General polytopic invariant sets

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***Abstract*— In this paper a method is described that allows for the computation of maximum volume feasible invariant sets. The method consists of an iterative sequence of LPs generating a sequence of feasible invariant sets w.r.t. a given stabilizing feedback gain K. Moreover at each iteration it is ensured the volumes of the resulting set increases monotomically untill convergence is achieved. The complexity of the obtained set is handled as a predefined parameter that can be chosen by the user in order to be able to make a trade-off between volume of the set and the number of constraints needed to define the set. The correctness of the algorithm is demonstrated by means of a numerical example.**

I. INTRODUCTION

Positively invariant sets are widely used in MPC for designing terminal constraint sets, also called target sets, as a tool for the guarantee of system closed-loop stability. These target sets are mainly used in dual mode MPC strategies. The main idea is to determine a set in the state space invariant for a certain feedback (usually a linear feedback) with the property that no constraints violation occurs as long as the state remains inside this set. A good overview of invariant sets can be found in the overview paper [4].

In literature two types of convex sets are essentialy used as candidate invariant sets, ellipsoidal and polyhedral sets. In this paper the focus will be on the computation of polyhedral sets. In [5] and [6] symmetrical polyhedral sets are discussed. In this paper the focus will be on polyhedral sets with a more general form because symmetrical polyhedral sets typically use to be very small in presence of assymetrical constraints. A systematic way for constructing such general polyhedral sets for linear time-invariant systems was proposed in [1]. The proposed algorithm constructs an invariant set by adding additional constraints untill an invariant set is obtained. In [2] this method was extended for uncertain systems with polytopic uncertainty. The contributions made in [1] and [2] ensured the computation of an invariant set with maximal volume satisfying all input and state constraints. A disadvantage of the proposed method, however, is that the number of iterations before an invariant set is obtained can be very large. Moreover, the complexity of the resulting set can be very large which can lead to very time consuming MPC algorithms when these sets are used as target sets.

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In this paper an algorithm is proposed that doesn't suffer from these drawbacks. The algorithm consists of an iterative procedure that ensures a feasible invariant set at the end of each iteration. Moreover, the complexity of the resulting set can be chosen beforehand as a predefined tuning parameter that allows the user to make a trade-off between set volume and set complexity.

This paper is organised as follows. In section II the problem is formulated. In section III the concept of the maximal admissable set (MAS) is explained as described in [1] and [2]. Further a method is described that allows to increase the volume of a polytope by optimizing only a single vertex at each iteration. This method is than used to define a sequence of LPs leading to a sequence of invariant feasible LPs with monotonic increasing volume. In section IV this method is applied on a numerical example. Section V gives some conclusions and ideas for future work.

II. PROBLEM FORMULATION

Consider the linear time-invariant system

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$

with $x_k \in \mathbb{R}^{n_x}$ denoting the state of the system at discrete time k and $u_k \in \mathbb{R}^{n_u}$ denoting the input at discrete time k. The inputs and the states are subject to linear constraints

$$u_k \in U = \{u | A_u u \leq b_u\}, k = 0, \ldots, \infty \tag{2}$$

$$x_k \in X = \{x | A_x x \leq b_x\}, k = 0, \ldots, \infty \tag{3}$$

with $0 \in U$ and $0 \in X$. We will assume the system is controllable and a robust stabilizing feedback $u_k = -Kx_k$ is applied to the system. The aim is to find a set S of initial states x_0 for which all corresponding states $x(0), \ldots, x(\infty)$ satisfy the state constraints X and all corresponding inputs $-Kx(0), \ldots, -Kx(\infty)$ satisfy constraints U. The set S is assumed to be polytopic and has a predetermined number of vertices.

Definition 1: (Positive Invariance): Given system (1) then $S \in \mathbb{R}^{n_x}$ is a positive invariant set w.r.t. the stabilizing feedback K if

$$\phi x \in S, \forall x \in S \tag{4}$$

with $\phi = A - BK$.

Definition 2: (Feasibility): An invariant set S for a system (1) is feasible with respect to constraints (2)-(3) if

$$x \in X, -Kx \in U, \forall x \in S. \tag{5}$$

It is clear that if a set is invariant and feasible, all initial states within that set guarantee the future states and inputs satisfy constraints (2)-(3).

The problem tackled in this paper is the following:

Problem 1: Given system (1) with robustly stabilizing feedback K, find a feasible and positive invariant set S with maximal volume and of polytopic form

$$S = \{x \in \mathbb{R}^{n_x} | A_S x \leq b_S\} \tag{6}$$

with $A_S \in \mathbb{R}^{n_p \times n}$, $b_S \in \mathbb{R}^{n_p \times 1}$ with dimension n_p predefined . In the following sections we refer to this problem as **P1**.

III. Results

A. Maximal Admissable set (MAS)

In [1] the concept of MAS is introduced for linear time-invariant systems under linear state feedback. In [2] this concept was extended to linear systems with polytopic uncertainty. Basically the MAS defines the maximum positively invariant feasible polytopic set. Using the MAS as target region for the end state typically leads to MPC algorithms with shorter horizons and therefore less calculation time. In the remainder some important points discussed in [2] are restated.

We first define the following set:

$$S^- = \{x | \phi x \in S\}. \tag{7}$$

S^- can be interpreted as the set of all previous states for which it is guaranteed that the current state lies inside S. The following lemma provides necessary and sufficient conditions for positive invariance of a set.

Lemma 1: A set S is a positive invariant set for the system (1) iff

$$S \subset S^- \tag{8}$$

In [2] this lemma is used to construct an algorithm to determine the MAS. The proposed algorithm starts with a set $S = \{x \in \mathbb{R}^{n_x} | A_{S_0} x \leq b_{S_0}\}$ with

$$A_{S_0} = \begin{bmatrix} A_x \\ -A_u K \end{bmatrix}, b_{S_0} = \begin{bmatrix} b_x \\ b_u \end{bmatrix} \tag{9}$$

and then iteratively adds constraints to S from S_1, S_2, ... with $S_i = \{x | A_{S_i} x \leq b_{S_i}\}$ and

$$A_{S_i} = A_{S_{i-1}} \phi, \quad b_{S_i} = b_{S_{i-1}}. \tag{10}$$

This iterative procedure is executed untill condition (9) is satisfied. It is also shown that if this procedure terminates in a finite number of iterations than the resulting MAS is a polytopic sets. The following theorem describes when the MAS S can be described by a finite number of constraints.

Theorem 1: Considering the following definitions

$$a = \|A_{S_0}\|, \tag{11}$$

$$b_{min} = \min_i b_{S_0(i)}, \tag{12}$$

$$\phi_{\max} = \max \|\phi\|, \tag{13}$$

$$c = \max_x \|x\| \quad s.t. \quad x \in S_0, \tag{14}$$

with $b_{S_0(i)}$ denoting the $i-th$ element of vector b_{S_0} and assuming $\phi_{max} < 1$, then the sets S and $\overline{S_n} \triangleq \cap_{i=0}^{n} S_i$, and n defined as

$$n = \left\lceil \frac{\ln b_{\min} - \ln a - \ln c}{\ln \phi_{\max}} \right\rceil, \tag{15}$$

are identical.

B. Volume Maximization

From Theorem 1 it can be seen that n increases proportional to $\frac{1}{1-\phi_{\max}}$ for values of ϕ_{max} close to 1. So for values of ϕ_{max} very close to 1 it is possible the algorithms proposed in [1] and [2] have to do a lot of iterations before they converge to a solution. Furthermore, the intermediate sets obtained during each iteration are not feasible positive invariant sets. Only the set obtained at the last iteration step of the algorithm is feasible positive invariant. This means that if the algorithm has to be aborted premature, because n is too big, the resulting set cannot be used as target set for a MPC algorithm. Another disadvantage is that although the MAS is the maximum feasible positive invariant set, the number of constraints needed to describe it can be very large. This reflects itself in the MPC algorithm by an increase in the calculation time. In the following a method is described that doesn't suffer from these disadvantages.

As stated in (6) we are looking for a set S with a predefined number of constraints equal to n_p. The purpose of fixing the number of constraints is to avoid the resulting set S is described by a large amount of constraints. The dimension n_p can be seen as a user defined parameter that allows to make a trade-off between volume of the set and calculation time needed to solve the MPC algorithm using the set S as target set. This approach eliminates one of the disadvantages of the MAS. In order to solve **P1** it is also necessary to maximize the volume of S. In fact, the solution of **P1** results into a reduced complexity approximation of the MAS. However, there doesn't exist a general formula for calculating the volumes of general polytopes. One of the techniques to calculate the volume of general polytopes is triangulation [3]. The main idea behind this technique is

to produce a dissection of polytope P into simplices, then compute the volumes of the individual simplices and add them up to find the volume of P. Suppose the n-simplex $T = Co\{v_0, \ldots, v_n\}$ with $v_0, \ldots, v_n$ affinely independent points of $\mathbb{R}^n$ then the volume $V(T)$ of T can be calculated as follows:

$$V(T) = \frac{1}{n!} \left| det(v_1 - v_0, \ldots, v_n - v_0) \right|. \tag{16}$$

Now, assume the general polytopic set S as defined in (6). The set S consists of n_p vertices $\{v_1, \ldots, v_{n_p}\}$. In a n-dimensional space each vertex of the set S has n neighbouring vertices. Consider now a specific vertex v_k with the following neighbouring vertices $\{v_{k+1}, \ldots, v_{k+n}\}$. The vertices $\{v_{k+1}, \ldots, v_{k+n}\}$ define a hyperplane that divide the set S in two sets S_0 and S_k (see Fig. 1). The volume of the set S can then be calculated as

$$V(S) = V(S_0) + V(S_k) \tag{17}$$

because $V(S_0 \cap S_k) = 0$, with $V(S)$ defining the volume of the set S. Remark that if all vertices are kept fix with exception from vertex v_k then $V(S_0)$ is fixed and $V(S)$ only depends of $V(S_k)$ as long as $V(S_0 \cap S_k) = 0$. So by increasing $V(S_k)$ the volume of S is also increased. Now note that the set S_k consists of the vertices $\{v_k, \ldots, v_{k+n}\}$ and that the set represents a n-simplex. This means that the volume of S_k can be calculated by (16). Because all vertices are considered fix with exception from vertex v_k this means that (16) turns into a linear function of v_k. This means that an optimization procedure increasing the volume of the set S by optimizing over a single vertex can be performed very efficiently as an LP. Therefore, in the remainder of the paper this single vertex optimization approach will be used to determine maximum volume set S defined by (6).

Remark that if an optimization over a single vertex is done in order to define an iterative sequence with increasing volume condition $V(S_0 \cap S_k) = 0$ must always be satisfied. If this is not satisfied it is possible that the volume decreases. Another important condition that must be satisfied is that after each volume optimization the resulting set is convex. This can be achieved by adding additional constraints. In order to see this in Fig. 1 an example is plotted for illustrative purposes. The initial set S is depicted as the polytope with the solid line. The set is divided in two subsets S_0 and S_k. The dotted line dividing both sets is the hyperplane containing the neighbouring vertices of v_k. It can be seen from the figure that restricting the search space for vertex v_k

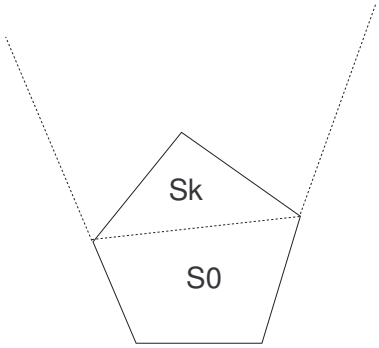

Fig. 1. Volume Maximization Strategy

to the proper half space created by this hyperplane ensures $V(S_0 \cap S_k) = 0$. In Fig. 1 two other dotted constraints are depicted. These constraints are necessary in order for the resulting set to be convex after the optimization. Remark that these constraints consist of the constraint set where all neighbouring vertices of v_k are active, but where v_k is inactive. So the dotted constraints in the figure define a search space for v_k that ensure optimizing $V(S_k)$ leads to an increase in the volume of S and also ensure convexity of the set. In the sequel this search space will be denoted by $\{x | V_c^k x \leq 1\}$. Remark that the index k is added to emphasize that these constraints are different for each vertex.

C. Feasible Positive Invariant Set

Condition (8) ensures the set S is invariant for system (1) w.r.t. the stabilizing feedback K. In [4] it is shown that in case of a polytopic set this condition can be ensured by only imposing invariance constraints on the vertices of the polytope. Following conditions are necessary and sufficient conditions for ensuring feasibility and invariance of the polytopic set S (6).

Theorem 2: A set S (6) with vertices $\{v_1, \ldots, v_{n_p}\}$ is positive invariant w.r.t. feedback K and feasible w.r.t. the constraints (2)-(3) iff $\forall i = 1, \ldots, n_p$

$$\phi v_i = \sum_{j=1}^{n_p} \lambda_j^i v_j \tag{18}$$

$$\sum_{j=1}^{n_p} \lambda_j^i = 1 \tag{19}$$

$$\lambda_j^i \geq 0 \tag{20}$$

$$\begin{bmatrix} A_x \\ -A_u K \end{bmatrix} v_i \leq \begin{bmatrix} b_x \\ b_u \end{bmatrix}. \tag{21}$$

Proof. Conditions (18),(19) and (20) state that $\phi v_i \in S$, which corresponds to invariance conditions imposed on the vertices of the set S. Condition (21) corresponds to a feasibility condition on all the vertices of the set S. Because the set S is convex this ensures each state in the set S satisfies the constraints (2)-(3).□

Combining the results from Theorem 2 with the volume maximization discussed in subsection III-B the following iterative procedure can be used in order to find the maximal volume feasible invariant polytopic set S defined by (6).

Theorem 3: The maximum volume invariant set S defined by (6) can be computed by solving the following LP successively for the individual vertices $v_k, k = 1, \ldots, n_p$:

$$\max_{Q, v_k} |\det([v_k - v_{k+1}, \cdots, v_{k+n} - v_{k+1}])| \tag{22}$$

$$\begin{aligned}
\phi \left[\begin{array}{ccc} v_1 & \ldots & v_{n_p} \end{array} \right] &= \left[\begin{array}{ccc} v_1 & \ldots & v_{n_p} \end{array} \right] Q && (23)\\
\mathbf{1}^T |Q e_j| &= 1 && (24)\\
Q &\geqslant 0 && (25)\\
\left[\begin{array}{c} A_x \\ -A_u K \end{array} \right] v_k &\leqslant \left[\begin{array}{c} b_x \\ b_u \end{array} \right] && (26)\\
e_k^T Q &= e_k^T Q^0 && (27)\\
v_j &= v_j^0, j \neq k && (28)\\
V_c^k v_k &\leq 1 && (29)\\
\{v_{k+1}, \ldots, v_{k+n}\} &\in N(v_k) && (30)
\end{aligned}$$

with $\left\{v_1^0, \ldots, v_{n_p}^0\right\}$, Q^0 defined by the previously computed LP solution and N_k the set containing the neighbouring vertices of v_k.

Proof. Cost function (22) ensures the volume of the n-simplex defined by the vertex v_k together with his neighbouring vertices $\{v_{k+1}, \ldots, v_{k+n}\}$ is increased. Because all the vertices $\{v_{k+1}, \ldots, v_{k+n}\}$ are fixed and v_k variable the cost function is a linear function in v_k. Conditions (23)-(24)-(25) ensure the conditions of Theorem 2 are satisfied meaning the set is feasible and positive invariant. Condition (29) ensures the resulting set S is convex. Conditions (26)-(27) ensure the cost function and the constraints are all linear. Also remark that the solution from the previous LP optimization is also a solution of the current LP. Therefore, this algorithm leads to a sequence of feasible positive invariant polytopes with monotonic increasing volumes. It follows that the sequence generated by the LP solutions is guaranteed to converge to a (possibly local) solution.□

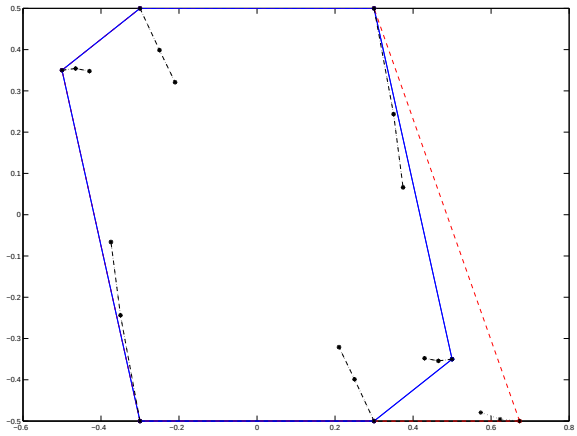

Fig. 2. Optimization of a single vertex. A vertex of the solid polytope is optimized. The new polytope is indicated by the dashed line. The resulting set is also invariant. The trajectory of the vertices are plotted as a dotted line.

Remark 1: Condition (23) imposes restrictions on all the vertices of the set S. But in fact it is sufficient to only impose them on the vertices $\{v_k, \ldots, v_{k+n}\}$. Because the other vertices are fixed , the constraints where they are active don't change and the static feedback gain K is fixed these vertices automatically satisfy the feasibility and invariance conditions.

IV. EXAMPLE

Our illustrative example concerns the following linear discrete time-system:

$$x_{k+1} = \left[\begin{array}{cc} 1 & 0.1 \\ 0 & 1 \end{array} \right] x_k + \left[\begin{array}{c} 0 \\ 1 \end{array} \right] u_k \tag{31}$$

with state constraints $-5 \leq x \leq 5$ and input constraints $-4 \leq u \leq 4$. The stabilizing feedback gain K is

$$K = \left[\begin{array}{cc} 0.2585 & 0.3575 \end{array} \right]. \tag{32}$$

We are looking for a set S of form (6) with predefined dimension $n_p = 6$. This means the set S will have 6 vertices. In Fig. 2 the optimization step of a single vertex is plotted. The polytope in the solid line consists of the following vertices

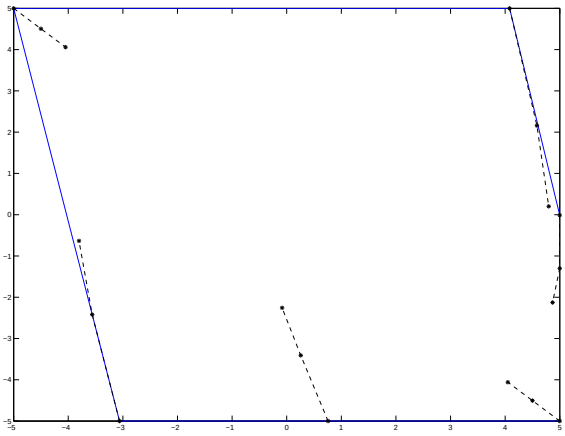

Fig. 3. The maximal volume feasible invariant set found after applying the iterative algorithm of Theorem 3.

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 & -0.3 & -0.5 & -0.3 & 0.3 \\ -0.35 & 0.5 & 0.5 & 0.35 & -0.5 & -0.5 \end{bmatrix}. \quad (33)$$

Vertex v_1 is optimized and a new vertex is obtained indicated by the crossing of the 2 dashed lines. It can be seen that by replacing v_1 with this new vertex the volume of the resulting polytope is increased. The new polytope is convex and invariant. In Fig. 3 the maximum volume polytope is displayed obtained after successively solving the LP sequences discussed in Theorem 3. The resulting set consists of the following vertices

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 4.08 & -5 & -3.06 & 0.76 \\ -5 & -0.01 & 5 & 5 & -5 & -5 \end{bmatrix}. \quad (34)$$

From the trajectories displayed in the figure it can be seen that the resulting set is invariant. Also note that the state and input constraints are satisfied.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

In this paper a method was described to calculate a maximal volume feasible positive invariant set for a linear time-variant system subject to a given stabilizing state feedback gain K. The described method optimizes the volume of the set by optimizing over a single vertex. The complexity of the resulting set was predefined by means of a tuning parameter n_p. This offers the possibility for the user to make a trade-off between volume of the set and the complexity of the set. The method also ensures that at the end of each LP optimization a feasible invariant set is obtained. So aborting the method before convergence of the LP sequences still gives raise to a feasible invariant set that can be used as target set for a MPC controller. An example showed that the method was capable of determining a maximum volume invariant set that satisfied the state and input constraints.

B. Future Works

Future works can be done by extending the algorithm to linear systems with polytopic uncertainty. Furthermore, in this work the static feedback gain K was assumed to be fixed. This can be restrictive with respect to the size of the resulting maximum volume invariant set. Future work could deal with handling K as an extra optimization variable in the algorithm.

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