Time-autonomy and time-controllability of $\mathcal{W}_{\mathfrak{s}}-$behaviours

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Abstract. This paper studies the notions of time-autonomy and (exact) time-controllability of behaviours with respect to a certain mild space which while being not too general encompasses natural partial differential equation solutions and excludes certain pathological ones. First we prove an iff test for time-autonomy. We then prove a sufficient condition for the time-controllability of mixed representations and give a necessary condition for the time-controllability of a kernel representation.

1 Introduction

The behavioural theory for systems described by a set of linear constant coefficient partial differential equations has been a challenging and fruitful area of research for quite some time (see for instance Pillai and Shankar [10], Oberst [8] and Wood et al. [9]). An excellent elementary introduction to the behavioural theory in the $1-$D case (corresponding to systems described by a bunch of linear constant coefficient ordinary differential equations) can be found in Polderman and Willems [11].

In [10], [8] and [9], the behaviours arising from systems of partial differential equations are studied in a general setting in which the time-axis does not play a distinguished role in the formulation of the definitions pertinent to control theory. Since in the study of systems with "dynamics", it is useful to give special importance to time in defining system theoretic concepts, recent attempts have been made in this direction (see for example Cotroneo and Sasane [3] and Sasane et al. [12]). The formulation of definitions with special emphasis on the time-axis is straightforward, since they can be seen quite easily as extensions of the pertinent definitions in the $1-$D case. However, the algebraic characterization of the properties of the behaviour, such as time-autonomy and time-controllability, turn out to be quite involved. In [12], an iff test for time-autonomy is given for the behaviour of a single scalar partial differential equation, in terms of the polynomial describing the behaviour. There the behaviour was assumed to comprise distributional solutions satisfying the partial differential equation. But, it was demonstrated that this is not a satisfactory treatment in the case of the diffusion equation

$$\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial^2 x}\right] w = 0,$$

in which case the space of distributional solutions also includes some "unphysical" solutions (such as ones which grow faster than Me^{Ax^2} as $x \to \pm\infty$, where M and A are positive). Thus, in the case of $\mathtt{n}-$D dynamical systems time-autonomy is sensitive to the underlying solution space and so the studies in [3] and [12] lead to diverse results.

In this paper, we work in a rather well-behaved space and study time-autonomy and time-controllability in this space (denoted by $\mathcal{W}_{\mathfrak{s}}$ in this paper). We claim that this is a mild space of solutions in which a lot of physical situations can be modelled. Furthermore, we consider only the $2-$D case, that is, when there is only one spatial axis. We derive a completely algebraic test on the polynomial that characterizes the time-autonomy of the corresponding behaviour.

We also study time-controllability, and give a few partial results in this direction, but an iff test for time-controllability remains an open problem. Thus from the systems and control point of view, the present paper can be seen as one that provides a motivation to study the space $\mathcal{W}_{\mathfrak{s}}$ in detail, and which paves the way to obtaining the elusive iff test for time-controllability. Although the traditional treatment of distributed parameter systems (in which one views them as an ordinary differential equation with an infinite-dimensional Hilbert space as the state-space) is quite successful, the study in the present paper will have its advantages, since it gives tests which are *algebraic* in nature (and hence computationally easy) for properties of the sets of trajectories (such as time-autonomy and time-controllability).

There is another motivation for this paper. There are some natural questions that arise here that might be missed from a purely mathematical point of view of the study of partial differential equations. For example, the theorem on time-autonomy in section could be interesting in studies on partial differential equations, since it gives an necessary algebraic test for the existence of nontrivial null solutions (see Hörmander [6]) in the space $\mathcal{W}_{\mathfrak{s}}$ and it gives a uniqueness test for global solutions with a specified action on one side of a characteristic half-plane. Also compare Theorem 3.1 with Corollary 8.6.9 on page 312 of Hörmander [7].

This paper is organized as follows. We first give the requisite preliminaries in the second section. In section 3 we succeed in obtaining an iff test characterizing the time-autonomy property of the $\mathcal{W}_{\mathfrak{s}}-$behaviour corresponding to a polynomial. In section 4 we will give a sufficient condition for the time-controllability of the $\mathcal{W}_{\mathfrak{s}}-$behaviour described by a simple mixed representation and a necessary condition for the time-controllability of the $\mathcal{W}_{\mathfrak{s}}-$behaviour described by a kernel representation. While in this section we give the main ingredients which could prove an iff test for time-controllability, however, owing to rather hard technical difficulties, we fall short of our intended goal. In the final section we elaborate on this and propose a few open questions.

2 Preliminaries

In this section, we recall some facts about distributions and dynamical systems. We shall tailor our choice of definitions and theorems to the needs of the remainder of this paper. We will stick to the following standard notation:

1. $\mathcal{D}(\mathbb{R})$ denotes the space of test functions, that is the set of compactly supported infinitely many times differentiable functions,
2. $\mathcal{D}'(\mathbb{R})$ denotes the space of distributions on $\mathbb{R}$,
3. $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of smooth functions,
4. $\mathcal{S}'(\mathbb{R})$ denotes the space of tempered distributions,
5. $\mathcal{F}$ and $\overline{\mathcal{F}}$ denote the operations of Fourier transformation and conjugate Fourier transformation, respectively.

Table 1 gives the distinction between the notation used for space and time.

	variable	differential operator	polynomial indeterminate	frequency domain (Fourier transform)
space	x	$\partial/\partial x$	η	y
time	t	$\partial/\partial t$	ξ	

Table 1: Notation.

This paper concerns $2-D$ *dynamical systems* $\Sigma = \left(\mathbb{R}^2, \mathbb{C}^{\mathbb{w}}, \mathfrak{B}\right)$, where $\mathbb{C}^{\mathbb{w}}$ is called the *signal space* and $\mathfrak{B} \subset \mathcal{D}'\left(\mathbb{R}^2\right)$ is called the *behaviour* of the system Σ (see for example Pillai and Shankar [10]).

Consider the polynomial matrix

$$R = \begin{bmatrix} r_{11} & \dots & r_{1\mathtt{w}} \\ \vdots & & \vdots \\ r_{\mathtt{g}1} & \dots & r_{\mathtt{g}\mathtt{w}} \end{bmatrix} \in \mathbb{C}[\eta,\xi]^{\mathtt{g}\times\mathtt{w}},$$

and consider each row of R as an element of the free module $\mathbb{C}[\eta,\xi]^{\mathtt{w}}$. Let $\langle R\rangle$ denote the $\mathbb{C}[\eta,\xi]-$submodule of $\mathbb{C}[\eta,\xi]^{\mathtt{w}}$ generated by the rows of the polynomial matrix R. The polynomial matrix R gives rise to a map $D_R : \mathcal{W}^{\mathtt{w}} \to \mathcal{D}'(\mathbb{R}^2)^{\mathtt{g}}$, where $\mathcal{W} \subset \mathcal{D}'\left(\mathbb{R}^2\right)$, which acts as follows:

$$D_R \begin{bmatrix} w_1 \\ \vdots \\ w_{\mathtt{w}} \end{bmatrix} = \begin{bmatrix} \sum_{\mathtt{k}=1}^{\mathtt{w}} r_{1\mathtt{k}}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right) w_{\mathtt{k}} \\ \vdots \\ \sum_{\mathtt{k}=1}^{\mathtt{w}} r_{\mathtt{g}\mathtt{k}}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right) w_{\mathtt{k}} \end{bmatrix}.$$

The $\mathcal{W}-$behaviour (described by a kernel representation) given by $R \in \mathbb{C}[\eta,\xi]^{\mathtt{g}\times\mathtt{w}}$, is defined to be $\mathfrak{B}^R_{\mathcal{W}} = \{w \in \mathcal{W}^{\mathtt{w}} \mid D_R w = 0\}$. The $\mathcal{W}-$behaviour (described by a mixed representation) given by (R,M), where $R \in \mathbb{C}[\eta,\xi]^{\mathtt{g}\times\mathtt{w}}$ and $M \in \mathbb{C}[\eta,\xi]^{\mathtt{g}\times\mathtt{l}}$ is defined to be

$$\mathfrak{B}^{(R,M)}_{\mathcal{W}} = \left\{w \in \mathcal{W}^{\mathtt{w}} \mid \text{there exists } \ l \in \mathcal{W}^{\mathtt{l}} \ \text{ such that } \ D_R w = D_M l\right\}.$$

Throughout this paper, the term "$\mathcal{W}-$behaviour" will refer to the $\mathcal{W}-$behaviour described by either a kernel representation or a mixed representation as defined above. Given a $\mathcal{W}-$behaviour, say $\mathfrak{B}_{\mathcal{W}}$, define

$$\mathcal{R}_{\mathfrak{B}_{\mathcal{W}}} = \{r \in \mathbb{C}[\eta,\xi]^{\mathtt{w}} \mid D_r w = 0 \ \text{ for all } \ w \in \mathfrak{B}_{\mathcal{W}}\}.$$

It was shown in Oberst [8] that given any $R \in \mathbb{C}[\eta,\xi]^{\mathtt{g}\times\mathtt{w}}$, $\mathcal{R}_{\mathfrak{B}^R_{\mathcal{W}}} = \langle R\rangle$, if $\mathcal{W}$ is $\mathcal{C}^\infty\left(\mathbb{R}^2,\mathbb{C}\right)$ or $\mathcal{D}'\left(\mathbb{R}^2\right)$. If $\mathcal{W} = \mathcal{D}(\mathbb{R}^2)$ or $\mathcal{S}(\mathbb{R}^2)$, then the equality $\mathcal{R}_{\mathfrak{B}^R_{\mathcal{W}}} = \langle R\rangle$ does not hold in general. If $\mathcal{R}_{\mathfrak{B}^R_{\mathcal{W}}} = \langle R\rangle$, then the module $\langle R\rangle$ is said to be *Willems with respect to* $\mathcal{W}$. This terminology was used earlier for example in Pillai and Shankar [10].
Let $\mathcal{W} \subset \mathcal{D}'\left(\mathbb{R}^2\right)$. A $\mathcal{W}-$behaviour, say $\mathfrak{B}_{\mathcal{W}}$, is said to be *time-autonomous* if

$$\left[\begin{array}{c} w \in \mathfrak{B}_{\mathcal{W}} \ \text{ and} \\ \langle w,\varphi\rangle = 0 \ \text{ for all } \ \varphi \in \mathcal{D}\left(\mathbb{R}^2\right) \ \text{ with } \operatorname{supp}(\varphi) \subset \mathbb{R}\times(-\infty,0) \end{array}\right] \Rightarrow [w = 0].$$

A $\mathcal{W}-$behaviour, say $\mathfrak{B}_{\mathcal{W}}$, is said to be *time-controllable* if for any w_1 and w_2 in $\mathfrak{B}_{\mathcal{W}}$, there exists a $w \in \mathfrak{B}_{\mathcal{W}}$ and a $\tau \geq 0$ such that

$$\langle w,\varphi\rangle = \begin{cases} \langle w_1,\varphi\rangle & \text{for all } \ \varphi \in \mathcal{D}\left(\mathbb{R}^2\right) \ \text{ with } \ \operatorname{supp}(\varphi) \subset \mathbb{R}\times(-\infty,0) \\ \langle \sigma_\tau w_2,\varphi\rangle & \text{for all } \ \varphi \in \mathcal{D}\left(\mathbb{R}^2\right) \ \text{ with } \ \operatorname{supp}(\varphi) \subset \mathbb{R}\times(\tau,\infty) \end{cases},$$

where σ_τ denotes translation by $(0,\tau) \in \mathbb{R}^2$:

$$\langle \sigma_\tau w_2,\varphi\rangle = \langle w,\varphi(\bullet,\bullet+\tau)\rangle \ \text{ for all } \ \varphi \in \mathcal{D}\left(\mathbb{R}^2\right).$$

w is then said to *concatenate* w_1 and w_2.

If $\mathcal{V}_1$ and $\mathcal{V}_2$ are topological vector spaces, then we denote by $\mathcal{L}\left(\mathcal{V}_1,\mathcal{V}_2\right)$ the space of continuous linear maps from $\mathcal{V}_1$ to $\mathcal{V}_2$. For example, $\mathcal{L}(\mathcal{D}(\mathbb{R}),\mathcal{D}'(\mathbb{R}))$ denotes the space of continuous linear maps from $\mathcal{D}(\mathbb{R})$ to $\mathcal{D}'(\mathbb{R})$. For a short introduction about vector-valued distributions, we refer the reader to Carroll [1]. If $T \in \mathcal{D}'\left(\mathbb{R}^2\right)$, then one can associate a continuous linear map $\iota T : \mathcal{D}(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ as follows: $\langle(\iota T)(\varphi),\psi\rangle = \langle T,\psi\otimes\varphi\rangle$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R})$. We recall below the Schwartz kernel theorem (see for instance page 128, Theorem 5.2.1, Hörmander [7]).

Lemma 2.1 (The Schwartz kernel theorem.) *The map $T \mapsto \iota T$ is an isomorphism from $\mathcal{D}'\left(\mathbb{R}^2\right)$ onto $\mathcal{L}\left(\mathcal{D}(\mathbb{R}),\mathcal{D}'\left(\mathbb{R}\right)\right)$.*

We note that $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R})) \subset \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{D}'(\mathbb{R}))$ and $\iota^{-1}\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R})) \subset \mathcal{D}'(\mathbb{R}^2)$. In the sequel, we will denote the space $\iota^{-1}\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ by $\mathcal{W}_{\mathtt{s}}$, and throughout this paper, we will study the behavioural trajectories that lie in this space $\mathcal{W}_{\mathtt{s}}$. Roughly speaking, one can think of $\mathcal{W}_{\mathtt{s}}$ to comprise those maps for which if one freezes a time-instant, then the resulting map (along the $x-$axis), is in $\mathcal{S}'(\mathbb{R})$. So a "wild" behaviour in the spatial direction as $x \to \pm\infty$ is ruled out.

We will now recall a few well-known technical results which we will use in the subsequent sections. For more details we refer the reader to Hörmander [7] and Schwartz [13].

Lemma 2.2 (The decomposition lemma.) *If φ is a test function in $\mathcal{D}(\mathbb{R})$ with supp(φ) contained in the union of two open sets $\mathcal{O}_1$ and $\mathcal{O}_2$, then there exist φ_1 and φ_2 in $\mathcal{D}(\mathbb{R})$ such that $\varphi = \varphi_1 + \varphi_2$, with supp$(\varphi_1) \subset \mathcal{O}_1$, and supp$(\varphi_2) \subset \mathcal{O}_2$.*

Lemma 2.3 (The approximation lemma.) *For every $\Phi \in \mathcal{D}\left(\mathbb{R}^2\right)$, there exists a sequence $\{\Phi_{\mathtt{n}}\}$ of functions that are finite sums of direct products of test functions (that is, $\sum_{\mathtt{k}} \psi_{\mathtt{k}}(x)\varphi_{\mathtt{k}}(t)$, where $\psi_{\mathtt{k}}$ and $\varphi_{\mathtt{k}}$ belong to $\mathcal{D}(\mathbb{R})$) such that $\Phi_{\mathtt{n}} \to \Phi$ in $\mathcal{D}\left(\mathbb{R}^2\right)$.*

We will now prove the following lemma on supports which we will use later.

Lemma 2.4 *If*

1. *$w = \sum_{\mathtt{j}=1}^{\mathtt{J}} S_{\mathtt{j}} \otimes T_{\mathtt{j}}$, where $S_{\mathtt{j}} \in \mathcal{D}'(\mathbb{R})$ and $T_{\mathtt{j}} \in \mathcal{D}'(\Omega)$, $\mathtt{j} \in \{1, \ldots, \mathtt{J}\}$, where $\Omega \subset \mathbb{R}$ is open,*
2. *$\{S_1, \ldots, S_{\mathtt{J}}\}$ is independent and*
3. *supp$(w) \subset \mathbb{R} \times X$, where $X \subset \Omega$ and $\Omega \setminus X$ is open in $\mathbb{R}$ and not empty,*

then $\cup_{\mathtt{j}=1}^{\mathtt{J}}$ supp $\left(T_{\mathtt{j}}\right) \subset X$.

Proof Suppose that for some $\mathtt{h} \in \{1, \ldots, \mathtt{J}\}$, supp $(T_{\mathtt{h}}) \not\subset X$. Thus there exists a $\varphi \in \mathcal{D}(\Omega)$ such that supp$(\varphi) \subset \Omega \setminus X$ and $\langle T_{\mathtt{h}}, \varphi \rangle \neq 0$. Hence the sum

$$\sum_{\mathtt{j}=1}^{\mathtt{J}} \langle T_{\mathtt{j}}, \varphi \rangle \, S_{\mathtt{j}}$$

is nonzero (for otherwise, it would contradict the independence of $\{S_1, \ldots, S_{\mathtt{J}}\}$). So there exists a $\psi \in \mathcal{D}(\mathbb{R})$ such that

$$\sum_{\mathtt{j}=1}^{\mathtt{J}} \langle T_{\mathtt{j}}, \varphi \rangle \, \langle S_{\mathtt{j}}, \psi \rangle \neq 0,$$

that is, $\langle w, \psi \otimes \varphi \rangle \neq 0$. However, since supp$(\psi \otimes \varphi) \subset \mathbb{R} \times (\Omega \setminus X)$, this is a contradiction to the fact that supp$(w) \subset \mathbb{R} \times X$. ■

Finally we prove the following diluted version of the fundamental principle for $\mathcal{W}_{\mathtt{s}}$, which will be used later in the proof of Theorem 4.1.

Theorem 2.5 *If $l \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ and $p \in \mathbb{C}[\eta] \setminus 0$, then there exists a $w \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ such that $D_p w = l$.*

Remark: If $w \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$, then $\frac{d}{dt}w \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ and $\frac{\partial}{\partial x}w \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ are defined as follows:

$$\begin{aligned} \left\langle \left(\frac{d}{dt}w\right)(\varphi), \psi \right\rangle &= -\left\langle w\left(\frac{d}{dt}\varphi\right), \psi \right\rangle, \\ \left\langle \left(\frac{\partial}{\partial x}w\right)(\varphi), \psi \right\rangle &= -\left\langle w(\varphi), \frac{\partial}{\partial x}\psi \right\rangle, \end{aligned}$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$ and all $\psi \in \mathcal{S}(\mathbb{R})$. Thus in the above statement of Theorem 2.5, by $D_p w$ we mean

$$p\left(\frac{\partial}{\partial x}, \frac{d}{dt}\right) w.$$

We also note that $D_p(\iota w) = \iota\,(D_p w)$.

Proof First we will solve the following problem of division: given a nonzero $p_* \in \mathbb{C}[y]$ and a $l_* \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$, find a $w_* \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ such that $p_* w_* = l_*$. We will first consider the case when $p_* = y - \lambda$.

`Case` $1°$: $\lambda \in \mathbb{C} \setminus \mathbb{R}$. It can be checked that if $\psi \in \mathcal{S}(\mathbb{R})$, then $\frac{1}{\bullet - \lambda}\psi(\bullet) \in \mathcal{S}(\mathbb{R})$, and the map $\psi \mapsto \frac{1}{\bullet - \lambda}\psi(\bullet) : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is continuous. Define

$$\langle w_*(\varphi), \psi \rangle = \left\langle l_*(\varphi), \frac{1}{\bullet - \lambda}\psi(\bullet) \right\rangle.$$

Then it is easy to check that $w_* \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ and $(y - a)w_* = l_*$.

`Case` $2°$: $\lambda \in \mathbb{R}$. We can reduce this to the case $\lambda = 0$ by means of a translation. Let $\psi_0 \in \mathcal{S}(\mathbb{R})$ be such that $\psi_0(0) = 1$. Then it can be shown that for every $\psi \in \mathcal{S}(\mathbb{R})$, there exists a unique function $\psi_1 \in \mathcal{S}(\mathbb{R})$ and a $\mu \in \mathbb{C}$ such that

$$\psi(y) = \mu\psi_0(y) + y\psi_1(y) \text{ for all } y \in \mathbb{R}.$$

Indeed, let $\chi(y) = \psi(y) - \mu\psi_0(y)$, with $\mu = \psi(0)$, and let $\psi_1(y) = \int_0^1 \chi'(ry)dr$. Also, the map $\psi \mapsto \psi_1$ is linear and continuous from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$. Defining

$$\langle w_*(\varphi), \psi \rangle = \langle l_*(\varphi), \psi_1 \rangle,$$

we see that $w_* \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ and $y w_* = l_*$.

That the aforementioned problem of division is solvable for all nonzero polynomials is now seen easily by employing induction on the degree of p_*.

We now return to our original problem of finding a w satisfying $D_p w = l$. For $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$, define

$$\left\langle l^{\mathsf{f}}(\varphi), \psi \right\rangle = \langle \mathcal{F}\left[l(\varphi)\right], \psi \rangle \text{ and } \left\langle l^{\overline{\mathsf{f}}}(\varphi), \psi \right\rangle = \langle \overline{\mathcal{F}}\left[l(\varphi)\right], \psi \rangle.$$

Then l^{f} and $l^{\overline{\mathsf{f}}}$ are in $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$. Now given a l, first find a w_* such that $p(2\pi i y)w_* = l^{\mathsf{f}}$. Next define $w = w_*^{\overline{\mathsf{f}}}$. We have

$$\begin{aligned}
\langle D_p w(\varphi), \psi \rangle &= \left\langle D_p w_*^{\overline{\mathsf{f}}}(\varphi), \psi \right\rangle \\
&= \left\langle D_p w_*^{\overline{\mathsf{f}}}(\varphi), \mathcal{F}\overline{\mathcal{F}}\psi \right\rangle \\
&= \left\langle \mathcal{F}\left[D_p w_*^{\overline{\mathsf{f}}}(\varphi)\right], \overline{\mathcal{F}}\psi \right\rangle \\
&= \left\langle p(2\pi i y)\mathcal{F}\left[w_*^{\overline{\mathsf{f}}}(\varphi)\right], \overline{\mathcal{F}}\psi \right\rangle \\
&= \left\langle w_*^{\overline{\mathsf{f}}}(\varphi), \mathcal{F}\left[p(2\pi i y)\overline{\mathcal{F}}\psi\right] \right\rangle \\
&= \langle \overline{\mathcal{F}}\left[w_*(\varphi)\right], \mathcal{F}\left[p(2\pi i y)\overline{\mathcal{F}}\psi\right] \rangle \\
&= \langle w_*(\varphi), p(2\pi i y)\overline{\mathcal{F}}\psi \rangle \\
&= \langle p(2\pi i y)w_*(\varphi), \overline{\mathcal{F}}\psi \rangle \\
&= \langle l^{\mathsf{f}}(\varphi), \overline{\mathcal{F}}\psi \rangle \\
&= \langle \mathcal{F}\left[l(\varphi)\right], \overline{\mathcal{F}}\psi \rangle \\
&= \langle l(\varphi), \psi \rangle,
\end{aligned}$$

and this completes the proof. ■

Remarks:

1. In fact the above Theorem 2.5 also holds for polynomials $p \in \mathbb{C}[\eta_1, \ldots, \eta_m] \setminus 0$: that is, if $l \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^m))$ and $p \in \mathbb{C}[\eta_1, \ldots, \eta_m] \setminus 0$, then there exists a $w \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^m))$ such that $D_p w = l$. The proof is essentially the same once we have the following result:

$$\begin{array}{l} \text{If } p \in \mathbb{C}[\eta_1, \ldots, \eta_m] \setminus 0, \text{ then the multiplication mapping} \\ f \mapsto pf : \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^m) \quad \text{has a continuous inverse.} \end{array} \tag{1}$$

   (1) is precisely Theorem 1 on page 556 of Hörmander [5].

2. Although we do not use it in this paper, Theorem 2.5 also holds for nonzero polynomials purely in ξ. We prove it below:

Theorem 2.6 *If $l \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ and $p \in \mathbb{C}[\xi] \setminus 0$, then there exists a $w \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ such that $D_p w = l$.*

Proof 1. First we assume that $p = \xi - \lambda$, for some $\lambda \in \mathbb{C}$. It is sufficient to prove the result for $\lambda = 0$. Indeed suppose that given any l there exists a w such that

$$\frac{d}{dt} w = l.$$

Then given a l_*, we first find a w_0 such that

$$\frac{d}{dt} w_0 = e^{-\lambda t} l_*.$$

Note here that if $\alpha \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ (for example if $\alpha(t) = e^{-\lambda t}$ as above) and $w \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$, then $\alpha w \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ is defined by $\langle (\alpha w)(\varphi), \psi \rangle = \langle w(\alpha\varphi), \psi \rangle$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$. Define $w_* = e^{\lambda t} w_0$. Clearly we have

$$\left(\frac{d}{dt} - \lambda\right) w_* = \left(\frac{d}{dt} - \lambda\right) \left(e^{\lambda t} w_0\right) = e^{\lambda t} \frac{d}{dt} w_0 + \lambda e^{\lambda t} w_0 - \lambda e^{\lambda t} w_0 = l_*.$$

2. We now prove that given a l, one can find a w such that $\frac{d}{dt} w = l$. Suppose that $\theta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is such that $0 \le \theta \le 1$ and

$$\theta(x) = \begin{cases} 0 & \text{for } x < -1 \\ 1 & \text{for } x > 1 \end{cases}$$

(for example, the primitive of an appropriate test function). Now define w as follows: for $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$,

$$\langle w(\varphi), \psi \rangle = \left\langle l\left(\theta(\bullet) \int_0^\infty \varphi(\bullet + y) dy\right), \psi \right\rangle - \left\langle l\left((1 - \theta(\bullet)) \int_{-\infty}^0 \varphi(\bullet + y) dy\right), \psi \right\rangle.$$

Then it can be checked that $w \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ and

$$\left\langle \left(\frac{d}{dt} w\right)(\varphi), \psi \right\rangle = \langle l(\varphi), \psi \rangle,$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$.
3. Finally, if $p = \prod_{k=1}^{N} (\xi - \lambda_k)$, then given a l, using the above, we succesively determine w_k ($k \in \{1, \ldots, N\}$) as follows: $w_0 := l$, $D_{\xi - \lambda_k} w_k = w_{k-1}$, $k \in \{1, \ldots, N\}$. Then it can be easily seen that with $w := w_N$, we have $D_p w = l$. ■

An immediate corollary of Theorems 2.5 and 2.6 is the following:

Corollary 2.7 *If $l \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ and $p = p_1 p_2$, where $p_1 \in \mathbb{C}[\eta] \setminus 0$ and $p_2 \in \mathbb{C}[\xi] \setminus 0$, then there exists a $w \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ such that $D_p w = l$.*

An interesting question that arises now is the following: does Theorem 2.5 hold for all nonzero polynomials in $\mathbb{C}[\eta, \xi]$?

3 Time-autonomy

In Sasane et al. [12] it is shown in Example 3.6 that the diffusion equation is not time-autonomous (with respect to $\mathcal{D}'(\mathbb{R}^2)$). This anomalous behaviour arises because at any point of time, the solution is allowed to grow faster than e^{Ax^2} with $A > 0$, which is unphysical. In this section we prove Theorem 3.1 below, which for $p = \xi - \eta^2$ includes the cases when we consider solutions which at each point of time are integrable in the spatial direction (the diffusion of matter) or when we consider solutions which at each point of time are bounded in the spatial direction (the diffusion of heat). This is the primary motivation for considering the space $\mathcal{W}_s$. Also, it is typical in physical situations that at any point of time, the profile in the spatial direction does not blow up exponentially at $\pm\infty$. Finally, we remark that this is in conformity with the successful traditional treatment of control-theoretic problems for systems described by partial differential equations in the infinite-dimensional system theory framework (see for example the book by Curtain and Zwart [4]), where the state-space is very often taken to be L_2 or a subspace thereof.

Theorem 3.1 *Let $p \in \mathbb{C}[\eta, \xi]$. The $\mathcal{W}_s-$behaviour given by p is time-autonomous iff p is not divisible by a polynomial in $\mathbb{C}[\eta]$ that has a purely imaginary root (that is, a root in $i\mathbb{R}$).*

Proof

'**If** ': We will first prove that the $\mathcal{W}_s-$behaviour $\mathfrak{B}_{\mathcal{W}_s}$ given by p is time-autonomous if no irreducible factor of p belongs to $\mathbb{C}[\eta]$. Let $p = a_0 + a_1\xi + \ldots + a_{\mathtt{N}}\xi^{\mathtt{N}}$, where $a_0, \ldots, a_{\mathtt{N}} \in \mathbb{C}[\eta]$. Suppose that $w \in \mathfrak{B}_{\mathcal{W}_s}$ is such that $\langle w, \varphi\rangle = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\operatorname{supp}(\varphi) \subset \mathbb{R} \times (-\infty, 0)$. Since $w \in \mathfrak{B}_{\mathcal{W}_s}$, we have

$$a_0\left(\frac{\partial}{\partial x}\right)w + a_1\left(\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}w + \ldots + a_{\mathtt{N}}\left(\frac{\partial}{\partial x}\right)\frac{\partial^{\mathtt{N}}}{\partial t^{\mathtt{N}}}w = 0.$$

Thus for all $\varphi \in \mathcal{D}(\mathbb{R})$ we obtain

$$a_0\left(\frac{\partial}{\partial x}\right)(\iota w)(\varphi) + a_1\left(\frac{\partial}{\partial x}\right)\left(\iota\frac{\partial}{\partial t}w\right)(\varphi) + \ldots + a_{\mathtt{N}}\left(\frac{\partial}{\partial x}\right)\left(\iota\frac{\partial^{\mathtt{N}}}{\partial t^{\mathtt{N}}}w\right)(\varphi) = 0.$$

Upon Fourier transformation we get

$$a_0\,(2\pi i y)\,\mathcal{F}\left[(\iota w)(\varphi)\right] + \ldots + a_{\mathtt{N}}\,(2\pi i y)\,\mathcal{F}\left[\left(\iota\frac{\partial^{\mathtt{N}}}{\partial t^{\mathtt{N}}}w\right)(\varphi)\right] = 0. \tag{2}$$

Define $w_{\mathfrak{f}} \in \mathcal{D}'(\mathbb{R}^2)$ by

$$\langle w_{\mathfrak{f}}, \psi \otimes \varphi\rangle = \langle \mathcal{F}\left[(\iota w)(\varphi)\right], \psi\rangle$$

(see Lemma 2.3). Then from (2), it is clear that $w_{\mathfrak{f}} \in \mathcal{D}'(\mathbb{R}^2)$ satisfies

$$a_0\,(2\pi i y)\,w_{\mathfrak{f}} + a_1\,(2\pi i y)\,\frac{\partial}{\partial t}w_{\mathfrak{f}} + \ldots + a_{\mathtt{N}}\,(2\pi i y)\,\frac{\partial^{\mathtt{N}}}{\partial t^{\mathtt{N}}}w_{\mathfrak{f}} = 0.$$

Since we assume that no irreducible factor of p belongs to $\mathbb{C}[\eta]$, it follows that $\mathtt{N} > 1$. From Holmgren's uniqueness theorem (see Theorem 8.6.5, page 309, Hörmander [7], and also the remark immediately following the statement of this theorem on the same page) with

$$X_1 := \complement\{(x,t) \in \mathbb{R}^2 \mid a_{\mathtt{N}}(2\pi i x) = 0\},$$

it follows that $\operatorname{supp}(w_{\mathfrak{f}}) \subset \complement X_1$, since $\operatorname{supp}(w_{\mathfrak{f}}) \subset \mathbb{R}\times[0,\infty)$. Thus $\operatorname{supp}(w_{\mathfrak{f}}) \subset \complement X_1 \cap (\mathbb{R} \times [0,\infty))$ (see Figure 1). If $a_{\mathtt{N}}$ is a constant ($\neq 0$), then we obtain $w_{\mathfrak{f}} = 0$. If $a_{\mathtt{N}}$ is not a constant then there exists a $\mathtt{k} < \mathtt{N}$ such that $(a_{\mathtt{k}}, a_{\mathtt{N}}) = 1$, that is, the greatest common divisor of $a_{\mathtt{k}}$ and $a_{\mathtt{N}}$ is 1. Each of half-lines in $\complement X_1 \cap (\mathbb{R} \times [0,\infty))$ carries a solution of the differential equation with polynomial coefficients and $w_{\mathfrak{f}}$ is the sum of these. We prove our claim for each of the summands, because each of these has support on the corresponding half-line. By means of a translation, we

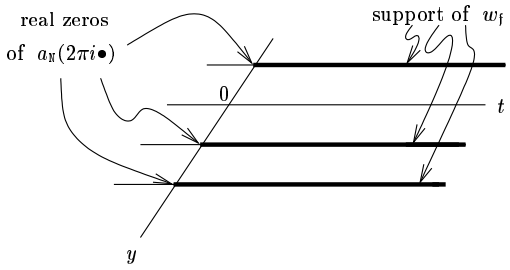


Figure 1: The support of $w_{\mathfrak{f}}$.

may assume that the half-line is $\{0\} \times [0, \infty)$. Let $T \in (0, \infty)$. Then $w_{\mathfrak{f}}$ is a distribution of finite order in $\mathbb{R} \times (-T, T)$. Applying Theorem 2.3.5 (page 47, Hörmander [7]), it follows that there exist distributions $T_1, \ldots, T_{\mathtt{J}} \in \mathcal{D}'(\mathbb{R})$, with $T_{\mathtt{J}} \neq 0$ such that

$$w_{\mathfrak{f}} = \sum_{\mathtt{j}}^{\mathtt{J}} \frac{\partial^{\mathtt{j}}}{\partial y^{\mathtt{j}}} \delta_y \otimes T_{\mathtt{j}} \tag{3}$$

in the strip $\mathbb{R} \times (-T, T)$. We note that from Lemma 2.4 it follows that $\operatorname{supp}\left(T_{\mathtt{J}}|_{(-T,T)}\right) \subset [0, T)$, since $\operatorname{supp}(w_{\mathfrak{f}}) \subset \mathbb{R} \times [0, \infty)$. From (3), we have

$$w = \sum_{\mathtt{j}}^{\mathtt{J}} (2\pi i x)^{\mathtt{j}} \otimes T_{\mathtt{j}}$$

in $\mathbb{R} \times (-T, T)$. Since $D_p w = 0$, we have

$$D_p \frac{\partial^{\mathtt{J}}}{\partial y^{\mathtt{J}}} w = 0,$$

and so $D_p \left(\mathtt{1}_x \otimes T_{\mathtt{J}}\right) = 0$, where $\mathtt{1}_x$ denotes the regular distribution corresponding to the constant function taking value 1 everywhere. But now we can drop all the terms in D_p which contain $\frac{\partial}{\partial x}$, leaving a linear ordinary differential operator, say D_{p_0}, in only t with constant coefficients. Owing to the assumption that $(a_{\mathtt{k}}, a_{\mathtt{N}}) = 1$, we obtain that $p_0 \neq 0$. From the fact that

$$D_{p_0} \left(\mathtt{1}_x \otimes T_{\mathtt{J}}\right) = 0$$

in $\mathbb{R} \times (-T, T)$, we have $D_{p_0} T_{\mathtt{J}} = 0$ in $(-T, T)$. Since $T_{\mathtt{J}}$ is zero in $(-T, 0)$, it follows that $T_{\mathtt{J}} = 0$ in $(-T, T)$ and so $w_{\mathfrak{f}} = 0$ in $\mathbb{R} \times (-T, T)$. But we recall that the choice of T was arbitrary. Hence $w_{\mathfrak{f}} = 0$ and finally $w = 0$. This completes the proof of the time-autonomy of the behaviour corresponding to a polynomial that has no irreducible factor belonging to $\mathbb{C}[\eta]$.

We now prove the result in full generality. Thus, let us assume that $p = p_1 p_2$ where $p_1 \in \mathbb{C}[\eta]$ is not divisible by a polynomial in $\mathbb{C}[\eta]$ having a purely imaginary root, and $p_2 \in \mathbb{C}[\eta, \xi]$ has no irreducible factor that belongs to $\mathbb{C}[\eta]$. Suppose that $w \in \mathfrak{B}_{\mathcal{W}_s}$ is such that $\langle w, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\operatorname{supp}(\varphi) \subset \mathbb{R} \times (-\infty, 0)$ and define $w_1 = D_{p_1} w$. Clearly $\langle w_1, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\operatorname{supp}(\varphi) \subset \mathbb{R} \times (-\infty, 0)$ and $D_{p_2} w_1 = 0$. Consequently, from the first part of the proof, we get $w_1 = 0$, and so $D_{p_1} w = 0$. But from Lemma 2.5 (Cotroneo and Sasane [3]), if $\lambda_{\mathtt{k}} \in \mathbb{C}$ ($\mathtt{k} \in \{1, \ldots, \mathtt{N}\}$) denote the distinct roots of p_1 of multiplicity $\mathtt{n_k}$:

$$p_1 = \prod_{\mathtt{k}=1}^{\mathtt{N}} \left(\eta - \lambda_{\mathtt{k}}\right)^{\mathtt{n_k}},$$

then

$$w = \sum_{\mathtt{k}=1}^{\mathtt{N}} \sum_{\mathtt{l}=0}^{\mathtt{n_k}-1} x^{\mathtt{l}} e^{\lambda_{\mathtt{k}} x} \otimes T_{\mathtt{kl}}, \tag{4}$$

for some $T_{\mathtt{kl}} \in \mathcal{D}'(\mathbb{R})$, $\mathtt{k} \in \{1, \ldots, \mathtt{N}\}$, $\mathtt{l} \in \{0, \ldots, \mathtt{n_k} - 1\}$. But since all the roots of p_1 have nonzero real part (otherwise p_1 would be divisible by a polynomial in $\mathbb{C}[\eta]$ having a purely imaginary root), it follows from (4) that $w \notin \imath^{-1}\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ unless all the $T_{\mathtt{kl}}$'s are zero. Thus $w = 0$. This completes the proof of the 'if' part.

'**Only if** ': Suppose on the contrary that p is divisible by $\eta - i\zeta$, where $\zeta \in \mathbb{R}$. Let $\theta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ be given by

$$\theta(t) = \begin{cases} 0 & t \leq 0, \\ e^{-\frac{1}{t}} & t > 0. \end{cases}$$

Then it is easy to see that if w is defined by $w := e^{i\zeta} \otimes \theta \in \mathcal{W}_{\mathfrak{s}}$, then w has zero past, nonzero future and it satisfies $D_p w = 0$. ■

Example *(The diffusion equation.)* The $\mathcal{W}_{\mathfrak{s}}$−behaviour corresponding to the polynomial $p = \xi - \eta^2$ is time-autonomous. Indeed, p is irreducible, and so it is not divisible by a polynomial in $\mathbb{C}[\eta]$ that has a purely imaginary root.

In particular, we recover the two known classical results that the diffusion equation is well-posed with respect to the spaces $\mathcal{C}^\infty(\mathbb{R}, L_\infty(\mathbb{R}))$ and $\mathcal{C}^\infty(\mathbb{R}, L_1(\mathbb{R}))$, since both these spaces are isomorphic to subspaces of $\mathcal{W}_{\mathfrak{s}}$.

Finally, we recall the smooth nonzero trajectory with zero past given in Sasane et al. [12]: $w \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{C})$ is given by

$$w(x,t) = \sum_{\mathtt{k}=0}^{\infty} f^{(\mathtt{k})}(t) \frac{x^{2\mathtt{k}}}{(2\mathtt{k})!}, \quad -\infty < x, t < \infty,$$

with

$$f(t) = \begin{cases} e^{-\frac{1}{t^2}} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t \leq 0. \end{cases}.$$

From Theorem 3.1 follows that $w \notin \mathcal{W}_{\mathfrak{s}}$. ◊

Example The following table gives a few simple instances of time-autonomous and non time-autonomous $\mathcal{W}_{\mathfrak{s}}$−behaviours.

Time-autonomous	**Not time-autonomous**
$1 + \xi\eta$	$\xi\eta$
$c + \eta^2$ with $c < 0$	$c + \eta^2$ with $c \geq 0$
$\xi^2 - \eta^2$	

◊

4 Time-controllability

In this section, first we will prove a sufficient condition for the time-controllability of mixed representations and give a necessary condition for the time-controllability of a kernel representation.

Theorem 4.1 *The $\mathcal{W}_{\mathfrak{s}}$−behaviour $\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}$ (described in terms of a mixed representation) given by $(pI_{\mathtt{w}}, M)$, where $p \in \mathbb{C}[\eta]$, and $M \in \mathbb{C}[\eta, \xi]^{\mathtt{w} \times \mathtt{l}}$, is time-controllable.*

Proof If $p = 0$, the result is obvious. So we assume $p \neq 0$ in the following. Let w_1 and w_2 be in $\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}$. Then there exist l_1 and l_2 in $\mathcal{W}_{\mathfrak{s}}^{\mathtt{l}}$ such that $D_p w_{\mathtt{k}} = D_M l_k$, $\mathtt{k} \in \{1, 2\}$. Let $\tau > 0$ and let $l \in \mathcal{W}_{\mathfrak{s}}^{\mathtt{l}}$ be defined by

$$l = (1_x \otimes \alpha(\bullet)) l_1 + (1_x \otimes \alpha(\tau - \bullet)) (\sigma_\tau l_2),$$

where $\alpha \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ (see Figure 2) is such that

$$\alpha(t) = \begin{cases} 1 & t \leq \frac{\tau}{4}, \\ 0 & t \geq \frac{\tau}{2}. \end{cases} \tag{5}$$

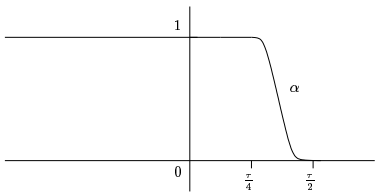


Figure 2: The cut-off curve α.

Then it is easy to see that l concatenates l_1 and l_2. Let $\chi_{\mathtt{k}} \in \mathcal{W}_{\mathtt{s}}$, $\mathtt{k} \in \{1, \ldots, \mathtt{w}\}$ be defined as follows:

$$\begin{bmatrix} \chi_1 \\ \vdots \\ \chi_{\mathtt{w}} \end{bmatrix} = D_M l.$$

From Theorem 2.5, it follows that there exist $\omega_1, \ldots, \omega_{\mathtt{w}}$ in $\mathcal{W}_{\mathtt{s}}$ such that $D_p \omega_{\mathtt{k}} = \chi_{\mathtt{k}}$, $\mathtt{k} \in \{1, \ldots, \mathtt{w}\}$. Let

$$\Omega := \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_{\mathtt{w}} \end{bmatrix} \in \mathcal{W}_{\mathtt{s}}^{\mathtt{w}}.$$

Furthermore, define $w \in \mathcal{W}_{\mathtt{s}}^{\mathtt{w}}$ as follows:

$$w = \Omega + \left[1\!\!1_x \otimes \beta(\bullet)\right](w_1 - \Omega) + \left[1\!\!1_x \otimes \beta(\tau - \bullet)\right](\sigma_\tau w_2 - \Omega),$$

where $\beta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is such that

$$\beta(t) = \begin{cases} 1 & t \leq \frac{\tau}{8}, \\ 0 & t \geq \frac{\tau}{4}. \end{cases} \tag{6}$$

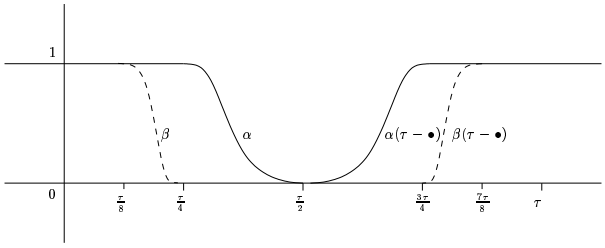


Figure 3: The cut-off curves.

We claim that (w, l) is a trajectory that concatenates (w_1, l_1) and (w_2, l_2), and $w \in \mathfrak{B}_{\mathcal{W}_{\mathtt{s}}}$. First we will show that

$$D_p\left[(1\!\!1_x \otimes \beta(\bullet))(w_1 - \Omega) + (1\!\!1_x \otimes \beta(\tau - \bullet))(\sigma_\tau w_2 - \Omega)\right] = 0. \tag{7}$$

Clearly, if $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\varphi) \subset \left(\frac{3}{8}\tau, \infty\right)$, then for any $\psi \in \mathcal{D}(\mathbb{R})$ we have

$$\left\langle D_p\left[(1\!\!1_x \otimes \beta(\bullet))(w_1 - \Omega) + (1\!\!1_x \otimes \beta(\tau - \bullet))(\sigma_\tau w_2 - \Omega)\right], \psi \otimes \varphi \right\rangle = 0. \tag{8}$$

Similarly, if $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\varphi) \subset \left(-\infty, \frac{5}{8}\tau\right)$, then for any $\psi \in \mathcal{D}(\mathbb{R})$, (8) holds.

Now since $\mathbb{R} \subset \left(-\infty, \frac{5}{8}\tau\right) \cup \left(\frac{3}{8}\tau, \infty\right)$ it follows from Lemma 2.2 that for every $\varphi \in \mathcal{D}(\mathbb{R})$ and every $\psi \in \mathcal{D}(\mathbb{R})$, (8) holds. Finally from Lemma 2.3, it follows that (7) holds.

Next, we have $D_p w = D_p \Omega = D_M l$, and so $w \in \mathfrak{B}_{\mathcal{W}_s}$. Finally, it can be verified easily that

$$\langle w, \varphi \rangle = \begin{cases} \langle w_1, \varphi \rangle & \text{for all } \varphi \in \mathcal{D}\left(\mathbb{R}^2\right) \text{ with } \operatorname{supp}(\varphi) \subset \mathbb{R} \times (-\infty, 0) \\ \langle \sigma_\tau w_2, \varphi \rangle & \text{for all } \varphi \in \mathcal{D}\left(\mathbb{R}^2\right) \text{ with } \operatorname{supp}(\varphi) \subset \mathbb{R} \times (\tau, \infty) \end{cases} .$$

■

Theorem 4.2 *Let $R \in \mathbb{C}[\eta, \xi]^{g \times w}$ and suppose that the $\mathbb{C}[\eta, \xi]-$module $\langle R \rangle$ is Willems with respect to $\mathcal{W}_s$. Then the behaviour $\mathfrak{B}_{\mathcal{W}_s}$ (described in terms of the kernel representation) given by R is time-controllable only if*

$$\neg \left[\begin{array}{l} \text{there exists } \chi \in \mathbb{C}[\eta, \xi]^w \setminus \langle R \rangle \text{ and a } p \in \mathbb{C}[\eta, \xi] \text{ such that} \\ p \text{ is not divisible by a polynomial in } \mathbb{C}[\eta] \text{ having a purely imaginary root} \\ \text{and } \ p \cdot \chi \in \langle R \rangle \end{array} \right] . \qquad (9)$$

Proof Suppose that there exists a $\chi \in \mathbb{C}[\eta, \xi]^w \setminus \langle R \rangle$ and a $p \in \mathbb{C}[\eta, \xi]$ such that p is not divisible by a polynomial in $\mathbb{C}[\eta]$ having a purely imaginary root, and furthermore, there holds $p \cdot \chi \in \langle R \rangle$.

Clearly the trivial zero trajectory $w_1 := 0$ belongs to $\mathfrak{B}_{\mathcal{W}_s}$. Since χ is not in $\langle R \rangle$ (which is Willems with respect to $\mathcal{W}_s$), it follows that it does not kill every element in $\mathfrak{B}_{\mathcal{W}_s}$. Let $w_2 \in \mathcal{W}_s^w$ be a trajectory such that $D_\chi w_2 \neq 0$.

Let w be a trajectory in $\mathfrak{B}_{\mathcal{W}_s}$ which concatenates w_1 and w_2. Define $u = D_\chi w$. Since $p \cdot \chi \in \langle R \rangle$, and $w \in \mathfrak{B}_{\mathcal{W}_s}$, it follows that $D_p u = 0$. But since $\langle w, \varphi \rangle = 0$ for all φ with support in $\mathbb{R} \times (-\infty, 0)$, it follows from the definition of u that $\langle u, \varphi \rangle = 0$ for for all such φ. Hence it follows from Theorem 3.1 that u must be zero. But this a contradiction since u matches the nonzero future of $D_\chi w_2$ (w_2 might have to be shifted in order to achieve this). This completes the proof. ■

It can be seen that (9) implies that the $\mathbb{C}(\eta)[\xi]-$module $\mathbb{C}(\eta)[\xi]^w / \mathbb{C}(\eta)[\xi]^g R$ is torsion free and so we have the following corollary.

Corollary 4.3 *Let $R \in \mathbb{C}[\eta, \xi]^{g \times w}$ and suppose that the $\mathbb{C}[\eta, \xi]-$module $\langle R \rangle$ is Willems with respect to $\mathcal{W}_s$. Then the behaviour $\mathfrak{B}_{\mathcal{W}_s}$ (described in terms of the kernel representation) given by R is time-controllable only if the $\mathbb{C}(\eta)[\xi]-$module $\mathbb{C}(\eta)[\xi]^w / \mathbb{C}(\eta)[\xi]^g R$ is torsion free.*

We remark that not every submodule of $\mathbb{C}[\eta, \xi]^w$ is Willems with respect to $\mathcal{W}_s$. Indeed, if we take $R = [1 + \eta]$, then the $\mathcal{W}_s-$behaviour given by R is zero, and so the set of all annihilators of this $\mathcal{W}_s-$behaviour is the full module $\mathbb{C}[\eta, \xi]$ which is not equal to $\langle 1 + \eta \rangle$.

Example *(The diffusion equation.)* We first show that the $C[\eta, \xi]-$module $\left\langle \xi - \eta^2 \right\rangle$ is Willems with respect to $\mathcal{W}_s$. If w is an element in the $\mathcal{W}_s-$behaviour, we have

$$\frac{\partial}{\partial t} w = \frac{\partial^2}{\partial x^2} w$$

and so if $q = q_0 + q_1 \xi + \ldots + q_N \xi^N \in \mathbb{C}[\eta, \xi]$ is such that $D_q w = 0$, then we have

$$\begin{aligned} 0 &= q_0 \left(\frac{\partial}{\partial x} \right) w + q_1 \left(\frac{\partial}{\partial x} \right) \frac{\partial}{\partial t} w + \ldots + q_N \left(\frac{\partial}{\partial x} \right) \frac{\partial^N}{\partial t^N} w \\ &= q_0 \left(\frac{\partial}{\partial x} \right) w + q_1 \left(\frac{\partial}{\partial x} \right) \frac{\partial^2}{\partial x^2} w + \ldots + q_N \left(\frac{\partial}{\partial x} \right) \frac{\partial^{2N}}{\partial x^{2N}} w. \end{aligned}$$

Using the notation from the proof of Theorem 3.1, we have

$$\left[q_0 (2\pi i y) + q_1 (2\pi i y) (2\pi i y)^2 + \ldots + q_N (2\pi i y) (2\pi i y)^{2N} \right] w_{\mathfrak{f}} = 0.$$

Let us define

$$\alpha(y) = q_0 (2\pi i y) + q_1 (2\pi i y) (2\pi i y)^2 + \ldots + q_N (2\pi i y) (2\pi i y)^{2N}$$

and let $R > 0$ be large enough such that the roots of α are all contained in the ball $B(0,R) = \{s \in \mathbb{C} \mid |s| < R\}$. Clearly the nonzero trajectory

$$w := e^{2\pi i R x} \otimes e^{-4\pi^2 R^2 t} \in \mathcal{W}_{\mathfrak{s}}$$

belongs to the behaviour. Furthermore,

$$w_{\mathfrak{f}} = \delta_R \otimes e^{-4\pi^2 R^2 t}$$

satisfies $0 = \alpha(y) w_{\mathfrak{f}} = \alpha(R) w_{\mathfrak{f}}$. Since $\alpha(R) \neq 0$, we have $w_{\mathfrak{f}} = 0$, which contradicts the fact that $w \neq 0$. Consequently the polynomial $\alpha \in \mathbb{C}[y]$ must be zero. Consider the unique ring homomorphism $\Theta : \mathbb{C}[\eta,\xi] \to \mathbb{C}[y]$ such that $\xi \mapsto (2\pi i y)^2$ and $\eta \mapsto 2\pi i y$. We claim that $\ker(\Theta) = \langle \xi - \eta^2 \rangle$. Clearly, $\langle \xi - \eta^2 \rangle \subset \ker(\Theta)$. To show the reverse inclusion, let $r \in \ker(\Theta)$, and let us write $r = r_0 + r_1 \xi + \ldots + r_{\mathtt{M}} \xi^{\mathtt{M}}$, for some $r_0, r_1, \ldots, r_{\mathtt{M}} \in \mathbb{C}[\eta]$. Since $\xi = \xi - \eta^2 + \eta^2$, we have $\xi^{\mathtt{k}} = (\xi - \eta^2 + \eta^2)^{\mathtt{k}} = (\xi - \eta^2) h_{\mathtt{k}} + \eta^{2\mathtt{k}}$, where $h_{\mathtt{k}} \in \mathbb{C}[\eta,\xi]$, $k \in \mathbb{N}$. Hence we obtain $r = \alpha + \beta(\xi - \eta^2)$ for some $\alpha \in \mathbb{C}[\eta]$ and $\beta \in \mathbb{C}[\eta,\xi]$. Finally, since $r \in \ker(\Theta)$, we have $\alpha(2\pi i y) = 0$ and so $\alpha = 0$. Consequently $r \in \langle \xi - \eta^2 \rangle$. Hence we have shown that q is divisible by p, that is $q \in \langle p \rangle$. So the module $\langle \xi - \eta^2 \rangle$ is Willems with respect to $\mathcal{W}_{\mathfrak{s}}$.

Furthermore since condition (9) is not satisfied ($\chi := 1$ and $p := \xi - \eta^2$ demonstrates this), it follows from Theorem 4.2 that the $\mathcal{W}_{\mathfrak{s}}-$behaviour is not time-controllable. Indeed this can be seen directly, since the constant trajectories 0 and 1 both lie in the $\mathcal{W}_{\mathfrak{s}}-$behaviour, but since the $\mathcal{W}_{\mathfrak{s}}-$behaviour is time-autonomous, it follows that no concatenating trajectory can exist in $\mathcal{W}_{\mathfrak{s}}$. However, a concatenating trajectory in $\mathcal{D}'(\mathbb{R}^2)$ exists (in fact it is in $\mathcal{C}^\infty(\mathbb{R}^2,\mathbb{C})$ owing to the hypoellipticity of the corresponding partial differential operator), see Çamlıbel and Sasane [2]. $\Diamond$

5 Epilogue

We end this paper with the following open questions:

1. Find an iff test on R to determine if the module $\langle R \rangle$ is Willems with respect to $\mathcal{W}_{\mathfrak{s}}$.
2. Does the time-controllability of a $\mathcal{W}_{\mathfrak{s}}-$behaviour imply that the $\mathcal{W}_{\mathfrak{s}}-$behaviour is then in fact described by a simple mixed representation as in Theorem 4.1?
3. Does the torsion freeness of the $\mathbb{C}(\eta)[\xi]-$module $\mathbb{C}(\eta)[\xi]^{\mathtt{w}} / \mathbb{C}(\eta)[\xi]^{\mathtt{g}} R$ imply that the $\mathcal{W}_{\mathfrak{s}}-$behaviour described by the kernel representation given by R is in fact the $\mathcal{W}_{\mathfrak{s}}-$behaviour described by some mixed representation as in Theorem 4.1?
4. Does Theorem 2.5 hold for all nonzero polynomials in $\mathbb{C}[\eta,\xi]$?

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