$

$$H_5 - H_6 = L P_k X \tag{50}$$

$$\begin{bmatrix} R_{1j} & R_{2j} \\ H_3 & H_4 \\ H_5 & H_6 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \leqslant \begin{bmatrix} X\mathbf{1} \\ \bar{u}\mathbf{1} \\ \mathbf{1} \end{bmatrix} \tag{51}$$

$$R_{1j}, R_{2j} \geqslant 0 \tag{52}$$

$$X = aI - D \tag{53}$$

$$a \geq 0 \tag{54}$$

$$\|D\|_\infty \leq a \tag{55}$$

$$D \geqslant 0 \tag{56}$$

$$X \succeq 0 \tag{57}$$

$$X = X^T \tag{58}$$

*leads to a sequence of feasible invariant sets* $P_0, P_1, \ldots$ *with volume* $(P_{k+1}) \geq$ *volume* $(P_k)$ *with starting set* $P_0$ *chosen such that* $\det(P_0) > 0$.

**Proof.** Condition (48) follows from condition (39) by assuming $P = P_k X$ and $R_{ij} = X H_{ij}$. The first constraint of condition (51) coincides with the first constraint of condition (43). In order to see this note that the first constraint of (51) can be re-written as

$$R_{1j}1 + R_{2j}1 \leq X1. \tag{59}$$

Due to condition (52) and the inverse positiveness conditions (53)-(56), this can be re-written as

$$X^{-1}\left(R_{1j}1 + R_{2j}1\right) \leq 1. \tag{60}$$

Because $R_{ij} = X H_{ij}$ it follows that this expression is the same as condition (43). Conditions (57) and (58) are necessary to ensure the cost function (47) is convex.

Also note that at iteration $k+1$ the set $P_k$ obtained at iteration $k$ still satisfies all the constraints imposed at iteration $k+1$ for $X = I$ with $I$ the unity matrix. This means that the set $P_k$ is a feasible solution for the optimization at time step $k+1$. Therefore, this iterative sequence generates a sequence of sets satisfying volume $(P_{k+1}) \geq$ volume $(P_k)$. Therefore the maximization of volume through (47) leads to a sequence of solutions corresponding to feasible invariant polytopes of monotonically increasing volume. It follows that the sequence of feasible invariant sets generated as solutions of the convex program of this theorem is guarenteed to converge to a (possibly local) solution of the nonlinear program defined in Theorem 3.5. □

**Remark 5 :** The procedure of Theorem 3.8 needs an initial feasible invariant set $P_0$ as starting

point. For the uncertain linear system (1) such a starting set can be found by means of Algorithm **1**. Also note that each set $P_{k+1}$ is related to the set $P_k$ from the previous iteration through following relationship

$$P_{k+1} = P_k X(k+1) \tag{61}$$

with $X(k+1)$ the solution of Theorem 3.8 at iteration $k+1$.

**Remark 6 :** Note that if $\det(P_0) > 0$ it then follows that $\det(P_1) = \det(P_0 X_1) = \det(P_0)\det(X_1) > 0$ since $\det(X_1) > 0$ because $X_1 \succeq 0$. By induction, it can be shown that each set $P_k$ in the sequence satisfies $\det(P_k) > 0$. Therefore the absolute value in the objective function (38) can be omitted which leads to the convex objective function (47). Also note that if an initial invariant set $V_0 = P_0^{-1}$ is found such that $\det(P_0) < 0$, then by switching two rows of $V_0$ a new value for $P_0$ can be obtained with $det(P_0) > 0$.

## 4 Examples

**Example 1** This example illustrates that the algorithms in this paper can lead to bigger feasible invariant sets than the methods described in Cannon et al. (2003). The example deals with the constrained control problem of a DC electric motor with independent excitation and is taken from Blanchini (1991). The system matrices of the linear continuous-time system are given by:

$$A = \begin{bmatrix} -0.07 & -0.86\,(1+q_1) \\ 0.06\,(1+q_1) & -q_2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{62}$$

The uncertainty is defined by

$$Q = \{\,(q_1, q_2) | -0.2 \leq q_1 \leq 0.2, 0.0085 \leq q_2 \leq 0.5\}\,. \tag{63}$$

Note that this implies that the uncertainty polytope $\Omega$ consists of 4 linear systems that define its vertices. The state constraints are given by

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} \leqslant x_k \leqslant \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{64}$$

and the input constraints by

$$-10 \leq u_k \leq 10. \tag{65}$$

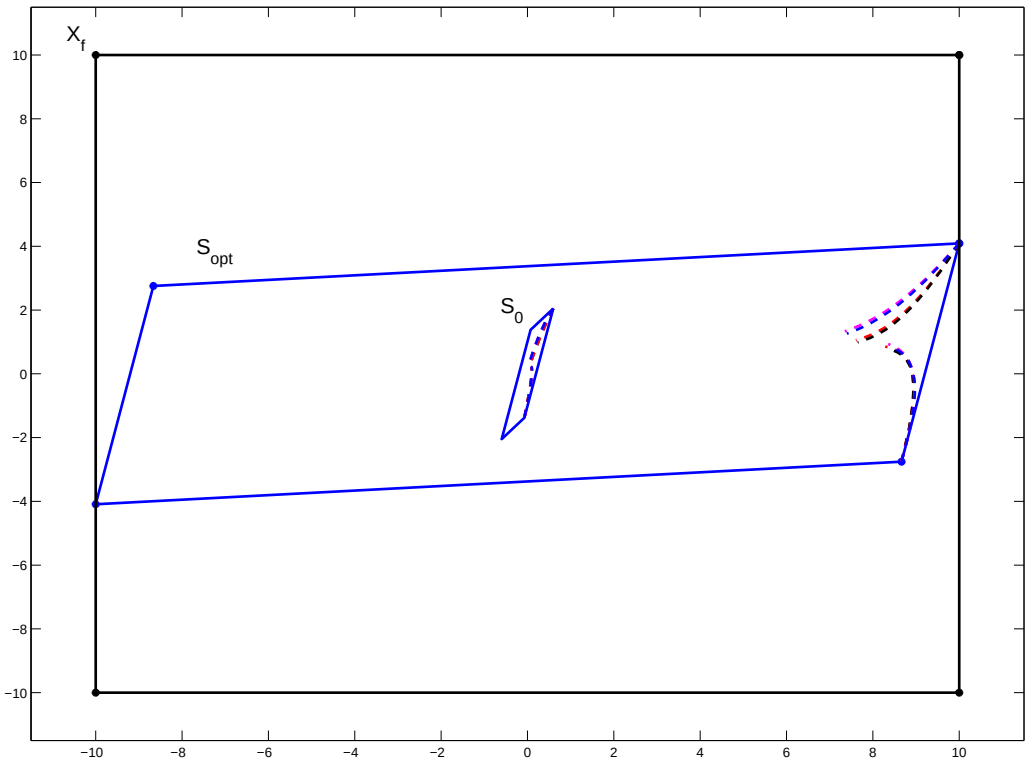


Figure 1. Feasible invariant polytopic sets for Example 1.

This system was discretized by the Euler discretization scheme using a sampling time $T_s = 0.1s$. The methods described in this paper are applied to this discretized system. The set $S_0$ in Figure 1 is the initial invariant feasible set obtained using Algorithm **1**. The primary vertices of this set are given by

$$v_1 = \begin{bmatrix} 0.598 \\ 2.049 \end{bmatrix}, v_2 = \begin{bmatrix} 0.071 \\ -1.380 \end{bmatrix} \tag{66}$$

and it is invariant (see different paths of the vertices in the figure) for the linear feedback gain

$$K = \begin{bmatrix} 15.475 & -7.020 \end{bmatrix}. \tag{67}$$

The set $S_{opt}$ is the resulting set after increasing the volume of $S_0$ with the sequence of convex programs described in Theorem 3.8. The set $S_{opt}$ consists of the primary vertices

$$v_1 = \begin{bmatrix} 10 \\ 4.092 \end{bmatrix}, v_2 = \begin{bmatrix} 8.663 \\ -2.756 \end{bmatrix} \tag{68}$$

and is invariant (see different paths of the vertices in the figure) for the linear feedback gain

$$K = \begin{bmatrix} 0.253 & -2.833 \end{bmatrix}. \tag{69}$$

Table 1. *Optimality and computational load for Example 2.*

| Algorithm | $\lvert det(P)\rvert$ | variables | time(s) |
|---|---|---|---|
| Algorithm **1** | $2.204 \times 10^3$ | 63 | 2.58 |
| Theorem 3.8 | $1.011 \times 10^6$ | 235 | 30.25 |
| Cannon et al. (2003) | $7.056 \times 10^4$ | 16514 | 2410.56 |

Note that the iterative method described in Cannon et al. (2003) is not capable of increasing the volume of the set $S_0$. Therefore, the method described in this work outperforms the one described in Cannon et al. (2003).

**Example 2** This example illustrates the advantage in computation time and memory usage of the methods in this paper compared with the methods of Cannon et al. (2003) due to the reduction in numbers of the optimization variables and constraints. The example consists of the control of the temperature profile of an one-dimensional bar Agudelo et al. (2006). In order to control the temperature profile 3 inputs are present which are the temperature of the bar at 3 locations (see Agudelo et al. (2006)). The model consists of parabolic Partial Differential Equations (PDE's). These PDE's are simulated by discretization in spatial and temporal domain. By dividing the one-dimensional bar into 400 sections (spatial discretization) and taking a sampling time of 1 sec (temporal discretization) a linear system was obtained in Agudelo et al. (2006) consisting of 398 states. In order to reduce this complexity a model reduction was performed by means of Proper Orthogonal Decomposition (POD) leading to following reduced order model:

$$A = \begin{bmatrix} 0.858 & -0.053 & 0.021 & 0.193 & -0.029 & 0.027 \\ -0.053 & 0.716 & 0.039 & 0.051 & 0.179 & 0.162 \\ 0.021 & 0.039 & 0.698 & 0.031 & 0.135 & -0.206 \\ 0.193 & 0.051 & 0.031 & 0.567 & 0.053 & -0.014 \\ -0.029 & 0.179 & 0.135 & 0.053 & 0.658 & -0.031 \\ 0.027 & 0.162 & -0.206 & -0.014 & -0.031 & 0.504 \end{bmatrix}, B = \begin{bmatrix} -0.438 & -1.607 & -0.474 \\ -0.074 & -1.479 & 1.389 \\ -1.647 & 1.011 & 0.204 \\ 0.866 & 1.869 & 0.821 \\ 0.633 & 0.004 & -1.364 \\ -0.997 & 1.185 & -0.554 \end{bmatrix} \tag{70}$$

subject to the following input constraints

$$\overline{u} = [1, 1, 1]^T, \underline{u} = [-1, -1, -1]^T. \tag{71}$$

An initial feasible invariant set is obtained using the LMI of Algorithm **1**. This set is then used as starting point for the optimization schemes of Theorem 3.8 and the iterative procedure of Cannon et al. (2003). The results of the different optimization schemes implemented on a P4-2.8 GHz PC with 3 GB RAM are summarized in table 1. From table 1 it can be seen that the optimization scheme of Theorem 3.8 needs significantly less memory and computation time than the one described in Cannon et al. (2003) and therefore outperforms the method of Cannon et al. (2003). Also note that for a dimension higher than 6 it is not possible to use the method of Cannon et al. (2003) on this platform due to the excessive memory requirements.

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