Checking $2 \times M$ separability via semidefinite programming

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In this paper we propose a sequence of tests which gives a definitive test for checking $2 \times M$ separability. The test is definitive in the sense that each test corresponds to checking membership in a cone, and that the closure of the union of all these cones consists exactly of *all* $2 \times M$ separable states. Membership in each single cone may be checked via semidefinite programming, and is thus a tractable problem. This sequential test comes about by considering the dual problem, the characterization of all positive maps acting $\mathbb{C}^{2\times 2} \to \mathbb{C}^{M\times M}$. The latter in turn is solved by characterizing all positive quadratic matrix polynomials in a complex variable.

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I. INTRODUCTION

In the last decade entanglement has been recognized to be a fundamental notion in quantum information processing. While initially entanglement lead to "paradoxes" in quantum mechanics, it was later discovered to be a useful tool in teleportation, secure key distribution (quantum cryptography), quantum computation, etc. (see, e.g., [1] and [2]). Despite its importance there does not yet exist a tractable conclusive test to check for entanglement or the lack thereof (separability). There are several partial tests the most famous of which is the Peres test [3] which says that a separable state remains positive under taking partial transposes. In some cases the Peres test is also conclusive, namely in the low dimensional cases $2 \times M$, $M = 2, 3$ (see [4]) and in cases where the density matrix is of low rank (see [5]). Recently, in [6] the partial positive test was combined with an extendibility property to obtain a sequence of tests stronger than the Peres test, that all separable matrices need to pass. It is not known whether failure to pass this sequence of tests yields entanglement. Our main result concerns a sequence of tests that for the $2 \times M$ separability problem do the opposite. Namely, passing one of the tests yields $2 \times M$ separability. This sequence of tests yields a definite answer to the $2 \times M$ separability problem in the sense that any $2 \times M$ separable matrix is arbitrarily close to a matrix satisfying one of the tests. Let us present the details.

Let $M_p(K)$ denote $p \times p$ matrices whose entries belong to K. Recall that a matrix $\rho \in M_2(M_M(\mathbb{C}))$ is called $2 \times M$ *separable* if there exists $x_i \in \mathbb{C}^2$ and $y_i \in \mathbb{C}^M$ so that

$$\begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix} = \sum_i x_i x_i^\dagger \otimes y_i y_i^\dagger.$$

Here † denotes the conjugate transpose and $\otimes$ denotes the Kronecker product. Let $n \geq 1$ and $\rho = \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix}$, where A, B, and D are $M \times M$ matrices. We introduce the following convex set, which may be empty. Let $\mathcal{G}(\rho; n) \subset M_{n+1}(M_2(M_M(\mathbb{C})))$ consist of all positive semidefinite block matrices $\Gamma = (\Gamma_{ij})_{i,j=0}^n$ (notation: $\Gamma \geq 0$) where

$$\Gamma_{ij} = \begin{pmatrix} \Gamma_{ij}^{(a)} & \Gamma_{ij}^{(b)} \\ \Gamma_{ij}^{(c)} & \Gamma_{ij}^{(d)} \end{pmatrix} \in M_2(M_M(\mathbb{C})), 0 \leq i, j \leq n,$$

satisfy the conditions

(i) $\sum_{i=0}^n \Gamma_{ii}^{(a)} = A$,

(ii) $\sum_{i=0}^n \Gamma_{ii}^{(d)} = D$,

(iii) $\sum_{i=0}^{n-1} \frac{1}{2}(\Gamma_{i,i+1}^{(c)} + \Gamma_{i,i+1}^{(b)}) = B$,

(iv) $\sum_{i=0}^n (\Gamma_{i,i}^{(c)} - \Gamma_{i,i}^{(b)}) = 0$, and

(v) $\sum_{i=0}^{n-k} (\Gamma_{i,i+k}^{(c)} + \Gamma_{i+k,i}^{(c)} - \Gamma_{i,i+k}^{(b)} - \Gamma_{i+k,i}^{(b)}) = 0$, $k = 1, \ldots, n$.

We will be interested in matrices ρ for which $\mathcal{G}(\rho; n)$ is nonempty. Therefore we define

$$\mathcal{A}_n := \{\rho \in M_2(M_M(\mathbb{C})) : \mathcal{G}(\rho; n) \neq \emptyset\}.$$

It is straightforward to check that $\mathcal{A}_n$ is a closed convex cone. Moreover, since $\mathcal{G}(\rho; n) \oplus 0_{2M} \subseteq \mathcal{G}(\rho; n+1)$ one obtains that $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, $n \geq 1$.

It is an important feature of the set $\mathcal{G}(\rho; n)$ that it is the intersection of the cone of positive semidefinite matrices (PSD) and an affine set. As such, the question of whether the set is empty or not, falls into a class of well studied problems, called semidefinite programming (SDP). Checking the nonemptyness of such an intersection is called the feasibility problem in SDP, and several software packages (e.g., [7], [8]) are readily available to solve the feasibility problem numerically. A good overview article on SDP is [9].

We now state the main result.

Theorem I.1 *The matrix $\rho \in M_2(M_M(\mathbb{C}))$ is $2 \times M$ separable if and only if*

$$\rho \in \overline{\cup_{n\geq 1} \mathcal{A}_n}.$$

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Based on the above theorem one may now formulate the following algorithm for checking for $2 \times M$ separability. Given a time limit in which the check needs to be performed, make n as large as possible so that $\rho \in \mathcal{A}_n$ may be checked via SDP within the given time limit. In case the test comes out affirmatively, the given matrix ρ is $2 \times M$ separable. When the test shows that $\rho \notin \mathcal{A}_n$ the test is inconclusive. Still, given the content of Theorem I.1, the negative outcome may be an indication that the matrix is entangled. Of course, the larger n was chosen, the stronger the indication is.

This is certainly not the first instance where SDP has been used for a quantum information problem. In fact, the earlier mentioned tests in [6] may be done by SDP. Moreover, in the papers [10], [11], [12], [13] connections have been made between several other problems in quantum information and the versatile SDP tool.

We will prove Theorem I.1 by characterizing positive maps (also known as entanglement witnesses) acting on 2×2 matrices. This characterization shall come about from the analysis of quadratic matrix polynomials in a complex variable.

II. POSITIVE MAPS

We say that a linear map $\Phi : \mathbb{C}^{2\times 2} \to \mathbb{C}^{M\times M}$ is *positive* when $\Phi(G) \geq 0$ whenever $G \geq 0$. By linearity it suffices to check this condition for rank 1 matrices G. Furthermore, we may assume that G has the form

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } G = \begin{pmatrix} z \\ 1 \end{pmatrix} \begin{pmatrix} \overline{z} & 1 \end{pmatrix}, z \in \mathbb{C}.$$

If we let $P = \Phi(E_{11})$, $Q = \Phi(E_{12})$ and $R = \Phi(E_{22})$, where E_{ij} is the 2×2 matrix with a 1 in entry (i, j) and zeros elsewhere, checking $\Phi(G) \geq 0$ for all G as above amounts to checking that $P \geq 0$ and

$$|z|^2 P + zQ + \overline{z}Q^\dagger + R \geq 0, \quad z \in \mathbb{C}. \tag{1}$$

Since $P \geq 0$ is automatically satisfied when (1) holds (let $|z| \to \infty$) it therefore suffices to check (1). In other words, we have the following lemma.

Lemma II.1 *The map $\Phi : \mathbb{C}^{2\times 2} \to \mathbb{C}^{M\times M}$ is positive if and only if $P = \Phi(E_{11})$, $Q = \Phi(E_{12})$ and $R = \Phi(E_{22})$ satisfy* (1).

We therefore need to study quadratic matrix polynomials in a complex variable.

III. QUADRATIC MATRIX POLYNOMIALS IN COMPLEX VARIABLES

For $M \times M$ matrices P, Q and R, consider the matrix inequalities (1). As observed, (1) implies $P \geq 0$, and clearly it also implies $R \geq 0$. Our analysis is based on the observation that we can eliminate $\arg z$ and subsequently $|z|$ in inequality (1). We do this as follows. Write $z = re^{j\theta}$ with $r, \theta \in \mathbb{R}$. Then (1) is equivalent to

$$r^2 P + r(e^{j\theta}Q + e^{-j\theta}Q^\dagger) + R \geq 0, \quad r, \theta \in \mathbb{R}. \tag{2}$$

We now use the following well-known fact. For completeness we provide a proof. Useful references on Toeplitz operators include [14, Chapter XXIII] and [15].

Lemma III.1 *Given is the operator-valued trigonometric polynomial $H(z) := z^{-1}S^\dagger + T + zS$, with S and T bounded linear operators on a Hilbert space $\mathcal{H}$. Then $H(z) \geq 0$, $|z| = 1$, if and only if the Toeplitz operator*

$$\begin{pmatrix} T & S^\dagger & 0 & 0 & \cdots \\ S & T & S^\dagger & 0 & \cdots \\ 0 & S & T & S^\dagger & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{3}$$

is positive semidefinite.

Proof. Consider the multiplication operator $g \to Hg$ on the Lebesgue space $L_2(\mathbb{T}, \mathcal{H})$ of square integrable Lebesgue measurable functions on the unit circle $\mathbb{T}$ with values in $\mathcal{H}$. By identifying $L_2(\mathbb{T}, \mathcal{H})$ with the Hilbert space $\ell_2(\mathbb{Z}, \mathcal{H})$ of square summable (in norm) sequences $(h_j)_{j=-\infty}^{\infty}$, $h_j \in \mathcal{H}$, the positive semidefiniteness of $H(z)$ for all $z \in \mathbb{T}$ is equivalent to the positive semidefiniteness of the doubly infinite Toeplitz matrix Λ with symbol $H(z)$ (that is, Λ is the doubly infinite version of (3)). As (3) is the restriction of this doubly infinite Toeplitz matrix to $\ell_2(\mathbb{N}, \mathcal{H})$, positive semidefiniteness of (3) follows.

Conversely, let $h = (h_j)_{j=-\infty}^{\infty} \in \ell_2(\mathbb{Z}, \mathcal{H})$ be so that $h_j = 0$, $j \geq B$. Then it follows from the positive semidefiniteness of (3) that $\langle \Lambda h, h \rangle \geq 0$. Since sequences h of the above form are dense in $\ell_2(\mathbb{Z}, \mathcal{H})$, $\Lambda \geq 0$ follows. But since $H(z)$ is the symbol of this multiplication operator Λ, it follows that $H(z) \geq 0$, $|z| = 1$. □

Fixing r and applying Lemma III.1 to (2), we obtain that (2) holds for all θ if and only if the following infinite block Toeplitz matrix is positive semidefinite:

$$\begin{pmatrix} R + r^2 P & rQ^\dagger & 0 & 0 & \cdots \\ rQ & R + r^2 P & rQ^\dagger & 0 & \cdots \\ 0 & rQ & R + r^2 P & rQ^\dagger & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \geq 0, r \in \mathbb{R}. \tag{4}$$

Notice that we may write (4) as

$$r^2 K + rL + N \geq 0, \quad r \in \mathbb{R}, \tag{5}$$

where K, L and N are infinite self-adjoint block Toeplitz matrices. Theorem 6.7 in [16] states that an operator polynomial of even degree, $2q$ say, that is positive semidefinite for all $r \in \mathbb{R}$ allows a factorization $T(r)T(r)^\dagger$, where $T(r)$ is an operator polynomial of degree q. In our case we obtain that

$$r^2 K + rL + N = (rT + S)(rT + S)^\dagger.$$

This yields that $K = TT^\dagger$, $L = TS^\dagger + ST^\dagger$ and $N = SS^\dagger$. Consequently, if we let $X = TS^\dagger - \frac{L}{2}$ we get that X is skew-adjoint ($X = -X^\dagger$) and

$$\begin{pmatrix} K & \frac{L}{2} + X \\ \frac{L}{2} + X^\dagger & N \end{pmatrix} \geq 0. \tag{6}$$

Thus (5) implies the existence of $X = -X^\dagger$ such that (6) holds. The converse is also valid, as

$$r^2K + rL + N = \begin{pmatrix} rI & I \end{pmatrix} \begin{pmatrix} K & \frac{L}{2} + X \\ \frac{L}{2} - X & N \end{pmatrix} \begin{pmatrix} rI \\ I \end{pmatrix}.$$

Applying this to our particular choice of K, L and N in terms of P, Q and R, we get that (1) holds if and only if there exists a skew-adjoint $X = (X_{ij})_{i,j\geq 0}$ so that

$$\begin{pmatrix} \begin{pmatrix} P & 0 & \cdots \\ 0 & P & \cdots \\ \vdots & & \ddots \end{pmatrix} & \begin{pmatrix} X_{00} & \frac{Q^\dagger}{2} + X_{01} & X_{02} & \cdots \\ \frac{Q}{2} + X_{10} & X_{11} & \frac{Q^\dagger}{2} + X_{12} & \cdots \\ \vdots & & \ddots & \end{pmatrix} \\ \begin{pmatrix} X_{00}^\dagger & \frac{Q^\dagger}{2} + X_{10}^\dagger & X_{20}^\dagger & \cdots \\ \frac{Q}{2} + X_{01}^\dagger & X_{11}^\dagger & \frac{Q^\dagger}{2} + X_{21}^\dagger & \cdots \\ \vdots & & \ddots & \end{pmatrix} & \begin{pmatrix} R & 0 & \cdots \\ 0 & R & \cdots \\ \vdots & & \ddots \end{pmatrix} \end{pmatrix} \geq 0. \tag{7}$$

Performing a permutation, we may rewrite (7) as

$$\Lambda_0 := \begin{pmatrix} \begin{pmatrix} P & X_{00} \\ X_{00}^\dagger & R \end{pmatrix} & \begin{pmatrix} 0 & \frac{Q^\dagger}{2} + X_{01} \\ \frac{Q^\dagger}{2} + X_{10}^\dagger & 0 \end{pmatrix} & \begin{pmatrix} 0 & X_{02} \\ X_{20}^\dagger & 0 \end{pmatrix} & \cdots \\ \begin{pmatrix} 0 & \frac{Q}{2} + X_{10} \\ \frac{Q}{2} + X_{01}^\dagger & 0 \end{pmatrix} & \begin{pmatrix} P & X_{11} \\ X_{11}^\dagger & R \end{pmatrix} & \begin{pmatrix} 0 & \frac{Q^\dagger}{2} + X_{12} \\ \frac{Q^\dagger}{2} + X_{21}^\dagger & 0 \end{pmatrix} & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \geq 0. \tag{8}$$

Next we want to show that we may in fact choose $X_{ij} = X_{i-j}$ for all i and j. Perhaps the quickest way to do this is by using Banach limits (for the definition, see [17, Section III.7]). Carrying out some of the ideas related to Banach limits directly to the current situation, we obtain the following argument. Let Λ_n be the infinite principal submatrix obtained from Λ_0 by omitting the first n block rows and columns, each of size $2M$. Thus Λ_n has $\begin{pmatrix} P & X_{nn} \\ X_{nn}^\dagger & R \end{pmatrix}$ in the top right corner. Of course, $\Lambda_n \geq 0$ for all n, and the sequence $\{\Lambda_n\}_{n\geq 0}$ is bounded (by $\|\Lambda_0\|$). Consider now the bounded sequence $\{\Xi_n\}_{n\geq 0}$ of averages defined via

$$\Xi_n := \frac{1}{n+1}(\Lambda_0 + \cdots + \Lambda_n) \geq 0, \ n \in \mathbb{N}_0.$$

This sequence has a convergent subsequence $\{\Xi_{n_k}\}_{k\geq 0}$ in the weak operator topology. Notice that Ξ_{n_k} has the same form as Λ_0; only the operators X_{ij} are different. Therefore, its limit Ξ_∞, which is positive semidefinite, must have the same form as Λ_0 as well, with X_{ij} replaced by Y_{ij}, say. We claim that $Y_{ij} = Y_{i+1,j+1}$ for all i and j. Indeed, since $\lim_{k\to\infty} \frac{X_{ij}}{n_k+1} = 0$ and $\lim_{k\to\infty} \frac{X_{i+n_k+1,j+n_k+1}}{n_k+1} = 0$, we get that

$$Y_{ij} - Y_{i+1,j+1} = \lim_{k\to\infty} \frac{X_{ij} + \cdots + X_{i+n_k,j+n_k}}{n_k+1} -$$

$$\lim_{k\to\infty} \frac{X_{i+1,j+1} + \cdots + X_{i+n_k+1,j+n_k+1}}{n_k+1} =$$

$$\lim_{k\to\infty} \frac{X_{ij}}{n_k+1} - \frac{X_{i+n_k+1,j+n_k+1}}{n_k+1} = 0.$$

In conclusion, we have obtained that (1) holds if and only if there exist $X_i = -X_{-i}^\dagger$, $i \in \mathbb{N}$ so that

$$\begin{pmatrix} \begin{pmatrix} P & X_0 \\ -X_0 & R \end{pmatrix} & \begin{pmatrix} 0 & \frac{Q^\dagger}{2} + X_{-1} \\ \frac{Q^\dagger}{2} - X_{-1} & 0 \end{pmatrix} & \begin{pmatrix} 0 & X_{-2} \\ -X_{-2} & 0 \end{pmatrix} & \cdots \\ \begin{pmatrix} 0 & \frac{Q}{2} + X_1 \\ \frac{Q}{2} - X_1 & 0 \end{pmatrix} & \begin{pmatrix} P & X_0 \\ -X_0 & R \end{pmatrix} & \begin{pmatrix} 0 & \frac{Q^\dagger}{2} + X_{-1} \\ \frac{Q^\dagger}{2} - X_{-1} & 0 \end{pmatrix} & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \geq 0. \tag{9}$$

Let us denote the principal $(n+1) \times (n+1)$ block submatrix of (9) by $\mathcal{D}(X_0, \ldots, X_n; P, Q, R)$. Then we obtain the following result.

Proposition III.2 *The matrices P, Q and R satisfy* (1) *if and only if there exist $X_i = -X_{-i}^\dagger$, $i = 0, 1, \ldots$ so that $\mathcal{D}(X_0, \ldots, X_n; P, Q, R) \geq 0$ for all n.*

Let $\mathcal{C}_n$ denote the cone of all matrices $\sigma = \begin{pmatrix} P & Q \\ Q^\dagger & R \end{pmatrix}$ so that there exist $X_i = -X_{-i}^\dagger$, $i = 0, \ldots, n$, so that $\mathcal{D}(X_0, \ldots, X_n; P, Q, R) \geq 0$. Using this notation we obtain the following.

Proposition III.3 *The matrices P, Q and R satisfy* (1) *if and only if*

$$\sigma = \begin{pmatrix} P & Q \\ Q^\dagger & R \end{pmatrix} \in \cap_{n \geq 1} \mathcal{C}_n.$$

Combining Lemma II.1 and Proposition III.3 we now obtain the following description of positive maps acting $\mathbb{C}^{2\times 2} \to \mathbb{C}^{M \times M}$.

Proposition III.4 *The map $\Phi : \mathbb{C}^{2\times 2} \to \mathbb{C}^{M\times M}$ is positive if and only if*

$$\begin{pmatrix} \Phi(E_{11}) & \Phi(E_{12}) \\ \Phi(E_{21} & \Phi(E_{22}) \end{pmatrix} \in \cap_{n \geq 1} \mathcal{C}_n.$$

Next we need duality to get back to the separability problem. We first prove the following auxiliary result.

Lemma III.5 *The cones $\mathcal{A}_n$ and $\mathcal{C}_n$ are one another's dual.*

Proof. Let $\rho \in \mathcal{A}_n$. In order to show that $\rho \in \mathcal{C}_n^*$ we need to show that $\mathrm{trace}(\rho\sigma) \geq 0$ for all $\sigma \in \mathcal{C}_n$. Thus, let $\sigma \in \mathcal{C}_n$, and let $X_i = -X_{-i}^\dagger$, $i = 0, \ldots, n$, be so that $D := \mathcal{D}(X_0, \ldots, X_n; P, Q, R) \geq 0$. Since $\rho \in \mathcal{A}_n$ there exists a $\Gamma \in \mathcal{G}(\rho; n)$. It is now straightforward to check that $\mathrm{trace}(\rho\sigma) = \mathrm{trace}(\Gamma D)$, and since D and Γ are positive semidefinite, $\mathrm{trace}(\rho\sigma) \geq 0$ follows. This shows that $\mathcal{A}_n \subset \mathcal{C}_n^*$.

For the converse inclusion, observe that when $\rho = \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix} \notin \mathcal{A}_n$, then for all Γ_{ij} satisfying (i)-(v) we have that $(\Gamma_{ij})_{i,j=0}^n \not\geq 0$. Notice that the block matrices $(\Gamma_{ij})_{i,j=0}^n$ satisfying (i)-(v) describe a affine space, which we may denote as $G + \mathcal{L}$ where G is a fixed matrix and $\mathcal{L}$ is a linear subspace (described by all matrices satisfying (i)-(v) with $A = D = B = 0$). This affine space $G + \mathcal{L}$ is separated from the cone PSD of positive semidefinites. Thus by the Hahn-Banach theorem and the selfduality of PSD there exist a positive semidefinite W so that $\mathrm{trace}(W(G + L)) < 0$ for all $L \in \mathcal{L}$. But then $W \in \mathcal{L}^\perp$ and thus W is of the form $\mathcal{D}(X_0, \ldots, X_n; P, Q, R)$. Let $\sigma = \begin{pmatrix} P & Q \\ Q^\dagger & R \end{pmatrix}$, which belongs to $\mathcal{C}_n$. As $\mathrm{trace}(\sigma\rho) = \mathrm{trace}(WG) < 0$, it follows that $\rho \notin \mathcal{C}_n^*$. Thus $\mathcal{A}_n = \mathcal{C}_n^*$.

Since $\mathcal{C}_n$ is closed, $\mathcal{A}_n^* = \mathcal{C}_n^{**} = \mathcal{C}_n$ follows also. □

We are now ready to prove the main result.

Proof of Theorem I.1 By [4] we have that the cone of $2 \times M$ separable matrices has the dual

$$\{ \begin{pmatrix} \Phi(E_{11}) & \Phi(E_{12}) \\ \Phi(E_{21} & \Phi(E_{22}) \end{pmatrix} : \Phi : \mathbb{C}^{2\times 2} \to \mathbb{C}^{M\times M} \text{ is positive } \},$$

which by Proposition III.4 equals $\cap_{n=1}^\infty \mathcal{C}_n$. The cone of $2 \times M$ separable matrices therefore equals

$$(\cap_{n=1}^\infty \mathcal{C}_n)^* = \overline{\cup_{n=1}^\infty \mathcal{C}_n^*} = \overline{\cup_{n=1}^\infty \mathcal{A}_n}.$$

In the last step we used Lemma III.5. □

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