Convergence Study of Block Singular Value Maximization Methods for Rank-1 Approximation to Higher Order Tensors

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Abstract

The convergence of the higher order power method (HOPM) for rank-1 approximation to tensors was systematically studied recently. In this paper, another block coordinate ascent method for solving the same problem, termed block singular value maximization method (BSVMM) here, is revisited and studied. At each subproblem, BSVMM jointly updates two blocks, leading to a matrix singular value maximization subproblem. BSVMM has its own advantages; however, the non-uniqueness of the optimizer of each subproblem causes some troubles in studying the convergence. Thus the main concern in this paper is toward an understanding of the convergence of BSVMM. First, BSVMM is shown to have a $O(1/K)$ convergence rate in the ergodic sense under certain measurements. It is then proved that if a limit point admits certain regularities, then this limit point is a partial maximizer; the whole sequence converges to a partial maximizer, provided that all the limit points admit the regularities. Two variants of BSVMM are introduced then. Concerning the symmetric rank-1 approximation to symmetric tensors, along the line of BSVMM, a method based on eigenvalue maximization procedure and its convergence are investigated. Preliminary experiments are provided to show the effectiveness of the methods.

Key words: tensor; rank-1 approximation; block coordinate ascent; convergence; singular value; eigenvalue; conditional gradient method; Kurdyka-Łojasiewicz function

AMS subject classifications. 90C26, 15A18, 15A69, 41A50

1 Introduction

The topic of tensor methods and tensor decomposition has a long history, and has attracted much interest in the signal processing, machine learning as well as mathematical communities in recent years [11, 13, 29, 45]. In the context of tensor decomposition, the best rank-1 approximation problem, which amounts to find a projection of a tensor onto the manifold of rank-1 tensors, plays an important role of building block in (robust) tensor approximation, tensor completion and tensor problems in machine learning [1–3, 47, 56, 57]. Besides, the problem of best rank-1 approximation is of large interest in its own, which finds applications in e.g., independent component analysis, higher order graph matching, geometric measure of entanglement for a symmetric pure state , and hypergraph theory [10, 12, 26, 27]. On the other hand, it is also equivalent to computing the tensor spectral norm, which is dual to the nuclear norm of $\mathcal{A}$; see e.g., [34].

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It is known that the best rank-1 approximation to a d-th order tensor $\mathcal{A} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ is equivalent to maximizing a multilinear form over unit spherical constraints [16], i.e.,

$$\max \ F(\boldsymbol{x}) := F(\mathbf{x}_1, \ldots, \mathbf{x}_d) = \mathcal{A} \times_1 \mathbf{x}_1^\top \times_2 \mathbf{x}_2^\top \cdots \times_d \mathbf{x}_d^\top \ \text{ s.t. } \ \|\mathbf{x}_i\| = 1, \ \mathbf{x}_i \in \mathbb{R}^{n_i}. \tag{1.1}$$

Here $\boldsymbol{x} := (\mathbf{x}_1, \ldots, \mathbf{x}_d)$, the notation $\times_m$ denotes the mode-m product, and $\mathcal{A} \times_m \mathbf{x}_m^\top$ is a $(d-1)$-th order tensor given by

$$(\mathcal{A} \times_m \mathbf{x}_m^\top)_{i_1 \cdots i_{m-1} i_{m+1} \cdots i_d} = \sum\nolimits_{i_m=1}^{n_m} a_{i_1 \cdots i_{m-1} i_m i_{m+1} \cdots i_d} \mathbf{x}_{m, i_m}.$$

Since the objective of (1.1) admits a block form and the constraints are decoupled, the conventional block coordinate ascent method can be applied; this leads to the well-known HOPM [16] whose iterative scheme simply reads as follows: At the $(k+1)$-th iteration, for $i = 1, \ldots, d$, compute

$$\mathbf{x}_i^{(k+1)} = \frac{\mathbf{y}_i^{(k+1)}}{\|\mathbf{y}_i^{(k+1)}\|}, \ \text{ where } \ \mathbf{y}_i^{(k+1)} = \mathcal{A} \times_1 \mathbf{x}_1^{(k+1)\top} \cdots \times_{i-1} \mathbf{x}_{i-1}^{(k+1)\top} \times_{i+1} \mathbf{x}_{i+1}^{(k)\top} \cdots \times_d \mathbf{x}_d^{(k)\top}.$$

HOPM is a special case of the alternating least squares (ALS). Besides HOPM, a large body of methods have been proposed to (approximately) solve (1.1) [15, 20, 23, 24, 40, 46, 55, 58], just to name a few. Note that computing the spectral norm when $d \geq 3$ (solving (1.1)) is NP-hard [25], and for any nonzero rational number α, deciding whether α is the spectral norm of $\mathcal{A}$ is also NP-hard.

Alternative to HOPM, by noticing the special structure of the multilinear form, one can also jointly update two blocks at a time; such an update scheme has been introduced in the same paper of HOPM; see [16, Section 3.3]. Specifically, suppose that $\mathbf{x}_j, j \neq i, i+1$ are fixed; one finds the leading left and right singular vectors of the matrix $\mathcal{A} \times_1 \mathbf{x}_1^\top \cdots \times_{i-1} \mathbf{x}_{i-1}^\top \times_{i+2} \mathbf{x}_{i+2}^\top \cdots \times_d \mathbf{x}_d^\top \in \mathbf{R}^{n_i \times n_{i+1}}$, i.e.,

$$(\mathbf{x}_i, \mathbf{x}_{i+1}) \in \arg\max \ \mathbf{y}^\top (\mathcal{A} \times_1 \mathbf{x}_1^\top \cdots \times_{i-1} \mathbf{x}_{i-1}^\top \times_{i+2} \mathbf{x}_{i+2}^\top \cdots \times_d \mathbf{x}_d^\top) \mathbf{z} \ \text{ s.t. } \|\mathbf{y}\| = \|\mathbf{z}\| = 1.$$

It turns out that with the Gauss-Seidel and cyclic manner, the above updating formula gives rise to another method for solving (1.1). Specifically, starting from a feasible point $\boldsymbol{x}^{(0)} = (\mathbf{x}_1^{(0)}, \ldots, \mathbf{x}_d^{(0)})$, at the $(k+1)$-th iteration, for $i = 1, \ldots, \lceil \frac{d}{2} \rceil$, compute

$$\begin{aligned}(\mathbf{x}_{2i-1}^{(k+1)}, \mathbf{x}_{2i}^{(k+1)}) \in \arg\max \ & \mathbf{y}^\top (\mathcal{A} \times_1 \mathbf{x}_1^{(k+1)\top} \cdots \times_{2i-2} \mathbf{x}_{2i-2}^{(k+1)\top} \times_{2i+1} \mathbf{x}_{2i+1}^{(k)\top} \cdots \times_d \mathbf{x}_d^{(k)\top}) \mathbf{z} \\ & \text{s.t. } \|\mathbf{y}\| = \|\mathbf{z}\| = 1. \end{aligned} \tag{1.2}$$

We call the above method block singular value maximization method (BSVMM) in the present paper, as at each subproblem of each iteration (1.2), it boils down to solving a singular value maximization problem. Note that BSVMM does not exclude the case that d is odd, because whenever the order d of a tensor is odd, one can impose a slack variable $\mathbf{x}_{d+1} \in \mathbb{R}$ with $(\mathbf{x}_{d+1})^2 = 1$ to (1.1) (in fact, it always holds $\mathbf{x}_{d+1} = 1$), so that the update (1.2) is consistent. In view of this, throughout this paper, we can without loss of generality assume that d of (1.1) is always even.

While this paper is mainly focused on the convergence study of BSVMM in a cyclic manner, it is possible to consider updating the blocks overlappingly, which is similar to [20, Section 3]. For example, consider the third order case: one can update the blocks as follows:

$$(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)}, \mathbf{x}_3^{(k)}) \to (\mathbf{x}_1^{(k+0.5)}, \mathbf{x}_2^{(k+0.5)}, \mathbf{x}_3^{(k)}) \to (\mathbf{x}_1^{(k+0.5)}, \mathbf{x}_2^{(k+1)}, \mathbf{x}_3^{(k+0.5)}) \to (\mathbf{x}_1^{(k+1)}, \mathbf{x}_2^{(k+1)}, \mathbf{x}_3^{(k+1)}),$$

where "$^{(k+0.5)}$" denotes the intermediate variables. That is, one first updates $(\mathbf{x}_1, \mathbf{x}_2)$; then $(\mathbf{x}_2, \mathbf{x}_3)$; finally $(\mathbf{x}_1, \mathbf{x}_3)$. Concerning the convergence, in fact, there is not much difference between the overlapping and the cyclic updates, as will be briefly discussed in Appendix D.

The motivation of studying BSVMM and its convergence is three-fold: Firstly, we find that BSVMM has its own advantages when comparing with HOPM, as will be discussed later; secondly, the convergence of

BSVMM cannot be covered by existing convergence results, and in order to understand its convergence, it is necessary to exploit the special structure of the problem; thirdly, BSVMM and its convergence give help to study a method based on eigenvalue maximization for the problem of symmetric rank-1 approximation to a symmetric tensor, which will be introduced and studied in Section 4.

Although the update scheme (1.2) has been introduced in [16, Section 3.3], it seems that it draws less attention than HOPM and lacks a systematic understanding of its convergence. Therefore, the first concern of this work attempts to give an understanding of the convergence rate, subsequence convergence as well as global convergence of BSVMM and its variants. *Subsequence convergence* means that a limit point is a singular vector tuple or a partial maximizer (see Section 2.1 for the definitions of singular vector tuple and partial maximizer of a tensor), while *global convergence* indicates that the whole sequence converges to a singular vector tuple or a partial maximizer. In what follows, we first give some comparisons between BSVMM and HOPM.

1.1 Comparisons with HOPM: Connections, advantages, and difficulties arising from convergence study

BSVMM has close connections with HOPM. First, BSVMM can be reduced to HOPM in the following sense: Consider the 4-th order case (and it can be generalized to higher order cases naturally): when computing the subproblem (1.2), if one applies the matrix power method to $\mathcal{A} \times_3 \mathbf{x}_3^{(k)} \times_4 \mathbf{x}_4^{(k)}$, starting from the initial guess $(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)})$, and terminating within one iteration, namely,

$$\begin{aligned} \mathbf{x}_1^{(k+1)} &= \mathbf{y}^{(k+1)}/\|\mathbf{y}^{(k+1)}\|, \quad \text{where} \quad \mathbf{y}^{(k+1)} = \mathcal{A} \times_2 \mathbf{x}_2^{(k)\top} \times_3 \mathbf{x}^{(k)\top} \times_4 \mathbf{x}_4^{(k)\top} \quad \text{and} \\ \mathbf{x}_2^{(k+1)} &= \mathbf{y}^{(k+1)}/\|\mathbf{y}^{(k+1)}\|, \quad \text{where} \quad \mathbf{y}^{(k+1)} = \mathcal{A} \times_1 \mathbf{x}_1^{(k+1)\top} \times_3 \mathbf{x}_3^{(k)\top} \times_4 \mathbf{x}_4^{(k)\top}, \end{aligned}$$

then BSVMM boils down to HOPM. On the other hand, HOPM can be seen as a sort of special case of BSVMM, in that $\mathbf{x}_i^{(k+1)}$ in the subproblem of HOPM scheme can be seen as the leading singular vector of the "matrix" $\mathbf{y}_i^{(k+1)}$.

It seems it is too early to argue the advantages of BSVMM here. In fact, discussing its advantages is merely to motivate our convergence study and to show that BSVMM can also be an alternative method for solving (1.1). In the sequel, we attempt to discuss some aspects on which BSVMM might be superior.

The first aspect can be seen as follows: Suppose HOPM and BSVMM start from the same initial guess $\boldsymbol{x}^{(0)}$, which "happens" to be a singular vector tuple of $\mathcal{A}$, with singular value $\sigma_0 > 0$. Moreover, $\boldsymbol{x}^{(0)}$ "happens" to have the following property: Denote $A_i := \mathcal{A} \times_1 \mathbf{x}_1^{(0)\top} \cdots \times_{2i-2} \mathbf{x}_{2i-2}^{(0)\top} \times_{2i+1} \mathbf{x}_{2i+1}^{(0)\top} \cdot \times_d \mathbf{x}_d^{(0)\top}$; there is at least a pair $(\mathbf{x}_{2i-1}^{(0)}, \mathbf{x}_{2i}^{(0)})$ which is not a leading singular vector pair of A_i, i.e., σ_0 is not the leading singular value of A_i. Since $\boldsymbol{x}^{(0)}$ is already a singular vector tuple and $\sigma_0 > 0$, HOPM gets stuck at $\boldsymbol{x}^{(0)}$. Considering BSVMM, since $(\mathbf{x}_{2i-1}^{(0)}, \mathbf{x}_{2i}^{(0)})$ is not a leading singular vector pair, it will find a different pair $(\mathbf{x}_{2i-1}^{(1)}, \mathbf{x}_{2i}^{(1)})$ and yield a strictly larger objective value than σ_0 until convergence. To be more concrete, consider a third order 2 dimensional tensor $\mathcal{A}$ given by

$$a_{111} = 10, a_{112} = 10, a_{221} = 1, a_{222} = 1, \text{ and } a_{ijk} = 0 \text{ otherwise.}$$

Let $\mathbf{x}_1^{(0)} = [0,1]^\top$, $\mathbf{x}_2^{(0)} = [0,1]^\top$ and $\mathbf{x}_3^{(0)} = [1/\sqrt{2}, 1/\sqrt{2}]^\top$. Then it can be verified that $\boldsymbol{x}^{(0)}$ is a singular vector tuple of $\mathcal{A}$, with $\sqrt{2}$ being the associated singular value. If HOPM starts from $\boldsymbol{x}^{(0)}$, then it gets stuck at that point. While for BSVMM, since $\mathcal{A} \times_3 \mathbf{x}_3^{(0)\top} = \left(\begin{smallmatrix} 10\sqrt{2} & 0 \\ 0 & 2/\sqrt{2} \end{smallmatrix}\right)$, we easily get $\mathbf{x}_1^{(1)} = \mathbf{x}_2^{(1)} = [1,0]^\top$ as the leading singular vector pair of $\mathcal{A} \times_3 \mathbf{x}_3^{(0)\top}$; then we get $\mathbf{x}_3^{(1)} = \mathbf{x}_3^{(0)}$, from which we see that $\cdots = \boldsymbol{x}^{(K)} = \cdots = \boldsymbol{x}^{(2)} = \boldsymbol{x}^{(1)}$ is the singular vector tuple generated by BSVMM, with the associated singular value being $10\sqrt{2}$, which is in fact the global optimum of the problem.

Another aspect concerns missing entries. Suppose that a fourth order tensor $\mathcal{A}$ is given, and that for instance, its entry a_{1111} is missing. Then $\mathcal{A} \times \mathbf{x}_3^\top \times \mathbf{x}_4^\top$ is a matrix with the first entry missing, and $\mathbf{x}_1$, $\mathbf{x}_2$ can

be determined by solving the matrix completion problem $\min_{\|\mathbf{x}_1\|=\|\mathbf{x}_2\|=1,\lambda\in\mathbb{R}} \sum_{i,j\neq 1}(B_{ij}-\lambda \mathbf{x}_{1,i}\mathbf{x}_{2,j})^2$, where $B=\mathcal{A}\times \mathbf{x}_3^\top \times \mathbf{x}_4^\top$. $\mathbf{x}_3$ and $\mathbf{x}_4$ can be computed in a similar manner. On the other hand, in HOPM we have to normalize a vector that is by itself incomplete. In this case, BSVMM can be used in the context of tensor completion while HOPM cannot.

The last aspect that BSVMM might be superior comes from our empirical study. Empirically, we observe that BSVMM yields larger objective value than HOPM in an average sense, while its efficiency is comparable or better than HOPM, especially when d is high and n_i's are not too large. More details are left to the experimental section.

Before discussing the difficulties in the convergence study of BSVMM, we shall review two recent works by Wang and Chu [53] and Uschmajew [51] on the global convergence of HOPM. In [53], it is shown that singular vector tuples of a generic tensor are isolated, which, together with that the consecutive differences of the generated rank-1 tensors converge to zero, gives the global convergence of HOPM for a generic tensor. [51] studied a reformulation of HOPM, showing that based on the Łojasiewicz inequality, the consecutive differences are absolutely summable, finally implying the global convergence.

However, considering BSVMM, even proving its subsequence convergence is nontrivial. This is because unlike HOPM, the subproblem (1.2) of BSVMM might have infinitely many global solutions. In coordinate descent, the lack of the uniqueness of the global solution to the subproblem might result in the failure of the convergence. While in [6, Proposition 2.7.1], it is shown that if the uniqueness property holds at each iteration, then subsequence convergence can be possessed. Although we have not found counterexamples for the failure of the subsequence convergence of BSVMM, the lack of the uniqueness property indeed leads to some difficulties in the analysis of BSVMM. Besides [6], a variety of research has also been focused on the convergence of block coordinate ascent/descent under various assumptions, e.g., the strong convexity of the objective [36], convexity [35,49], strict quasiconvexity with respect to $d-2$ blocks or pseudoconvexity [22], and pseudoconvexity in every pair of blocks among $d-1$ blocks or the uniqueness of the global solutions in $d-1$ blocks [50]. Unfortunately, these results cannot be applied, as BSVMM does not meet the aforementioned assumptions.

The lack of such property also causes troubles in the global convergence analysis. Without this property, it is almost impossible to obtain the decreasing of the consecutive differences, which is crucial in proving the global convergence.

To overcome these difficulties, assuming the uniqueness for each subproblem over all iterates seems to be necessary. Nonetheless, by exploiting the properties of the subproblem (1.2), the assumption can be weakened to that only assuming such uniqueness for the subproblems associated with limit points is enough. Besides, considering variants of BSVMM can overcome the difficulties as well. Details are left to Section 3.

1.2 Main results and organizations

The following results are obtained in the present paper:

- Concerning the convergence rate, BSVMM is shown to have an $O(1/K)$ rate of convergence in an ergodic sense under certain measurements (Theorem 3.1).
- Concerning the subsequence convergence, it is shown that if the subproblem (1.2) associated with a certain limit point has a unique solution, then this limit point is a partial maximizer (Theorem 3.2).
- Concerning the global convergence, under the above uniqueness property, for a generic tensor, the whole sequence generated by BSVMM is shown to converge to a partial maximizer (Theorem 3.3); if all the

limit points have the above uniqueness property, then the global convergence holds for all the tensors (Theorem 3.5).

- By imposing perturbations to the objective of (1.1), and by applying the block conditional gradient method with unit step-size, a variant of BSVMM is introduced (Scheme 3.21), with its convergence being studied for a more general class of problems. Specifically, it is shown that if maximizing a block strongly convex function by the block conditional gradient method with unit step-size, then under certain assumptions, the method globally converges (Theorem 3.6). Another variant of BSVMM, obtained by carefully choosing the global solution to (1.2), is introduced and the convergence is analyzed as well (Scheme 3.27).
- The convergence of an eigenvalue maximization method and its variant for symmetric rank-1 approximation to symmetric tensors is studied.

The remainder of this paper is organized as follows. In the next section, some definitions, basic facts as well as equivalent reformulations of (1.1) and BSVMM are introduced. The convergence of BSVMM and its variants are conducted in Section 3. The method for finding symmetric rank-1 approximation is introduced and analyzed in Section 4. Numerical experiments are carried out in Section 5, with conclusions drawn in Section 6.

2 Preliminaries

2.1 Optimality conditions

By using Lagrangian multipliers, the first order optimality condition of (1.1) is

$$\begin{cases} \mathcal{A} \times_2 \mathbf{x}_2^\top \cdots \times_d \mathbf{x}_d^\top & = \sigma \mathbf{x}_1, \\ \qquad\qquad \vdots & \\ \mathcal{A} \times_1 \mathbf{x}_1^\top \cdots \times_{d-1} \mathbf{x}_{d-1}^\top & = \sigma \mathbf{x}_d, \\ \sigma \in \mathbb{R}, \|\mathbf{x}_i\| = 1, \mathbf{x}_i \in \mathbb{R}^{n_i}, 1 \le i \le d. & \end{cases}$$

Here σ is defined as a singular value of $\mathcal{A}$, with $\boldsymbol{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_d)$ being a singular vector tuple associated with σ [33].

Fact 1. *$\boldsymbol{x}$ is a singular vector tuple of $\mathcal{A}$, associated with singular value σ, iff for each i, $(\mathbf{x}_{2i-1}, \mathbf{x}_{2i})$ is a pair of left and right singular vectors corresponding to the singular value σ of the matrix $\mathcal{A} \times_1 \mathbf{x}_1^\top \cdots \times_{2i-2} \mathbf{x}_{2i-2}^\top \times_{2i+1} \mathbf{x}_{2i+1}^\top \cdots \times_d \mathbf{x}_d^\top$.*

In view of the structure of BSVMM, it is not only expected that the sequence generated by BSVMM converges to a singular vector tuple, but also converges to a partial maximizer, which is defined as follows:

Definition 2.1 (Partial maximizer)**.** *A tuple $\boldsymbol{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_d)$ is called a partial maximizer of $\mathcal{A}$, if for $i = 1, \ldots, d/2$,*

$$\begin{aligned} (\mathbf{x}_{2i-1}, \mathbf{x}_{2i}) \in \arg\max \ & \mathbf{y}^\top (\mathcal{A} \times_1 \mathbf{x}_1^\top \cdots \times_{2i-2} \mathbf{x}_{2i-2}^\top \times_{2i+1} \mathbf{x}_{2i+1}^\top \cdots \times_d \mathbf{x}_d^\top) \mathbf{z} \\ & \text{s.t. } \|\mathbf{y}\| = \|\mathbf{z}\| = 1. \end{aligned} \tag{2.3}$$

Fact 2. *If $\boldsymbol{x}$ is a partial maximizer of $\mathcal{A}$, then it must be a singular vector tuple.*

2.2 Equivalent reformulations of (1.1) and BSVMM

To give another viewpoint of BSVMM, we consider equivalent reformulations of (1.1) and BSVMM.

One notices that the matrix singular value problem $(\mathbf{x},\mathbf{y}) \in \arg\max_{\|\mathbf{x}\|=\|\mathbf{y}\|=1} \mathbf{x}^\top A\mathbf{y}$ can be written as $X \in \arg\max_{\|X\|_F=1,\mathrm{rank}(X)=1}\langle A, X\rangle$. By denoting $(\mathbf{x}_{2i-1},\mathbf{x}_{2i})$ as $X_{2i-1,2i} := \mathbf{x}_{2i-1}\mathbf{x}_{2i}^\top \in \mathbb{R}^{n_{2i-1}\times n_{2i}}$, (1.1) can be equivalently written as

$$\begin{aligned} \max\ & F(X_{1,2},\ldots,X_{d-1,d}) := \mathcal{A}\times_{1,2} X_{1,2}\times_{3,4} X_{3,4}\cdots\times_{d-1,d} X_{d-1,d} \\ \text{s.t.}\ & \|X_{2i-1,2i}\|_F = 1,\ \mathrm{rank}(X_{2i-1,2i}) = 1,\ 1\le i\le d/2. \end{aligned} \tag{1.1$'$}$$

Here we denote

$$\mathcal{A}\times_{2i-1,2i} X_{2i-1,2i} := \mathcal{A}\times_{2i-1}\mathbf{x}_{2i-1}^\top\times_{2i}\mathbf{x}_{2i}^\top.$$

Denote $\boldsymbol{X} := (X_{1,2},\ldots,X_{d-1,d})$ and

$$\nabla_{X_{2i-1,2i}}F(\boldsymbol{X}) := \mathcal{A}\times_{1,2} X_{1,2}\cdots\times_{2i-3,2i-2} X_{2i-3,2i-2}\times_{2i+1,2i+2} X_{2i+1,2i+2}\cdots\times_{d-1,d} X_{d-1,d}.$$

The first order optimality condition of (1.1$'$) can be characterized as

$$\left\langle \nabla_{X_{2i-1,2i}}F(\boldsymbol{X}^*), X^*_{2i-1,2i} - X\right\rangle \ge 0,\quad \forall X\in C_{2i-1,2i},\ 1\le i\le d/2, \tag{2.4}$$

where

$$C_{2i-1,2i} := \{X\in\mathbb{R}^{n_{2i-1}\times n_{2i}} \mid \|X_{2i-1,2i}\|_F = 1,\ \mathrm{rank}(X_{2i-1,2i}) = 1\}.$$

This is exactly the same as the definition of partial maximizer, i.e., $\boldsymbol{X}^*$ satisfies (2.4) $\Leftrightarrow$ $(\mathbf{x}_1^*,\ldots,x_d^*)$ is a partial maximizer of $\mathcal{A}$. Throughout this paper, we also call $\boldsymbol{X}^*$ a partial maximizer if it meets (2.4).

With the above notations, and further denote

$$\boldsymbol{X}^{(k+1)}_{2i-1,2i} := (X^{(k+1)}_{1,2},\ldots,X^{(k+1)}_{2i-1,2i},X^{(k)}_{2i+1,2i+2},\ldots,X^{(k)}_{d-1,d}) \text{ and } \boldsymbol{X}^{(k+1)}_{-1,0} := \boldsymbol{X}^{(k)}_{d-1,d} = \boldsymbol{X}^{(k)};$$

BSVMM can be rewritten as follows: At the $(k+1)$-th iteration, for $i = 1,\ldots,d/2$, compute

$$X^{(k+1)}_{2i-1,2i} \in \arg\max \left\langle \nabla_{X_{2i-1,2i}}F(\boldsymbol{X}^{(k+1)}_{2i-3,2i-2}), X\right\rangle \quad \text{s.t.}\quad X\in C_{2i-1,2i}. \tag{1.2$'$}$$

From this point of view, at each subproblem, BSVMM (1.2$'$) solves a linear oracle, and thus it can be seen as a *multi block* version of the *conditional gradient method with unit step-size*[1] (CG-US for short), which is a generalization of the single block CG-US [38]. This viewpoint, together with the reformulation (1.1$'$), gives rise to a variant of BSVMM, as will be introduced in the next section.

Problem (1.1$'$) can also be written as

$$\max_{\boldsymbol{X}}\ G(\boldsymbol{X}) := F(\boldsymbol{X}) - \sum\nolimits_{i=1}^{d/2} I_{C_{2i-1,2i}}(X_{2i-1,2i}), \tag{1.1$''$}$$

where $I_{C_{2i-1,2i}}(\cdot)$ denotes the indicator function of $C_{2i-1,2i}$

$$I_{C_{2i-1,2i}}(X) = \begin{cases} 0 & \text{if } X\in C_{2i-1,2i},\\ +\infty & \text{otherwise.}\end{cases}$$

$\overline{\boldsymbol{X}}$ is said to be a critical point of (1.1$''$) if $0\in\partial G(\overline{\boldsymbol{X}})$, i.e.,

$$\nabla_{X_{2i-1,2i}}F(\overline{\boldsymbol{X}}_{2i-1,2i}) \in \partial I_{C_{2i-1,2i}}(\overline{X}_{2i-1,2i}),\qquad 1\le i\le d/2,$$

where $\partial G(\cdot)$ denotes the limiting subdifferential (see e.g. [39, Chapter 1]). From the definition of the limiting subdifferential, it can be seen that if $\boldsymbol{X}^*$ is a partial maximizer, then $0\in\partial G(\boldsymbol{X}^*)$. The notations and reformulations introduced in this paragraph is useful for studying the global convergence of BSVMM and its variants.

[1] Briefly speaking, the conditional gradient method, also known as the Frank-Wolfe method, is originally proposed to solve linearly constrained quadratic programs [19]. Given the problem $\min_{\mathbf{x}\in C} F(\mathbf{x})$, the method generates the sequence as $\mathbf{x}^{(k+1)} = (1-\alpha^k)\mathbf{x}^{(k)} + \alpha^k\mathbf{d}^{(k)}$, $\alpha^k\in(0,1]$, where $\mathbf{d}^{(k)}\in\arg\min_{\mathbf{d}\in C}\langle\nabla F(\mathbf{x}^{(k)}),\mathbf{d}\rangle$.

3 Convergence Study of BSVMM and its Variants

3.1 BSVMM: Convergence rate

As an ascent method, the objective value of the sequence generated by BSVMM is not decreasing, i.e.,

$$\cdots \le F(\boldsymbol{X}^{(k-1)}) \le F(\boldsymbol{X}^{(k)}_{1,2}) \le F(\boldsymbol{X}^{(k)}_{3,4}) \le \cdots \le F(\boldsymbol{X}^{(k)}_{d-1,d}) = F(\boldsymbol{X}^{(k)}) \le F(\boldsymbol{X}^{(k+1)}_{1,2}) \le \cdots .$$

We note that throughout the analysis, $\boldsymbol{X}^{(0)}$ is chosen with $F(\boldsymbol{X}^{(0)}) \neq 0$ (in fact, it can be chosen with $F(\boldsymbol{X}^{(0)}) > 0$ by reversing the sign of one $X^{(0)}_{2i-1,2i}$ in $\boldsymbol{X}^{(0)}$ if $F(\boldsymbol{X}^{(0)}) < 0$), to exclude the cases $F(\boldsymbol{X}^{(k)}) \equiv 0$ that we are not interested in.

By considering the relation between spectral norm and nuclear norm, we can say some more on the convergence rate.

Theorem 3.1. *Let $\{\boldsymbol{X}^{(k)}\}_{k=0}^{\infty}$ be generated by BSVMM (1.2′). Then*

1. $$\left(\left\|X^{(k+1)}_{2i-1,2i} + X^{(k)}_{2i-1,2i}\right\|_* - 1\right) F(\boldsymbol{X}^{(k+1)}_{2i-1,2i}) \ge F(\boldsymbol{X}^{(k+1)}_{2i-3,2i-2}), \quad \forall k,\ 1 \le i \le d/2,$$
   where $\|\cdot\|_*$ denotes the nuclear norm of a matrix, defined as the sum of its singular values.

2. $$\left\|X^{(k+1)}_{2i-1,2i} + X^{(k)}_{2i-1,2i}\right\|_* \le 2 \quad \text{and} \quad \lim_{k\to\infty} \left\|X^{(k+1)}_{2i-1,2i} + X^{(k)}_{2i-1,2i}\right\|_* = 2.$$

3. *There holds*
   $$\min_{k=0,\ldots,K} \left(1 - \prod_{i=1}^{d/2} \left(\left\|X^{(k+1)}_{2i-1,2i} + X^{(k)}_{2i-1,2i}\right\|_* - 1\right)\right) \le \frac{F(\boldsymbol{X}^{(K)}) - F(\boldsymbol{X}^{(0)})}{F(\boldsymbol{X}^{(0)})K},$$
   i.e., $\|X^{(k+1)}_{2i-1,2i} + X^{(k)}_{2i-1,2i}\|_*$ converges to 2 with rate $O(1/K)$ in an ergodic sense.

Proof. 1. From scheme (1.2′), it holds that $F(\boldsymbol{X}^{(k+1)}_{2i-1,2i}) = \|\nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^{(k+1)}_{2i-3,2i-2})\|_2$, where $\|\cdot\|_2$ is the spectral norm of a matrix. Using the duality between spectral norm and nuclear norm, and noticing the definition of $X^{(k+1)}_{2i-1,2i}$, we have

$$\begin{aligned}
\|\nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^{(k+1)}_{2i-1,2i})\|_2 &= \left\langle \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^{(k+1)}_{2i-3,2i-2}), X^{(k+1)}_{2i-1,2i} \right\rangle = \max_{\|X\|_*=1} \left\langle \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^{(k+1)}_{2i-3,2i-2}), X \right\rangle \\
&\ge \left\langle \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^{(k+1)}_{2i-3,2i-2}), \frac{X^{(k+1)}_{2i-1,2i} + X^{(k)}_{2i-1,2i}}{\left\|X^{(k+1)}_{2i-1,2i} + X^{(k)}_{2i-1,2i}\right\|_*} \right\rangle.
\end{aligned}$$

Rearranging the terms and noticing that $F(\boldsymbol{X}^{(k+1)}_{2i-3,2i-2}) = \langle \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^{(k+1)}_{2i-3,2i-2}), X^{(k)}_{2i-1,2i} \rangle$, it follows

$$\left(\left\|X^{(k+1)}_{2i-1,2i} + X^{(k)}_{2i-1,2i}\right\|_* - 1\right) F(\boldsymbol{X}^{(k+1)}_{2i-1,2i}) \ge F(\boldsymbol{X}^{(k+1)}_{2i-3,2i-2}). \tag{3.5}$$

Thus the first relation is verified.

2. From the definition of the nuclear norm

$$\|X\|_* = \inf\{\textstyle\sum_{j=1} \sigma_j \mid X = \sum_{j=1} \sigma_j X_j,\ \|X_j\|_F = 1,\ \mathrm{rank}(X_j) = 1\},$$

it follows that since $Z := X^{(k+1)}_{2i-1,2i} + X^{(k)}_{2i-1,2i}$ is the sum of two normalized rank one matrices, and it holds that

$$\|X^{(k+1)}_{2i-1,2i} + X^{(k)}_{2i-1,2i}\|_* = \|Z\|_* \le 2. \tag{3.6}$$

In fact, the equality holds only if $\langle X_{2i-1,2i}^{(k+1)}, X_{2i-1,2i}^{(k)}\rangle = 0$ or $X_{2i-1,2i}^{(k+1)} = X_{2i-1,2i}^{(k)}$. Since the sequence $\{F(\boldsymbol{X}_{2i-1,2i}^{(k)})\}$ is non-decreasing and bounded, this together with (3.5) and (3.6) implies that

$$\lim_{k\to\infty} \left\| X_{2i-1,2i}^{(k+1)} + X_{2i-1,2i}^{(k)} \right\|_* = 2. \tag{3.7}$$

3. Multiplying (3.5) from $i = 1$ to $d/2$, we get

$$\prod_{i=1}^{d/2} \left(\left\| X_{2i-1,2i}^{(k+1)} + X_{2i-1,2i}^{(k)} \right\|_* - 1 \right) \geq \frac{F(\boldsymbol{X}^{(k)})}{F(\boldsymbol{X}_{1,2}^{(k+1)})} \cdots \frac{F(\boldsymbol{X}_{d-3,d-2}^{(k+1)})}{F(\boldsymbol{X}^{(k+1)})} = \frac{F(\boldsymbol{X}^{(k)})}{F(\boldsymbol{X}^{(k+1)})}. \tag{3.8}$$

Then,

$$1 - \prod_{i=1}^{d/2} \left(\left\| X_{2i-1,2i}^{(k+1)} + X_{2i-1,2i}^{(k)} \right\|_* - 1 \right) \leq \frac{F(\boldsymbol{X}^{(k+1)}) - F(\boldsymbol{X}^{(k)})}{F(\boldsymbol{X}^{(k+1)})} \leq \frac{F(\boldsymbol{X}^{(k+1)}) - F(\boldsymbol{X}^{(k)})}{F(\boldsymbol{X}^{(0)})},$$

summing up of which from $k = 0$ to K yields

$$\sum_{k=0}^{K} \left(1 - \prod_{i=1}^{d/2} \left(\left\| X_{2i-1,2i}^{(k+1)} + X_{2i-1,2i}^{(k)} \right\|_* - 1 \right) \right) \leq \frac{F(\boldsymbol{X}^{(K)}) - F(\boldsymbol{X}^{(0)})}{F(\boldsymbol{X}^{(0)})},$$

and so

$$\min_{k=0,\ldots,K} \left(1 - \prod_{i=1}^{d/2} \left(\left\| X_{2i-1,2i}^{(k+1)} + X_{2i-1,2i}^{(k)} \right\|_* - 1 \right) \right) \leq \frac{F^* - F(\boldsymbol{X}^{(0)})}{F(\boldsymbol{X}^{(0)})K};$$

namely, $\|X_{2i-1,2i}^{(k+1)} + X_{2i-1,2i}^{(k)}\|_*$ converges to 2 with rate $O(1/K)$ in an ergodic sense. This completes the proof. □

3.2 BSVMM: Subsequence convergence

As mentioned in the introduction, the non-uniqueness of the global solution of (1.2′) causes troubles in the convergence analysis. Therefore, throughout this subsection, we assume that there is a limit point satisfying the following assumption.

Assumption 3.1. *For a tuple $\boldsymbol{X}$, we assume that the leading singular value of each matrix $\nabla_{X_{2i-1,2i}} F(\boldsymbol{X})$ is a simple root, $1 \leq i \leq d/2$. In other words, let $\sigma_1, \ldots, \sigma_r$ be singular values of $\nabla_{X_{2i-1,2i}} F(\boldsymbol{X})$; then $\sigma_1 > \sigma_2$.*

Theorem 3.2. *Let $\{\boldsymbol{X}^{(k)}\}$ be generated by BSVMM and let $\boldsymbol{X}^*$ be a limit point of $\{\boldsymbol{X}^{(k)}\}$. If $\boldsymbol{X}^*$ satisfies Assumption 3.1, then $\boldsymbol{X}^*$ meets the first order optimality condition (2.4) and hence its associated vector tuple $(\mathbf{x}_1^*, \ldots, \mathbf{x}_d^*)$ is a partial maximizer.*

Let $\{\boldsymbol{X}^{(k)}\}_{k\in\kappa} \to \boldsymbol{X}^*$, where κ denotes a subset of the index set. The key step is to prove that $\{\boldsymbol{X}^{(k+1)}\}_{k\in\kappa}$, which consists of the points next to those belonging to κ, also converges to $\boldsymbol{X}^*$. To the end of this we first prove the following lemma.

Lemma 3.1. *Let $\boldsymbol{X}^*$ be a limit point of $\{\boldsymbol{X}^{(k)}\}$ and $\{\boldsymbol{X}^{(k)}\}_{k\in\kappa} \to \boldsymbol{X}^*$. Then there exists a constant $c_0 > 0$, such that when $k(\in \kappa)$ is sufficiently large, there holds*

$$F(\boldsymbol{X}_{1,2}^{(k+1)}) - F(\boldsymbol{X}^{(k)}) \geq \frac{c_0}{2} \|X_{1,2}^{(k+1)} - X_{1,2}^{(k)}\|_F^2, \quad k \in \kappa. \tag{3.9}$$

Proof. Since $\boldsymbol{X}^*$ satisfies Assumption 3.1, by matrix theory, there must exist a positive number ϵ and a ball $\mathbb{B}(\boldsymbol{X}^*, \epsilon) := \{\boldsymbol{X} \mid \|\boldsymbol{X} - \boldsymbol{X}^*\|_F \leq \epsilon\}$, such that for all $\boldsymbol{X} \in \mathbb{B}(\boldsymbol{X}^*, \epsilon)$, Assumption 3.1 still holds for these $\boldsymbol{X}$. Since $\{\boldsymbol{X}^{(k)}\}_{k\in\kappa} \to \boldsymbol{X}^*$, there must exist a K_0 such that for all $k > K_0$, $\{\boldsymbol{X}^{(k)}\}_{k\in\kappa, k>K_0} \subset B(\boldsymbol{X}^*, \epsilon)$.

Let $\boldsymbol{X}^{(k_0)}$ be one of these points. Let $\nabla_{X_{1,2}} F(\boldsymbol{X}_{-1,0}^{(k_0+1)}) = U\Lambda V^\top$ be its SVD, with $U = [\mathbf{u}_1, \ldots, \mathbf{u}_r] \in \mathbb{R}^{n_1 \times r}$, $V = [\mathbf{v}_1, \ldots, \mathbf{v}_r] \in \mathbb{R}^{n_2 \times r}$, and $\Lambda = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$. Further denote $\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_{n_1}\}$ and $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_{n_2}\}$ as orthonormal basis of $\mathrm{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}^\perp$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}^\perp$, respectively.

Since $X_{1,2}^{(k_0)} \in C_{1,2}$, we denote $X_{1,2}^{(k_0)} = \mathbf{u}\mathbf{v}^\top$, $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$, and let $\mathbf{u} = \sum_{j_1=1}^{n_1} \alpha_{j_1} \mathbf{u}_{j_1}$ and $\mathbf{v} = \sum_{j_2=1}^{n_2} \beta_{j_2} \mathbf{v}_{j_2}$ be the representations of $\mathbf{u}$ and $\mathbf{v}$ under the basis $\{\mathbf{u}_{j_1}\}_{j_1=1}^{n_1}$ and $\{\mathbf{v}_{j_2}\}_{j_2=1}^{n_2}$. In particular, it holds $\sum_{j_1=1}^{n_1} \alpha_{j_1}^2 = 1$ and $\sum_{j_2=1}^{n_2} \beta_{j_1}^2 = 1$. On the other hand, since $\boldsymbol{X}^{k_0} \in \mathbb{B}(\boldsymbol{X}^*, \epsilon)$, the leading singular value of $\nabla_{X_{1,2}} F(\boldsymbol{X}^{(k_0)})$ is simple, and so $X_{1,2}^{(k_0+1)} = \mathbf{u}_1 \mathbf{v}_1^\top$ is the unique maximizer.

We then compare $F(\boldsymbol{X}_{1,2}^{(k_0+1)}) - F(\boldsymbol{X}^{(k_0)})$ and $\|X_{1,2}^{(k_0+1)} - X_{1,2}^{(k_0)}\|_F^2$.

$$\begin{aligned} F(\boldsymbol{X}_{1,2}^{(k_0+1)}) - F(\boldsymbol{X}^{(k_0)}) &= \left\langle X_{1,2}^{(k_0+1)}, \nabla_{X_{1,2}} F(\boldsymbol{X}^{(k_0)}) \right\rangle - \left\langle X_{1,2}^{(k_0)}, \nabla_{X_{1,2}} F(\boldsymbol{X}^{(k_0)}) \right\rangle \\ &= \sigma_1 - \sum_{j_1=1, j_2=1} \alpha_{j_1} \beta_{j_2} \mathbf{u}_{j_1}^\top \nabla_{X_{1,2}} F(\boldsymbol{X}^{(k_0)}) \mathbf{v}_{j_2} \\ &= \sigma_1 - \sum_{j=1}^{r} \alpha_j \beta_j \sigma_j. \end{aligned} \tag{3.10}$$

On the other hand,

$$\begin{aligned} \|X_{1,2}^{(k_0+1)} - X_{1,2}^{(k_0)}\|_F^2 &= 2 - 2\langle X_{1,2}^{(k_0+1)}, X_{1,2}^{(k_0)} \rangle \\ &= 2 - 2\mathbf{u}_1^\top \mathbf{u} \cdot \mathbf{v}_1^\top \mathbf{v} = 2 - 2\mathbf{u}_1^\top (\sum_{j_1=1} \alpha_{j_1} \mathbf{u}_{j_1}) \cdot \mathbf{v}_1^\top (\sum_{j_2=1} \beta_{j_2} \mathbf{v}_{j_2}) \\ &= 2(1 - \alpha_1 \beta_1). \end{aligned} \tag{3.11}$$

Comparing (3.10) with (3.11) we have

$$F(\boldsymbol{X}_{1,2}^{(k_0+1)}) - F(\boldsymbol{X}^{(k_0)}) - \frac{\sigma_1 - \sigma_2}{2} \|X_{1,2}^{(k_0+1)} - X_{1,2}^{(k_0)}\|_F^2 = \sigma_2(1 - \alpha_1\beta_1) - \sum_{j=2}^{r} \alpha_j \beta_j \sigma_j \geq \sigma_2 + \min_{\|\overline{\alpha}\| = \|\overline{\beta}\| = 1} \overline{\alpha}^\top B \overline{\beta}, \tag{3.12}$$

where $\overline{\alpha} \in \mathbb{R}^{n_1}$, $\overline{\beta} \in \mathbb{R}^{n_2}$, and

$$B := [\mathrm{diag}(-\sigma_2, -\sigma_2, \ldots, -\sigma_r, 0, \ldots, 0)\ \mathbf{0}] \in \mathbb{R}^{n_1 \times n_2}, \qquad \text{(w.l.o.g. assume that } n_1 \leq n_2\text{)}.$$

It is then clear that $\min_{\|\overline{\alpha}\| = \|\overline{\beta}\| = 1} \overline{\alpha}^\top B \overline{\beta} = -\sigma_2$, and so

$$F(\boldsymbol{X}_{1,2}^{(k_0+1)}) - F(\boldsymbol{X}^{(k_0)}) \geq \frac{\sigma_1 - \sigma_2}{2} \|X_{1,2}^{(k_0+1)} - X_{1,2}^{(k_0)}\|_F^2. \tag{3.13}$$

Finally, it is clear that for all $\boldsymbol{X} \in \mathbb{B}(\boldsymbol{X}^*, \epsilon)$, there exists a constant $c_0 > 0$, such that $\inf_{\boldsymbol{X} \in \mathbb{B}(\boldsymbol{X}^*, \epsilon)} \left(\sigma_1(\nabla_{X_{1,2}} F(\boldsymbol{X})) - \sigma_2(\nabla_{X_{1,2}} F(\boldsymbol{X}))\right) \geq c_0 > 0$, where $\sigma_1(\nabla_{X_{1,2}} F(\boldsymbol{X}))$ and $\sigma_2(\nabla_{X_{1,2}} F(\boldsymbol{X}))$ denote the largest and second largest singular values of $\nabla_{X_{1,2}} F(\boldsymbol{X})$, respectively. This combining with (3.13) yields that when $k \in \kappa$, $k > K_0$, (3.9) holds. □

Since $F(\boldsymbol{X}_{1,2}^{(k+1)}) - F(\boldsymbol{X}^{(k)}) \to 0$, the above lemma implies that $\lim_{k(\in\kappa)\to\infty} X_{1,2}^{(k+1)} = X_{1,2}^*$. Continuing this argument we can prove $\boldsymbol{X}^{(k+1)} \to \boldsymbol{X}^*$, as shown in the following.

Proof of Theorem 3.2. Since $X_{1,2}^{(k+1)} \to X_{1,2}^*$ as $k(\in \kappa) \to \infty$, there must exist a K_1, such that when $k > K_1$, $\{\boldsymbol{X}_{1,2}^{(k+1)}\}_{k \in \kappa, k > K_1} \subset B(\boldsymbol{X}^*, \epsilon)$. Therefore, as the proof of Lemma 3.1, there must exist a $c_1 > 0$, such that when $k \in \kappa,\ k > K_1$, the following relation is true

$$F(\boldsymbol{X}_{3,4}^{(k+1)}) - F(\boldsymbol{X}_{1,2}^{(k+1)}) \geq \frac{c_1}{2} \|X_{3,4}^{(k+1)} - X_{3,4}^{(k)}\|_F^2, \qquad k \in \kappa.$$

Proceeding similarly, there must exist positive constants $c_2, c_3, \ldots, c_{d/2-1}$, such that when $k \in \kappa$, $k > K_i$, $i = 2, \ldots, d/2-1$, there holds

$$F(\boldsymbol{X}_{2i+1,2i+2}^{(k+1)}) - F(\boldsymbol{X}_{2i-1,2i}^{(k+1)}) \geq \frac{c_i}{2}\|X_{2i+1,2i+2}^{(k+1)} - X_{2i+1,2i+2}^{(k)}\|_F^2.$$

Let $K := \max_{0\leq i\leq d/2-1}\{K_i\}$ and $c := \min_{0\leq i\leq d/2-1}\{c_i\}$. Then, summing up the above inequalities yields

$$F(\boldsymbol{X}^{(k+1)}) - F(\boldsymbol{X}^{(k)}) \geq \frac{c}{2}\|\boldsymbol{X}^{(k+1)} - \boldsymbol{X}^{(k)}\|_F^2, \quad k \in \kappa,\ k > K, \tag{3.14}$$

which demonstrates that

$$\lim_{k(\in\kappa)\to\infty} \boldsymbol{X}^{(k+1)} = \boldsymbol{X}^*. \tag{3.15}$$

From the definition of $X_{2i-1,2i}^{(k+1)}$, we have

$$\left\langle \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)}), X - X_{2i-1,2i}^{(k+1)} \right\rangle \leq 0, \quad \forall\ X \in C_{2i-1,2i}.$$

Passing to the limit over κ and taking into account (3.15), it holds that for $i = 1, \ldots, d/2$,

$$\left\langle \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^*), X - X_{2i-1,2i}^* \right\rangle \leq 0, \quad \forall\ X \in C_{2i-1,2i},$$

which is exactly the optimality condition (2.4). This completes the proof. □

In general, it is unclear whether the Assumption 3.1 can be weakened or not. However, when the tensor is of order 3 or 4, without Assumption 3.1 the subsequence convergence can be established as well.

Proposition 3.1. *Let $\mathcal{A}$ be of order 3 or 4. Then every limit point of $\{\boldsymbol{X}^{(k)}\}$ satisfies (2.4).*

Proof. In this setting, BSVMM (1.2′) has two blocks. Assume that $\{\boldsymbol{X}^{(k)}\}_{k\in\kappa} \to \boldsymbol{X}^*$. For the first block and for any $X \in C_{1,2}$,

$$\left\langle \nabla_{X_{1,2}} F(\boldsymbol{X}^{(k)}), X - X_{1,2}^{(k)} \right\rangle \leq \left\langle X_{1,2}^{(k+1)} - X_{1,2}^{(k)}, \nabla_{X_{1,2}} F(\boldsymbol{X}^{(k)}) \right\rangle = F(\boldsymbol{X}_{1,2}^{(k+1)}) - F(\boldsymbol{X}^{(k)}) \to 0;$$

for the second block,

$$\left\langle \nabla_{X_{3,4}} F(\boldsymbol{X}^{(k)}), X - X_{3,4}^{(k)} \right\rangle \leq 0, \quad \forall X \in C_{3,4}.$$

Passing to the limit over the subsequence in the above two inequalities yields the result. □

3.3 BSVMM: Global convergence

Some definitions are introduced first.

Definition 3.1. *Let $\omega(\boldsymbol{X}^{(0)})$ denote the set of limit points of $\{\boldsymbol{X}^{(k)}\}$, started from $\boldsymbol{X}^{(0)}$. A point $\boldsymbol{X}$ is said to be isolated in $\omega(\boldsymbol{X}^{(0)})$, if there exists a number $\epsilon > 0$ such that $\mathbb{B}(\boldsymbol{X}, \epsilon) \cap \omega(\boldsymbol{X}^{(0)}) \setminus \{\boldsymbol{X}\} = \emptyset$.*

Definition 3.2. *Let $M_{\mathcal{A}}$ denote the set of partial maximizers of $\mathcal{A}$. $M_{\mathcal{A}}$ is called level-discrete if for any $c \in \mathbb{R}$, the set $L_{c,M_{\mathcal{A}}} := \{\boldsymbol{X} \in M_{\mathcal{A}} \mid F(\boldsymbol{X}) = c\}$ is either empty or discrete.*

The following proposition, which partly borrows from [18], is crucial in the next theorem.

Proposition 3.2. (c.f. [18, Proposition 8.3.10]) *Let $\{\mathbf{x}^{(k)}\}$ be a sequence with a limit point $\mathbf{x}^*$ which is isolated in $\omega(\mathbf{x}^{(0)})$. If for any subsequence $\{\mathbf{x}^{(k)}\}_{k\in\kappa} \to \mathbf{x}^*$, there holds $\lim_{k(\in\kappa)\to\infty} \|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| = 0$, then the whole sequence $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}^*$.*

With the above definitions and properties, we have the first theorem of this subsection.

Theorem 3.3. *Let $\{\boldsymbol{X}^{(k)}\}$ be generated by BSVMM (1.2′). Then the whole sequence converges to a partial maximizer if*

1. *either there is a limit point isolated in $\omega(\boldsymbol{X}^{(0)})$ and satisfies Assumption 3.1,*
2. *or there is a limit point satisfying Assumption 3.1, and $M_{\mathcal{A}}$ is level-discrete.*

Proof. 1. Under the assumptions, assume that $\boldsymbol{X}^*$ is such a limit point with a subsequence $\{\boldsymbol{X}^{(k)}\}_{k\in\kappa} \to \boldsymbol{X}^*$. According to Theorem 3.2, $\boldsymbol{X}^*$ is a partial maximizer, and $\{\boldsymbol{X}^{(k)}\}_{k\in\kappa}$ satisfies (3.14) when k is sufficiently large, which together with the isolatedness of $\boldsymbol{X}^*$ meets the requirements of Proposition 3.2. Hence $\{\boldsymbol{X}^{(k)}\}$ converges to a partial maximizer.

2. We first show that $F(\cdot)$ is constant on $\omega(\boldsymbol{X}^{(0)})$. Otherwise, assume that $\boldsymbol{X}_1^*, \boldsymbol{X}_2^* \in \omega(\boldsymbol{X}^{(0)})$, with $F(\boldsymbol{X}_1^*) < F(\boldsymbol{X}_2^*)$. Denote $\alpha := (F(\boldsymbol{X}_2^*) - F(\boldsymbol{X}_1^*))/4$. On the other hand assume that $\{\boldsymbol{X}^{(k_1)}\}_{k_1\in\kappa_1} \to \boldsymbol{X}_1^*$ and $\{\boldsymbol{X}^{(k_2)}\}_{k_2\in\kappa_2} \to \boldsymbol{X}_2^*$. By the continuity of $F(\cdot)$, there is a K_1 such that when $k_1 > K_1$, $|F(\boldsymbol{X}^{(k_1)}) - F(\boldsymbol{X}_1^*)| \le \alpha$; similarly there is a K_2 such that when $k_2 > K_2$, $|F(\boldsymbol{X}^{(k_2)}) - F(\boldsymbol{X}^*)| \le \alpha$. Let $\boldsymbol{X}^{(\bar{k}_1)} \in \{\boldsymbol{X}^{(k_1)}\}$ and $\boldsymbol{X}^{(\bar{k}_2)} \in \{\boldsymbol{X}^{(k_2)}\}$, with $\bar{k}_1 > \bar{k}_2 \ge \max\{K_1, K_2\}$. Since $F(\boldsymbol{X}^{(k)})$ is nondecreasing, $F(\boldsymbol{X}^{(\bar{k}_1)}) \ge F(\boldsymbol{X}^{(\bar{k}_2)})$. However,

$$F(\boldsymbol{X}^{(\bar{k}_1)}) - F(\boldsymbol{X}^{(\bar{k}_2)}) = F(\boldsymbol{X}^{(\bar{k}_1)}) - F(\boldsymbol{X}_1^*) + F(\boldsymbol{X}_1^*) - F(\boldsymbol{X}_2^*) + F(\boldsymbol{X}_2^*) - F(\boldsymbol{X}^{(\bar{k}_2)}) \le -2\alpha,$$

which deduces a contradiction. Thus $F(\cdot)$ is constant on $\omega(\boldsymbol{X}^{(0)})$. Combining this conclusion with that $M_{\mathcal{A}}$ is level-discrete and conclusion 1 yields the desired results. □

The above theorem relies on the isolatedness of the partial maximizers, where the assumption holds if the singular vector tuples of a tensor are isolated. Fortunately, the result was proved to hold for a generic tensor; see Friedland and Ottaviani [21] (their paper calculates the exact number of singular vector tuples corresponding to nonzero singular values) and also Wang and Chu [53]. Thanks to their results, the following can be deduced.

Theorem 3.4. *Let $\{\boldsymbol{X}^{(k)}\}$ be generated by BSVMM. For a generic tensor, either none of the points in $\omega(\boldsymbol{X}^{(0)})$ satisfy Assumption 3.1, or the whole sequence converges to a partial maximizer.*

Proof. It suffices to show that if there is a limit point $\boldsymbol{X}^*$ satisfying Assumption 3.1, then $\omega(\boldsymbol{X}^{(0)})$ is a singleton. Suppose not; we consider two cases. Suppose first that $\boldsymbol{X}^*$ is isolated in $\omega(\boldsymbol{X}^{(0)})$. by conclusion 1 of Theorem 3.3, the whole sequence converges to $\boldsymbol{X}^*$ and hence $\omega(\boldsymbol{X}^*)$ is a singleton. Otherwise if $\boldsymbol{X}^*$ is not isolated in $\omega(\boldsymbol{X}^{(0)})$, then there is an infinite sequence of limit points $\{\boldsymbol{Y}^{(k)}\} \in \omega(\boldsymbol{X}^{(0)}) \to \boldsymbol{X}^*$. Since $\boldsymbol{X}^*$ satisfies Assumption 3.1, there is a sufficiently large K such that when $k > K$, all the elements in $\{\boldsymbol{Y}^{(k)}\}_{k>K}$ also satisfy Assumption 3.1. By Theorem 3.2, all the vector tuples associated with $\{\boldsymbol{Y}^{(k)}\}_{k>K}$ are partial maximizers and so are singular vector tuples. However, this contradicts with the fact that singular vector tuples of a generic tensor are isolated. As a result, the second case cannot happen, either. So if there is a limit point $\boldsymbol{X}^*$ satisfying Assumption 3.1, then $\omega(\boldsymbol{X}^{(0)})$ is a singleton, and the whole sequence converges to that partial maximizer. □

The above discussions on the global convergence are based on the generic property of a tensor. To establish the global convergence for all tensors, we use an abstract convergence result proven in [4, Theorem 2.9] (see Theorem A.1) for descent methods for semi-algebraic problems. Such result establishes the global convergence under quite general scenarios, stating that if the function in consideration is a Kurdyka-Łojasiewicz (KL) function (see Appendix B for the definition), and three assumptions (Assumption A.1) are satisfied, then the global convergence holds. The semi-algebraic function, which is closely related to Theorem A.1, is also introduced in Appendix B.

In what follows, we mainly examine that BSVMM fulfills Assumption A.1, based on which the global convergence can be proved.

The first inequality to be verified is an enhancement of (3.14), which holds for the whole sequence, instead of for a subsequence.

Lemma 3.2. *Let $\{\boldsymbol{X}^{(k)}\}$ be generated by BSVMM. If all the points in $\omega(\boldsymbol{X}^{(0)})$ satisfy Assumption 3.1, then there exists a constant $c > 0$ and a sufficiently large K, such that*

$$F(\boldsymbol{X}^{(k+1)}) - F(\boldsymbol{X}^{(k)}) \geq \frac{c}{2}\|\boldsymbol{X}^{(k+1)} - \boldsymbol{X}^{(k)}\|_F^2, \quad \forall k > K. \tag{3.16}$$

Proof. This lemma is proved by contradiction. Suppose the conclusion does not hold. The for any $c > 0$, there exists at least an index k such that (3.16) does not hold, i.e., $F(\boldsymbol{X}^{(k+1)}) - F(\boldsymbol{X}^{(k)}) < c/2\|\boldsymbol{X}^{(k+1)} - \boldsymbol{X}^{(k)}\|_F^2$. In particular, for an infinitely decreasing sequence $\{c_l\}_{l=1}^{\infty}$ with $c_1 > c_2 > \cdots c_n \to 0$, and for each c_l, there is an index k associated with c_l, denoted as k_l, such that

$$F(\boldsymbol{X}^{(k_l+1)}) - F(\boldsymbol{X}^{(k_l)}) < \frac{c_l}{2}\|\boldsymbol{X}^{(k_l+1)} - \boldsymbol{X}^{(k_l)}\|_F^2. \tag{3.17}$$

Note that $\{\boldsymbol{X}^{(k_l)}\}$ is an infinite sequence. By the boundedness of $\{\boldsymbol{X}^{(k_l)}\}$, there exists a convergent subsequence $\{\boldsymbol{X}^{(k)}\}_{k\in\kappa} \subseteq \{\boldsymbol{X}^{(k_l)}\}$. According to the assumption and the proofs of Lemma 3.1 and Theorem 3.2, there exists a constant $\underline{c} > 0$, such that when $k(\in \kappa)$ is sufficiently large, it holds that

$$F(\boldsymbol{X}^{(k+1)}) - F(\boldsymbol{X}^{(k)}) \geq \frac{\underline{c}}{2}\|\boldsymbol{X}^{(k+1)} - \boldsymbol{X}^{(k)}\|_F^2,$$

which contradicts with (3.17). As a consequence, (3.16) holds. □

Another inequality to be verified is proved in the following, where $G(\cdot)$ has been defined in Section 2.2.

Lemma 3.3. *There exists a constant $b > 0$ such that for any k, there is a $\boldsymbol{W}^{(k+1)} \in \partial G(\boldsymbol{X}^{(k+1)})$ satisfying*

$$\|\boldsymbol{W}^{(k+1)}\|_F \leq b\|\boldsymbol{X}^{(k+1)} - \boldsymbol{X}^{(k)}\|_F. \tag{3.18}$$

Proof. Since $X_{2i-1,2i}^{(k+1)}$ is a maximizer of each subproblem, it then follows from the definition of the limiting subdifferential that

$$\nabla_{X_{2i-1,2i}} F(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)}) \in \partial I_{C_{2i-1,2i}}(X_{2i-1,2i}^{(k+1)}), \tag{3.19}$$

Denote $\boldsymbol{W}^{(k+1)} := (\ldots, \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^{(k+1)}) - \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)}), \ldots)$. Then (3.19) tells us $\boldsymbol{W}^{(k+1)} \in \partial G(\boldsymbol{X}^{(k+1)})$.

We then have $\|\boldsymbol{W}^{(k+1)}\|_F \leq \sum_{i=1}^{d/2} \|\nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^{(k+1)}) - \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)})\|_F$, and

$$\begin{aligned}
&\|\nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^{(k+1)}) - \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)})\|_F \\
\leq\ & \tilde{b}\|X_{1,2}^{(k+1)} \otimes \cdots \otimes X_{2i-3,2i-2}^{(k+1)} \otimes X_{2i+1,2i+2}^{(k+1)} \otimes \cdots \\
&\qquad \otimes X_{d-1,d}^{(k+1)} - X_{1,2}^{(k+1)} \otimes \cdots \otimes X_{2i-3,2i-2}^{(k+1)} \otimes X_{2i+1,2i+2}^{(k)} \otimes \cdots \otimes X_{d-1,d}^{(k)}\|_F \\
\leq\ & \tilde{b}d/2\|\boldsymbol{X}^{(k+1)} - \boldsymbol{X}^{(k)}\|_F,
\end{aligned}$$

where $\tilde{b} = \|A\|_2$ and A refers to a certain unfolding matrix of $\mathcal{A}$, and the last inequality is due to Lemma C.1. Combining the above pieces yields

$$\|\boldsymbol{W}^{(k+1)}\|_F \leq \tilde{b}(d/2)^2\|\boldsymbol{X}^{(k+1)} - \boldsymbol{X}^{(k)}\|_F,$$

as desired. □

With the above two lemmas as well as the KL property of $G(\cdot)$ verified in Proposition C.1, we arrive at the following theorem.

Theorem 3.5. *Let $\{\boldsymbol{X}^{(k)}\}$ be generated by the BSVMM (1.2′). If all the points in $\omega(\boldsymbol{X}^{(0)})$ satisfy Assumption 3.1, then $\omega(\boldsymbol{X}^{(0)})$ is a singleton and the whole sequence converges to a partial maximizer of (1.1′).*

Proof. For any subsequence $\{\boldsymbol{X}^{(k)}\}_{k\in\kappa} \to \overline{\boldsymbol{X}}$, it is easy to see that $G(\boldsymbol{X}^{(k)}) \to G(\overline{\boldsymbol{X}})$ as $k(\in\kappa)\to\infty$, together with inequalities (3.16) and (3.18) showing that Assumption A.1 is met. On the other hand, Proposition C.1 proves that $-G(\cdot)$ is a lower semicontinuous (l.s.c.) and KL function. Therefore, the assumptions of Theorem A.1 are fulfilled, deducing that the whole sequence $\{\boldsymbol{X}^{(k)}\}$ converges and so $\omega(\boldsymbol{X}^{(0)})$ is a singloten. Combining with Theorem 3.2, it follows that the whole sequence converges to a partial maximizer of (1.1′). □

Before ending this subsection, we show that the class of positive tensors fulfills the assumption of Theorem 3.5. A tensor is called positive if all of its entries are positive, denoted as $\mathcal{A} > 0$. Let $\{\boldsymbol{X}^{(k)}\}$, starting from $\boldsymbol{X}^{(0)} > 0$, be a sequence generated by BSVMM for solving (1.1) with $\mathcal{A} > 0$. Then for any $\boldsymbol{X}^* \in \omega(\boldsymbol{X}^{(0)})$, $\boldsymbol{X}^*$ consists of nonnegative and nonzero matrices. This leads to the fact that $\nabla_{X_{2i-1,2i}}F(\boldsymbol{X}^*)$ is a positive matrix. According to Perron-Frobenius theorem for positive matrices, the leading eigenvalue of $\nabla_{X_{2i-1,2i}}F(\boldsymbol{X}^*)^\top\nabla_{X_{2i-1,2i}}F(\boldsymbol{X}^*)$ is simple, and so is the leading singular value of $\nabla_{X_{2i-1,2i}}F(\boldsymbol{X}^*)$ (The Perron-Frobenius theorem also implies that the leading left and right singular vectors are positive, and so $\boldsymbol{X}^*$ in fact consists of positive matrices). As a result, all the points in $\omega(\boldsymbol{X}^{(0)})$ satisfy Assumption 3.1, and the global convergence of BSVMM naturally holds for positive tensors.

3.4 The first variant and convergence

A variant of BSVMM, designed towards the goal that $\|\boldsymbol{X}^{(k+1)} - \boldsymbol{X}^{(k)}\|_F \to 0$, is introduced in this subsection. The variant is still based on the method of block CG-US, but it modifies the objective of (1.1′) slightly to avoid any assumption for convergence.

To be more specific, we consider the following modified problem of (1.1′):

$$\max \overline{F}(\boldsymbol{X}) := F(\boldsymbol{X}) + \epsilon/2\sum\nolimits_{i=1}^{d/2}\|X_{2i-1,2i}\|_F^2 \text{ s.t. } X_{2i-1,2i}\in C_{2i-1,2i},\ 1\le i\le d/2, \tag{3.20}$$

where the fixed $\epsilon > 0$ can be arbitrarily small, which will be further discussed at the end of this subsection. The difference between $\overline{F}$ and F is the squares $\sum\|X_{2i-1,2i}\|_F^2$. The idea of imposing the squared terms is inspired by [17,30,43]. The squared terms give two advantages: Firstly, it converts $F(\boldsymbol{X})$ with respect to each block to a strongly convex function, leading to a better convergence result; see the results in Theorem 3.6; secondly, under the constraints $C_{2i-1,2i}$, the squares reduce to a constant $d\epsilon/4$, showing that in essence, there is by no means any difference between (1.1′) and (3.20).

The variant is executed as follows: At the $(k+1)$-th iteration, for $1\le i\le d/2$, compute

$$X_{2i-1,2i}^{(k+1)} \in \arg\max \left\langle \nabla_{X_{2i-1,2i}}\overline{F}(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)}), X\right\rangle \text{ s.t. } X\in C_{2i-1,2i}, \tag{3.21}$$

Concerning the convergence, we attempt to study it for a more general class of problems, which includes (3.20) as a special case. Specifically, we study problems of the form

$$\max\ F(\boldsymbol{x}) = F(\mathbf{x}_1,\ldots,\mathbf{x}_d)\ \text{ s.t. }\ \mathbf{x}_i\in D_i,\ 1\le i\le d, \tag{3.22}$$

under the following assumptions on $F(\cdot)$ and D_i:

Assumption 3.2. *1. $F(\cdot)$ is a smooth and semi-algebraic function, and D_i's are compact and semi-algebraic sets, $1\le i\le d$.*

2. $F(\cdot)$ is strongly convex with respect to each block, i.e., for $1\le i\le d$ and $\forall\tilde{\mathbf{x}}_i,\mathbf{x}_i\in D_i$, there are constants $\underline{c}_i > 0$ such that

$$F(\mathbf{x}_1,\ldots,\mathbf{x}_{i-1},\tilde{\mathbf{x}}_i,\mathbf{x}_{i+1}\ldots,\mathbf{x}_d) \ge F(\boldsymbol{x}) + \langle\nabla_{\mathbf{x}_i}F(\boldsymbol{x}),\tilde{\mathbf{x}}_i-\mathbf{x}_i\rangle + \underline{c}_i/2\|\tilde{\mathbf{x}}_i-\mathbf{x}_i\|_F^2.$$

3. $\nabla F(\cdot)$ *is Lipschitz continuous on* $D_1 \times \cdots D_d$, *i.e., for any* $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in D_1 \times \cdots \times D_d$, *there are constants* $\bar{c}_i > 0$ *such that for* $1 \le i \le d$,

$$\|\nabla_{\mathbf{x}_i} F(\tilde{\boldsymbol{x}}) - \nabla_{\mathbf{x}_i} F(\boldsymbol{x})\|_F \le \bar{c}_i \|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_F.$$

The block CG-US for solving (3.22) is formally described as follows:

$$\mathbf{x}_i^{(k+1)} \in \arg\max \left\langle \nabla_{\mathbf{x}_i} F(\boldsymbol{x}_{i-1}^{(k+1)}), \mathbf{x} \right\rangle \quad \text{s.t.} \quad \mathbf{x} \in D_i, \tag{3.23}$$

where $\boldsymbol{x}_i^{(k+1)} := (\mathbf{x}_1^{(k+1)}, \ldots, \mathbf{x}_i^{(k+1)}, \mathbf{x}_{i+1}^{(k)}, \ldots, \mathbf{x}_d^{(k)})$ and $\boldsymbol{x}_0^{(k+1)} = \boldsymbol{x}_d^{(k)} = \boldsymbol{x}^{(k)}$.

We remark that the above assumptions are standard in recent convergence studies; see e.g. [4, 8, 54]. Although a plenty of research has been focused on the convergence of block coordinated descent/ascent methods, it seems that the global convergence of the block CG-US for maximizing a block strongly convex function has not been discussed in the literature[2]. Of course, the study is also based on the abstract convergence result (Theorem A.1).

Theorem 3.6. *Let* $\{\boldsymbol{x}^{(k)}\}$ *be generated by* (3.23) *for solving* (3.22). *If Assumption 3.2 is satisfied, then* $\{\boldsymbol{x}^{(k)}\}$ *converges to a partial maximizer of* (3.22), *where the definition of the partial maximizer is the same as the first order optimality condition* (2.4).

Proof. The proof is also focused on verifying the assumptions of Theorem A.1. According to the strong convexity and the definition of $\mathbf{x}_i^{(k+1)}$, there holds

$$\begin{aligned} F(\boldsymbol{x}_i^{(k+1)}) &\ge F(\boldsymbol{x}_{i-1}^{(k+1)}) + \langle \nabla_{\mathbf{x}_i} F(\boldsymbol{x}_{i-1}^{(k+1)}), \mathbf{x}_i^{(k+1)} - \mathbf{x}_i^{(k)} \rangle + \underline{c}_i/2 \|\mathbf{x}_i^{(k+1)} - \mathbf{x}_i^{(k)}\|_F^2 \\ &\ge F(\boldsymbol{x}_{i-1}^{(k+1)}) + \underline{c}_i/2 \|\mathbf{x}_i^{(k+1)} - \mathbf{x}_i^{(k)}\|_F^2. \end{aligned}$$

Summing the above inequalities over i and letting $\underline{c} := \min_i \underline{c}_i$ yields

$$F(\boldsymbol{x}^{(k+1)}) - F(\boldsymbol{x}^{(k)}) \ge \underline{c}/2 \|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\|_F^2. \tag{3.24}$$

Since D_i's are compact, the above relation shows that $\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)} \to 0$. Let $\{\boldsymbol{x}^{(k)}\}_{k \in \kappa} \to \boldsymbol{x}^*$ be a convergent subsequence. Then it holds $\{\boldsymbol{x}^{(k+1)}\}_{k \in \kappa} \to \boldsymbol{x}^*$. Passing to the subsequence in (3.23) and letting $k (\in \kappa) \to \infty$, it follows

$$\langle \nabla_{\mathbf{x}_i} F(\boldsymbol{x}^*), \mathbf{x} - \mathbf{x}_i^* \rangle \le 0, \qquad \forall \mathbf{x} \in D_i,$$

showing that $\boldsymbol{x}^*$ is a partial maximizer.

Denote $G(\boldsymbol{x}) := F(\boldsymbol{x}) - \sum_{i=1}^d I_{D_i}(\mathbf{x}_i)$. We show that there is a $\boldsymbol{w}^{(k+1)} \in \partial G(\boldsymbol{x}^{(k+1)})$ and a constant $b > 0$ such that

$$\|\boldsymbol{w}^{(k+1)}\|_F \le b \|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\|_F. \tag{3.25}$$

The definition of $\mathbf{x}_i^{(k+1)}$ leads to $\nabla_{\mathbf{x}_i} F(\boldsymbol{x}_{i-1}^{(k+1)}) \in \partial I_{D_i}(\mathbf{x}_i^{(k+1)})$. Therefore, similar to the proof of Proposition 3.3, denote $\boldsymbol{w}^{(k+1)} := (\ldots, \nabla_{\mathbf{x}_i} F(\boldsymbol{x}^{(k+1)}) - \nabla_{\mathbf{x}_i} F(\boldsymbol{x}_{i-1}^{(k+1)}), \ldots)$; then $\boldsymbol{w}^{(k+1)} \in \partial G(\boldsymbol{x}^{(k+1)})$, and

$$\begin{aligned} \|\boldsymbol{w}^{(k+1)}\|_F &\le \sum_{i=1}^d \|\nabla_{\mathbf{x}_i} F(\boldsymbol{x}^{(k+1)}) - \nabla_{\mathbf{x}_i} F(\boldsymbol{x}_{i-1}^{(k+1)})\|_F \\ &\le \sum_{i=1}^d \bar{c}_i \|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}_{i-1}^{(k+1)}\|_F \le \bar{c} d \|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\|_F, \end{aligned}$$

where the second inequality is due to the Lipschitz continuity of $\nabla F(\cdot)$, and $\bar{c} := \max_i \bar{c}_i$.

[2]Recently, research was focused on the convergence rate of conditional gradient method for minimizing a strongly convex function [32]. Clearly, our results are different in terms of the target and the problem under consideration.

On the other hand, for any convergent subsequence $\{\boldsymbol{x}^{(k)}\}_{k\in\kappa} \to \overline{\boldsymbol{x}}$, it is easy to see that $G(\boldsymbol{x}^{(k)}) \to G(\overline{\boldsymbol{x}})$ as $k(\in \kappa) \to \infty$. So far Assumption A.1 has been verified.

Finally, since D_i's are compact and semi-algebraic sets, $I_{D_i}(\cdot)$'s are semi-algebraic functions, and it holds that $-G(\cdot)$ is a proper, l.s.c. and KL function. As a result, the global convergence of $\{\boldsymbol{x}^{(k)}\}$ follows directly from Theorem A.1. This combining with the fact that every limit point is a partial maximizer gives the desired result. □

Before stating the global convergence of BSVMM variant (3.21), we discuss the behavior of its limit points.

Proposition 3.3. *Let $\{\boldsymbol{X}^{(k)}\}$ be generated by the BSVMM variant (3.21) and let $\boldsymbol{X}^*$ be a limit point, with $\boldsymbol{X}^* = (\mathbf{u}_1\mathbf{v}_1^\top, \ldots, \mathbf{u}_{d/2}\mathbf{v}_{d/2}^\top)$. Then $(\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{v}_{d/2})$ is a singular vector tuple of $\mathcal{A}$. Furthermore, if ϵ specified in problem (3.20) is sufficiently small, say,*

$$0 < \epsilon \le \min_{1\le i\le d/2} \left(\sigma_{\max}(\nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^*)) - \sigma_{sl}(\nabla_{2i-1,2i} F(\boldsymbol{X}^*))\right)/2,$$

where $\sigma_{\max}(\cdot)$ and $\sigma_{sl}(\cdot)$ respectively denote the leading and the second largest (not the second one) singular values of a matrix. Then $\boldsymbol{X}^*$ is a partial maximizer of (1.1′).

Proof. Let $\{\boldsymbol{X}^{(k)}\}_{k\in\kappa} \to \boldsymbol{X}^*$. From the proof of the above theorem, we have for $1 \le i \le d/2$,

$$\langle \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^*) + \epsilon X^*_{2i-1,2i}, X - X^*_{2i-1,2i}\rangle \le 0, \quad \forall\ X \in C_{2i-1,2i}. \tag{3.26}$$

For ease of notations, denote $X^*_{2i-1,2i} := \mathbf{u}_i\mathbf{v}_i^\top$, $A_i := \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}^*)$, and $B_i := A_i + \epsilon\mathbf{u}_i\mathbf{v}_i^\top$. It follows

$$B_i v_i = \sigma_{\max}(B_i)\mathbf{u}_i \Leftrightarrow A_i\mathbf{v}_i + \epsilon\mathbf{u}_i = \sigma_{\max}(B_i)\mathbf{u}_i \Leftrightarrow A_i\mathbf{v}_i = (\sigma_{\max}(B_i) - \epsilon)\mathbf{u}_i;$$

similarly,

$$B_i^\top \mathbf{u}_i = \sigma_{\max}(B_i)\mathbf{v}_i \Leftrightarrow A_i^\top \mathbf{u}_i + \epsilon\mathbf{v}_i = \sigma_{\max}(B_i)\mathbf{v}_i \Leftrightarrow A_i^\top \mathbf{u}_i = (\sigma_{\max}(B_i) - \epsilon)\mathbf{v}_i,$$

which demonstrates that $\mathbf{u}_i, \mathbf{v}_i$ are left and right singular vectors of A_i, and $\sigma_{\max}(B_i) - \epsilon$ is a singular value of A_i. Moreover, note that

$$\sigma_{\max}(B_i) - \epsilon = \langle \nabla_{X_{2i-1,2i}} \overline{F}(\boldsymbol{X}^*), X^*_{2i-1,2i}\rangle - \epsilon = F(\boldsymbol{X}^*) =: \sigma.$$

We thus can deduce that $(\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_{d/2}, \mathbf{v}_{d/2})$ is a singular vector tuple of $\mathcal{A}$, with its associated singular value σ.

Assume that it is not the leading singular value of A_i, say, $\sigma_{\max}(B_i) - \epsilon \ne \sigma_{\max}(A_i)$; then $\sigma_{\max}(B_i) - \epsilon \le \sigma_{sl}(A_i)$. On the other hand, suppose $\mathbf{u}, \mathbf{v}$ are a pair of the leading left and right singular vectors of A_i. It is clear that $\mathbf{u}^\top\mathbf{u}_i = \mathbf{v}^\top\mathbf{v}_i = 0$, and then

$$\begin{aligned}
\langle B_i, \mathbf{u}_i\mathbf{v}_i^\top\rangle &= \sigma_{\max}(B_i) - \epsilon + \epsilon \\
&\le \sigma_{sl}(A_i) + \epsilon \\
&\le \sigma_{sl}(A_i) + (\sigma_{\max}(A_i) - \sigma_{sl}(A_i))/2 = (\sigma_{\max}(A_i) + \sigma_{sl}(A_i))/2 \\
&< \sigma_{\max}(A_i) = \langle A_i, \mathbf{u}\mathbf{v}^\top\rangle = \langle B_i, \mathbf{u}\mathbf{v}^\top\rangle,
\end{aligned}$$

which contradicts with (3.26). As a result, $\mathbf{u}_i, \mathbf{v}_i$ are also the leading singular vectors of A_i, and so $\boldsymbol{X}^*$ is a partial maximizer of (1.1′). □

Combining Theorem 3.6 and Proposition 3.3, we naturally have the global convergence of BSVMM variant (3.21), as stated in the following.

Theorem 3.7. *Let $\{\boldsymbol{X}^{(k)}\}$ be generated by BSVMM variant (3.21). Then the whole sequence converges to a singular vector tuple of $\mathcal{A}$. If ϵ of (3.20) is chosen as that in Proposition 3.3, then the whole sequence $\{\boldsymbol{X}^{(k)}\}$ converges to a partial maximizer of (1.1′).*

Proof. It is clear that $-\overline{G}(\boldsymbol{X}) := -\overline{F}(\boldsymbol{X}) + \sum_{i=1}^{d/2} I_{C_{2i-1,2i}}(X_{2i-1,2i})$ is a l.s.c. and KL function; for any fixed $\epsilon > 0$, $\overline{F}(\boldsymbol{X})$ is strongly convex with respect to each block, and $\nabla\overline{F}(\boldsymbol{X})$ is Lipschitz continuous on $C_{1,2} \times \cdots \times C_{d-1,d}$ (similar to the proof in Proposition 3.3). Therefore, according to Theorem 3.6, $\{\boldsymbol{X}^{(k)}\}$ globally converges. This together with Proposition 3.3 gives the results. □

Remark 3.1. *In practice, it is impossible to know the limit points of $\{\boldsymbol{X}^{(k)}\}$ a prior. However, since a sufficiently small ϵ is enough for the strong convexity of each block, for practical use we can set a sufficiently small ϵ, say, $\epsilon = 10^{-5}/\|\mathcal{A}\|_F$, to expect that the chosen ϵ meets the assumption in Proposition 3.3.*

3.5 The second variant and convergence

The variant in this subsection still generates $X_{2i-3,2i-2}^{(k+1)}$ corresponding to the leading singular vectors of $\nabla_{X_{2i-1,2i}} F(\boldsymbol{X}_{2i-1,2i}^{(k+1)})$; however, unlike BSVMM, this variant utilizes the leading singular vector subspace and builds connections between $X_{2i-1,2i}^{(k+1)}$ and $X_{2i-1,2i}^{(k)}$.

To this end, let $\nabla_{X_{2i-1,2i}} F(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)}) = U\Lambda V^\top$, $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n_{2i-1}}\}$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n_{2i}}\}$, $X_{2i-1,2i}^{(k)} = \mathbf{u}\mathbf{v}^\top$, $\mathbf{u} = \sum_{j_1=1}^{n_{2i-1}} \alpha_{j_1}\mathbf{u}_{j_1}$ and $\mathbf{v} = \sum_{j_2=1}^{n_{2i}} \beta_{j_2}\mathbf{v}_{j_2}$ be denoted similarly as those in the proof of Lemma 3.1. To compute $X_{2i-1,2i}^{(k+1)}$, assume that the dimension of the subspace spanned by the leading singular vectors of $\nabla_{X_{2i-1,2i}} F(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)})$ is of size $\tilde{n}$, i.e., $\sigma_1 = \cdots = \sigma_{\tilde{n}} > \sigma_{\tilde{n}+1} \geq \cdots \geq \sigma_r > 0$. At the $(k+1)$-th iteration, for $1 \leq i \leq d/2$, the proposed variant is executed as follows

$$\begin{aligned}
&X_{2i-1,2i}^{(k+1)} := \tilde{\mathbf{u}}\tilde{\mathbf{v}}^\top, \text{ where } \tilde{\mathbf{u}} = \|\alpha_{[1:\tilde{n}]}\|^{-1} \sum\nolimits_{j_1=1}^{\tilde{n}} \alpha_{j_1}\mathbf{u}_{j_1} \text{if } \ \alpha_{[1:\tilde{n}]} \neq 0 \text{ and } \tilde{\mathbf{u}} = \mathbf{u}_1 \text{ otherwise,} \\
&\tilde{\mathbf{v}} = \|\alpha_{[1:\tilde{n}]}\|^{-1} \sum\nolimits_{j_2=1}^{\tilde{n}} \alpha_{j_2}\mathbf{v}_{j_2}, \text{if } \ \alpha_{[1:\tilde{n}]} \neq 0 \text{ and } \tilde{\mathbf{v}} = \mathbf{v}_1 \text{ otherwise;} \\
&\text{and } \alpha_{[1:\tilde{n}]} \text{ denotes the first } \tilde{n} \text{ components of } \alpha = [\alpha_1, \ldots, \alpha_{n_{2i-1}}]^\top.
\end{aligned} \tag{3.27}$$

If for all the subproblems, $\tilde{n} = 1$, then the above variant reduces to BSVMM.

Denote $\sigma_{\max_i}^k := \sigma_1$ and $\sigma_{sl_i}^k := \sigma_{\tilde{n}+1}$. We assume that $\alpha_{[1:\tilde{n}]} \neq 0$, whereas the case when $\alpha_{[1:\tilde{n}]} = 0$ can be proved with a simpler argument similarly. With the above definitions, it turns out that $X_{2i-1,2i}^{(k+1)} \in C_{2i-1,2i}$ and

$$\left\langle X_{2i-1,2i}^{(k+1)}, \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)}) \right\rangle = \|\alpha_{[1:\tilde{n}]}\|^{-2} \left(\alpha_1^2\sigma_1 + \cdots + \alpha_{\tilde{n}}^2\sigma_{\tilde{n}}\right) = \sigma_{\max_i}^k.$$

We then consider $F(\boldsymbol{X}_{2i-1,2i}^{(k+1)}) - F(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)})$ and $\|X_{2i-1,2i}^{(k+1)} - X_{2i-1,2i}^{(k)}\|_F^2$.

$$\begin{aligned}
\|X_{2i-1,2i}^{(k+1)} - X_{2i-1,2i}^{(k)}\|_F^2 &= 2 - 2\langle X_{2i-1,2i}^{(k+1)}, X_{2i-1,2i}^{(k)} \rangle = 2 - 2\tilde{\mathbf{u}}^\top\mathbf{u} \cdot \tilde{\mathbf{v}}^\top\mathbf{v} \\
&= 2 - 2\|\alpha_{[1:\tilde{n}]}\|^{-2} \Big(\sum_{j_1=1}^{\tilde{n}} \alpha_{j_1}\mathbf{u}_{j_1}\Big)^\top \Big(\sum_{j_1=1}^{n_{2i-1}} \alpha_{j_1}\mathbf{u}_{j_1}\Big) \cdot \Big(\sum_{j_2=1}^{\tilde{n}} \alpha_{j_2}\mathbf{v}_{j_2}\Big)^\top \Big(\sum_{j_2=1}^{n_{2i}} \beta_{j_2}\mathbf{v}_{j_2}\Big) \\
&= 2\left(1 - \left\langle \alpha_{[1:\tilde{n}]}, \beta_{[1:\tilde{n}]} \right\rangle\right).
\end{aligned} \tag{3.28}$$

On the other side,

$$\begin{aligned}
F(\boldsymbol{X}_{2i-1,2i}^{(k+1)}) - F(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)}) &= \sigma_{\max_i}^k - \left\langle X_{2i-1,2i}^{(k)}, \nabla_{X_{2i-1,2i}} F(\boldsymbol{X}_{2i-3,2i-2}^{(k+1)}) \right\rangle \\
&= \sigma_{\max_i}^k - \sum_{j=1}^{r} \alpha_j\beta_j\sigma_j.
\end{aligned} \tag{3.29}$$

Combining (3.28) and (3.29) and similar to the proof of Lemma 3.1, we have

$$F(\boldsymbol{X}^{(k+1)}_{2i-1,2i}) - F(\boldsymbol{X}^{(k+1)}_{2i-3,2i-2}) - \frac{\sigma^k_{\max_i} - \sigma^k_{sl_i}}{2}\|X^{(k+1)}_{2i-1,2i} - X^{(k)}_{2i-1,2i}\|_F^2$$
$$= \sigma_{\tilde{n}} - \sigma_{\tilde{n}}\sum_{j=1}^{\tilde{n}}\alpha_j\beta_j - \sum_{j=\tilde{n}+1}^{r}\alpha_j\beta_j\sigma_j \geq \sigma_{\tilde{n}} - \sigma_{\tilde{n}} = 0.$$

Proceeding to all the blocks, and denoting $c^k := \min_{1\leq i\leq d/2}\{\sigma^k_{\max_i} - \sigma^k_{sl_i}\}$, we have

$$F(\boldsymbol{X}^{(k+1)}) - F(\boldsymbol{X}^{(k)}) \geq \frac{c^k}{2}\|\boldsymbol{X}^{(k+1)} - \boldsymbol{X}^{(k)}\|_F^2. \tag{3.30}$$

Note that we do not require the fixedness of the index $\tilde{n}$; instead, the only condition for the validation of (3.30) is that there is a positive gap between the leading and the second largest (not the second one) singular values of the matrices in consideration, and this in fact holds for any fixed matrices. To extend (3.30) to be a universal inequality, we make the following assumption:

Assumption 3.3. *There exists a constant $c > 0$, such that $\lim_{k\to\infty}\inf c^k \geq c > 0$.*

With the above assumption, (3.30) becomes

$$F(\boldsymbol{X}^{(k+1)}) - F(\boldsymbol{X}^{(k)}) \geq \frac{c}{2}\|\boldsymbol{X}^{(k+1)} - \boldsymbol{X}^{(k)}\|_F^2, \quad k \text{ sufficiently large.} \tag{3.31}$$

The proof of the global convergence of BSVMM variant (3.27) is exactly the same as Theorem 3.6. Thus we only present the result here and omit the proof.

Theorem 3.8. *Let $\{\boldsymbol{X}^{(k)}\}$ be generated by the BSVMM variant (3.27). If Assumption 3.3 is fulfilled, then the whole sequence converges to a partial maximizer of (1.1′).*

Remark 3.2. *The purpose of Assumption 3.3 is to exclude the extreme case that $\sigma^k_{sl_i} \to \sigma^k_{\max_i}$ as $k \to \infty$, which might rarely happen. It is not easy to compare this assumption with that in Theorem 3.5 and conclude which is weaker. Although Assumption 3.3 only requires a positive gap between the largest and second largest singular values, such assumption is made under a limit setting, which is different from that in Theorem 3.5. Of course, Considering the practical issue, the variant presented in this subsection is less efficient, as it has to compute the full SVD. Nonetheless, we introduce this variant here, so as to expect it at least has theoretical interest.*

4 Convergence Study of An Eigenvalue Maximization Method for Symmetric Rank-One Approximation to a Symmetric Tensor

4.1 Description of the problem and method

We are concerned with the following problem in this section:

$$\max\ F(\mathbf{x}) := \mathcal{A}\times_1\mathbf{x}^\top\times_2\mathbf{x}^\top\cdots\times_d\mathbf{x}^\top \ \text{ s.t. } \ \|\mathbf{x}\| = 1,\ \mathbf{x}\in\mathbb{R}^n, \tag{4.32}$$

where $\mathcal{A}\in\mathbb{R}^{n\times\cdots\times n}$ is a d-th order symmetric (sometimes it is called supersymmetric) tensor, i.e., every entry $a_{i_1\cdots i_d}$ is invariant under any permutation of $\{i_1,\ldots,i_d\}$. Such a problem is closely related to the best symmetric rank-1 approximation of a symmetric tensor [16]. For convenience, the objective can be written as [41]

$$F(\mathbf{x}) = \mathcal{A}\mathbf{x}^d = \sum\nolimits_{i_1,\ldots,i_d=1}^{n} a_{i_1\cdots i_d}\mathbf{x}_{i_1}\cdots\mathbf{x}_{i_d}, \tag{4.33}$$

denoting a degree-d homogeneous polynomial. Various methods have been proposed to solve (4.32); see, e.g. [9, 24, 28, 30, 31, 37, 42, 59], just to name a few.

The KKT system of (4.32) is given by

$$\mathcal{A} \times_1 \mathbf{x}^\top \cdots \times_{d-1} \mathbf{x}^\top = \lambda \mathbf{x}, \quad \|\mathbf{x}\| = 1, \ \lambda \in \mathbb{R}. \tag{4.34}$$

$(\mathbf{x}, \lambda)$ satisfying the above system is called a l^2-eigenpair by Lim [33] or Z-eigenpair by Qi [41]. Without any confusion, we omit the prefix and call it an eigenpair of $\mathcal{A}$. Similar to (4.33), the left hand-side of (4.34) can also be simplified as $\mathcal{A}\mathbf{x}^{d-1}$. Note that $\nabla F(\mathbf{x}) = d\mathcal{A}\mathbf{x}^{d-1}$. Denote $\mathcal{A}\mathbf{x}^{d-2} := \mathcal{A} \times_1 \mathbf{x}^\top \cdots \times_{d-2} \mathbf{x}^\top \in \mathbb{R}^{n\times n}$. Then $\nabla^2 F(\mathbf{x}) = d(d-1)\mathcal{A}\mathbf{x}^{d-2}$. With this notation, we have the following observation:

Fact 3. *$(\mathbf{x}, \lambda)$ is an eigenpair of $\mathcal{A}$ iff it is an eigenpair of the matrix $\mathcal{A}\mathbf{x}^{d-2}$.*

In the spirit of BSVMM, in this section, the following method for solving (4.32) is considered, whose $(k+1)$-th iteration is detailed as follows:

$$\boxed{\mathbf{x}^{(k+1)} \in \arg\max \ \mathbf{x}^\top \nabla^2 F(\mathbf{x}^{(k)})\mathbf{x} \quad \text{s.t. } \|\mathbf{x}\| = 1.} \tag{4.35}$$

That is, similar to BSVMM, the new trial $\mathbf{x}^{(k+1)}$ is a leading eigenvector of the symmetric matrix $\mathcal{A}(\mathbf{x}^{(k)})^{d-2}$. Due to this reason, the above method is called eigenvalue maximization method (EMM) in this section.

Note that in the context of blind source separation, a similar method was proposed in [9].

EMM is closely related to S-HOPM [16, 28], which generates the iterations via the following scheme:

$$\mathbf{x}^{(k+1)} = \mathcal{A}(\mathbf{x}^{(k)})^{d-1} / \|\mathcal{A}(\mathbf{x}^{(k)})^{d-1}\|. \tag{4.36}$$

To see the relation, when computing the subproblem (4.35) of EMM, if the matrix power method is applied, started from $\mathbf{x}^{(k)}$, terminated within one iteration, then EMM reduces to S-HOPM. The advantage of EMM is similar to that of BSVMM over HOPM: Let $\mathbf{x}^{(0)}$ be the initial guess, with $\lambda = \mathcal{A}(\mathbf{x}^{(0)})^d$ happening to be an eigenvalue of $\mathcal{A}$; moreover, λ is not a leading eigenvalue of $\mathcal{A}(\mathbf{x}^{(0)})^{d-2}$. Initialized with $\mathbf{x}^{(0)}$, S-HOPM may get stuck at $\mathbf{x}^{(0)}$, while EMM can jump out from this point. This implies that the latter might have a better opportunity to find a solution with larger objective value.

It was shown that S-HOPM converges when the objective function is convex or concave and the order of the tensor is even [28]. SS-HOPM, which is a modified version of S-HOPM, was shown to converge to an eigenpair provided that a suitable parameter is chosen [30]. However, for EMM, it seems that there is a lack of an understanding of its convergence in the literature. Therefore, the rest of this section is toward convergence issues.

In view of the structure of EMM, it is expected that the sequence converges to a partial maximizer of (4.32), defined as follows:

Definition 4.1 (Partial maximizer)**.** *$\mathbf{x}$ is called a partial maximizer of a symmetric tensor $\mathcal{A}$, if*

$$\mathbf{x} \in \arg\max \mathbf{y}^\top \nabla^2 F(\mathbf{x})\mathbf{y}, \quad \text{s.t. } \|\mathbf{y}\| = 1. \tag{4.37}$$

4.2 Convergence study

We study the convergence under different circumstances. When d is general, we are not very clear about the convergence; the only result obtained is:

Proposition 4.1. *Let $\{\mathbf{x}^{(k)}\}$ be generated by EMM (4.35). If $\langle \nabla^2 F(\mathbf{x}^{(k)}), \mathbf{x}^{(k+1)}\mathbf{x}^{(k+1)\top} - \mathbf{x}^{(k)}\mathbf{x}^{(k)\top} \rangle \to 0$, then*

1. every limit point $\mathbf{x}^*$ is a partial maximizer of $\mathcal{A}$;

2. $\mathbf{x}^*$ is a strict local maximizer, provided that $\lambda_{\max} > (d-1)\lambda_2$, where $\lambda_{\max} > \lambda_2 \geq \cdots$ denote the leading and the second eigenvalues of $\nabla^2 F(\mathbf{x}^*)$, respectively.

Proof. 1. From the definition of EMM (4.35), it follows

$$\langle \nabla^2 F(\mathbf{x}^{(k)}), \mathbf{x}\mathbf{x}^\top - \mathbf{x}^{(k)}\mathbf{x}^{(k)\top} \rangle \leq \langle \nabla^2 F(\mathbf{x}^{(k)}), \mathbf{x}^{(k+1)}\mathbf{x}^{(k+1)\top} - \mathbf{x}^{(k)}\mathbf{x}^{(k)\top} \rangle, \quad \forall\ \|\mathbf{x}\| = 1.$$

Let $\{\mathbf{x}^{(k)}\}_{k\in\kappa} \to \mathbf{x}^*$ be a convergence subsequence. Passing to this subsequence, letting $k(\in \kappa) \to \infty$ and taking into account the assumption, it holds that $\langle \nabla^2 F(\mathbf{x}^*), \mathbf{x}\mathbf{x}^\top - \mathbf{x}^*\mathbf{x}^{*\top} \rangle \leq 0, \forall\ \|\mathbf{x}\| = 1$, which shows that $\mathbf{x}^*$ is a partial maximizer of $\mathcal{A}$.

2. The assertion can be proved by directly verifying the second order sufficient conditions, e.g., [6, Section 3.3.3], or follows from [44, Theorem 2]. □

In what follows, we assume that d is even. In this case, it is convenient to rewrite (4.32) as problem (4.32′). Assume that $d \geq 4$. Denote

$$X = \mathbf{x}\mathbf{x}^\top,\ F(X) = \mathcal{A} \times_{1,2} X \cdots \times_{d-1,d} X = F(\mathbf{x}),\ C = \{X \in \mathbb{S}^{n\times n} \mid \|X\|_F = 1, \mathrm{rank}(X) = 1\},$$

where $\mathbb{S}^{n\times n}$ denotes the space of real symmetric matrices. Then (4.32) can be rewritten as

$$\max\ F(X) \quad \text{s.t. } X \in C. \tag{4.32′}$$

Furthermore, denote

$$\mathcal{A}X^{d/2-1} := \mathcal{A} \times_{1,2} X \cdots \times_{d-3,d-2} X \in \mathbb{R}^{n\times n};$$

then $\nabla F(X) = d/2\mathcal{A}X^{d/2-1}$, and EMM (4.35) amounts to

$$X^{(k+1)} \in \arg\max \left\langle \nabla F(X^{(k)}), X \right\rangle, \quad \text{s.t. } X \in C. \tag{4.35′}$$

In what follows, we assume that d is even.

4.2.1 Convex case

When $F(X)$ is convex, it holds that

$$F(X^{(k+1)}) - F(X^{(k)}) \geq \langle \nabla F(X^{(k)}), X^{(k+1)} - X^{(k)} \rangle \geq 0,$$

and so $F(X^{(k+1)}) - F(X^{(k)}) \to 0$, yielding $\langle \nabla F(X^{(k)}), X^{(k+1)} - X^{(k)} \rangle \to 0$ as $k \to \infty$. Therefore, following Proposition 4.1 we have

Theorem 4.1. *Let $\{X^{(k)}\}$ be generated by EMM (4.35′). If $F(X)$ is convex, then every limit point is a partial maximizer, which turns out to be a strict local maximizer if the assumptions in Proposition 4.1 holds.*

To study the global convergence, we follow the ideas in the previous section. The following assumption is made:

Assumption 4.1. *Let $\omega(X^{(0)})$ be the set of limit points of $\{X^{(k)}\}$. For any $X^* \in \omega(X^{(0)})$, we assume that the leading eigenvalue of the matrix $\nabla F(X^*)$ is simple.*

The following lemma is parallel to Lemma 3.2.

Lemma 4.1. *Let $\{X^{(k)}\}$ be generated by EMM (4.35′). If Assumption 4.1 is satisfied, then there exists a constant $c > 0$ and a sufficiently large K, such that*

$$F(X^{(k+1)}) - F(X^{(k)}) \geq \frac{c}{2}\|X^{(k+1)} - X^{(k)}\|_F^2, \quad k > K. \tag{4.38}$$

Proof. Let $\{X^{(k)}\}_{k\in\kappa} \to X^*$ be a convergent subsequence. We first prove that there is a consant $c_0 > 0$, such that when $k \in \kappa$ is sufficiently large, there holds

$$F(X^{(k+1)}) - F(X^{(k)}) \geq \frac{c_0}{2}\|X^{(k+1)} - X^{(k)}\|_F^2, \quad k \in \kappa, \tag{4.39}$$

and then use (4.39) to prove (4.38).

The following argument is similar to that of Lemma 3.1. Since X^* satisfies Assumption 4.1, there must exist a ball $\mathbb{B}(X^*, \epsilon)$, such that for all $X \in \mathbb{B}(X^*, \epsilon)$, Assumption 4.1 holds, and for this ϵ, there exists a K_0 such that $\{X^{(k)}\}_{k\in\kappa,k>K_0} \subset \mathbb{B}(X^*, \epsilon)$. For any $X^{(k_0)} \in \{X^{(k)}\}_{k\in\kappa,k>K_0}$, since $\nabla F(X^{(k_0)})$ is symmetric, let $\nabla F(X^{(k_0)}) = U\Lambda U^\top$ be an EVD, with $U = [\mathbf{u}_1, \ldots, \mathbf{u}_r] \in \mathbb{R}^{n\times r}$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_r)$. Denote $\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\}$ as orthonormal basis of $\mathrm{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}^\perp$.

Since $X^{(k_0)} \in C$, $X^{(k_0)} = uu^\top$, with $\mathbf{u} = \sum_{j=1}^n \alpha_j \mathbf{u}_j$ and $\sum_{j=1}^n \alpha_j^2 = 1$. And $X^{(k_0+1)} = \mathbf{u}_1\mathbf{u}_1^\top$ because it is the unique maximizer. Considering $\langle \nabla F(X^{(k_0)}), X^{(k_0+1)} - X^{(k_0)}\rangle$ and $\|X^{(k_0+1)} - X^{(k_0)}\|_F^2$, we have

$$\langle \nabla F(X^{(k_0)}), X^{(k_0+1)} - X^{(k_0)}\rangle = \lambda_1 - \sum_{j=1}^r \alpha_j^2\lambda_j,$$

and

$$\|X^{(k_0+1)} - X^{(k_0)}\|_F^2 = 2 - 2\langle X^{(k_0+1)}, X^{(k_0)}\rangle = 2(1 - \alpha_1^2),$$

which together with the convexity of $F(\cdot)$ yields

$$\begin{aligned}
&F(X^{(k_0+1)}) - F(X^{(k_0)}) - \frac{\lambda_1 - \lambda_2}{2}\|X^{(k_0+1)} - X^{(k_0)}\|_F^2 \\
&\geq \langle \nabla F(X^{(k_0)}), X^{(k_0+1)} - X^{(k_0)}\rangle - \frac{\lambda_1 - \lambda_2}{2}\|X^{(k_0+1)} - X^{(k_0)}\|_F^2 \\
&= \lambda_2(1 - \alpha_1^2) - \sum_{j=2}^r \alpha_j^2\lambda_j \geq 0.
\end{aligned} \tag{4.40}$$

Finally, it follows that there exists a constant $c_0 > 0$, such that $\inf_{X\in\mathbb{B}(X^*,\epsilon)}(\lambda_1(X) - \lambda_2(X)) \geq c_0 > 0$. This combining with (4.40) yields that when $k \in \kappa$, $k > K_0$, (4.39) holds.

With (4.39) at hand, and similar to the proof of Lemma 3.2, we conclude that there exists a constant $c > 0$, such that when k is sufficiently large, (4.38) holds. This completes the proof. □

Theorem 4.2. *Let $\{X^{(k)}\}$ be generated by EMM (4.35′). If Assumption 4.1 is satisfied, then the sequence globally converges to a partial maximizer, which turns out to be a strict local maximizer if the assumptions in Proposition 4.1 holds.*

Proof. To apply Theorem A.1, it remains to show that there is a $W^{(k+1)} \in \partial G(X^{(k+1)})$ such that $\|W^{(k+1)}\|_F \leq b\|X^{(k+1)} - X^{(k)}\|_F^2$, whereas in the current setting, $G(X) = F(X) - I_C(X)$. In fact, it follows from (4.35′) that $\nabla F(X^{(k)}) \in \partial I_C(X^{(k+1)})$, and so $W^{(k+1)} := \nabla F(X^{(k+1)}) - \nabla F(X^{(k)}) \in \partial G(X^{(k+1)})$. Invoking Lemma C.2, we have

$$\|W^{(k+1)}\|_F \leq \tilde{b}d/2\|X^{(k+1)} \otimes \cdots \otimes X^{(k+1)} - X^{(k)} \otimes \cdots \otimes X^{(k)}\|_F \leq \tilde{b}cd/2\|X^{(k+1)} - X^{(k)}\|_F,$$

where $\tilde{b}$ denotes the spectral norm of a certain unfolding matrix of $\mathcal{A}$. Therefore, applying Theorem A.1 to EMM (4.35′) and recalling Theorem 4.1, the assertion holds. □

It shall be mentioned that the convexity of $F(X)$ only happens when $d/2$ is even, which limits the convergence study. Thus in what follows, we consider a variant of EMM, which does not require $d/2$ to be even.

4.2.2 Forcing $F(X)$ to be strongly convex

Motivated by [17, 30, 43], we impose a shift term to the objective function. For any integer $d/2 \geq 2$ ($d/2$ not necessary being even), the new objective function becomes

$$\overline{F}(X) := F(X) + \alpha \|X\|_F^{d/2}.$$

Following [30], it can be shown that when $\alpha > (d/2-1)\max_{\|X\|_F=1} \rho(\overline{\mathcal{A}}\mathrm{vec}(X)^{d/2-2})$, $\overline{F}(X)$ is convex. Here $\mathrm{vec}(\cdot)$ denotes the vectorization of a matrix, and $\overline{\mathcal{A}}$ represents the associated unfolding of $\mathcal{A}$, whereas in this sense, $\overline{\mathcal{A}}\mathrm{vec}(X)^{d/2-2} \in \mathbb{R}^{n^2 \times n^2}$; $\rho(\cdot)$ denotes the spectral radius of a matrix.

When $\overline{F}(X)$ becomes convex, we further add a quadratic term such that it is strongly convex, i.e.,

$$\overline{\overline{F}}(X) := \overline{F}(X) + \epsilon/2\|X\|_F^2,$$

where ϵ can be any positive scalar. So far, the problem to be solved becomes

$$\max \ \overline{\overline{F}}(X) \quad \text{s.t. } X \in C, \tag{4.32''}$$

and it can be seen that (4.32″) is in fact equivalent to (4.32′), as $\max_{X\in C} \overline{\overline{F}}(X) = \max_{X\in C} F(X) + \alpha + \epsilon/2$.

Applying the CG-US to (4.32″) yields the following variant scheme of EMM

$$X^{(k+1)} \in \arg\max \left\langle \nabla\overline{\overline{F}}(X^{(k)}), X \right\rangle, \quad \text{s.t. } X \in C, \tag{4.41}$$

where $\nabla\overline{\overline{F}}(X^{(k)}) = \nabla F(X^{(k)}) + \alpha d/2 \|X^{(k)}\|_F^{d/2-2} X^{(k)} + \epsilon X^{(k)} = \nabla F(X^{(k)}) + (\alpha d/2 + \epsilon) X^{(k)}$.

Theorem 4.3. *Let $\{X^{(k)}\}$ be generated by EMM variant (4.41). Then the whole sequence converges to an X^* with $X^* = \mathbf{x}\mathbf{x}^\top$, where $\mathbf{x}$ is an eigenvector of $\mathcal{A}$. Moreover, if α and ϵ satisfy that*

$$\alpha d/2 + \epsilon \leq \left(\lambda_{\max}(\nabla F(X^*)) - \lambda_{sl}(\nabla F(X^*))\right)/2,$$

where $\lambda_{sl}(\cdot)$ denotes the second largest eigenvalue of a matrix, then $\mathbf{x}$ is a partial maximizer, which turns out to be a strict local maximizer if the assumptions in Proposition 4.1 holds.

Proof. It follows that problem (4.32″) is the single block version of the more general problem (3.22) and moreover, $\overline{\overline{F}}(X)$ and C satisfy Assumption 3.2. Thus the whole sequence converges to a point X^*. Then, using a similar argument as that in Proposition 3.3, we can show the remaining assertions. □

Remark 4.1. *EMM variant (4.41) can be viewed as a projected gradient method (PGM)*

$$X^{(k+1)} = P_C\left(X^{(k)} + \beta^{-1}\nabla F(X^{(k)})\right),$$

where $\beta = \alpha d/2 + \epsilon$, and $P_C(\cdot)$ denotes the projection onto C. Commonly, the convergence of PGM requires that $\beta > L$, where L is the Lipschitz constant of $\nabla F(\cdot)$ on C. However, considering the efficiency, a too large β is not preferred. So it is an important issue to choose the step-size in practice. Usually, a line-search strategy can be employed to choose the step-size in order to accelerate the method.

BSVMM variant (3.21) can also be regarded as a block version of the PGM.

5 Illustrative Experiments

This section gives some understandings of BSVMM, BSVMM variant (3.21) and EMM from numerical experiments. All the computations are conducted on an Intel i7-3770 CPU desktop computer with 16 GB of RAM. The supporting software is MATLAB R2013a. The MATLAB toolbox Tensorlab [52] is employed to implement the tensor computations.

5.1 Setup

In each experiment, all the algorithms start from the same initial guess. The stopping criteria for BSVMM, BSVMM variant (3.21) and HOPM are respectively $\|\boldsymbol{X}^{(k+1)} - \boldsymbol{X}^{(k)}\|_F \leq 10^{-4}$ and $\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\|_F \leq 10^{-4}$; those for EMM and SS-HOPM are $\|X^{(k+1)} - X^{(k)}\|_F \leq 10^{-4}$ and $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_F \leq 10^{-4}$.

BSVMM, its variant and EMM involve matrix singular value/eigenvalue problems as subproblems. These subproblems can be solved by, e.g., the power iteration (PI) and the Rayleigh Quotient Iteration (RQI) [48]. Note that RQI enjoys cubic convergence rate. To gain efficiency, the subproblems are solved inexactly, i.e., the algorithm is stopped within 10 iterations, or whenever the consecutive difference is less than 10^{-3}. Warm start is used, that is, the initial guess for solving the current subproblem is the output of the last subproblem.

In BSVMM and its variant, the matrices associated with subproblems are prepared in a hierarchical way. For instance, for an 8-th order tensor, we first compute $\mathcal{A} \times_5 \mathbf{x}_5^{(k)\top} \cdots \times_8 \mathbf{x}_8^{(k)\top}$, based on which we compute two matrices $\mathcal{A} \times_3 \mathbf{x}_3^{(k)\top} \cdots \times_8 \mathbf{x}_8^{(k)\top}$ and $\mathcal{A} \times_1 \mathbf{x}_1^{(k+1)\top} \times_2 \mathbf{x}_2^{(k+1)\top} \times_5 \mathbf{x}_5^{(k)\top} \cdots \times_8 \mathbf{x}_8^{(k)\top}$; we then compute the tensor $\mathcal{A} \times_1 \mathbf{x}_1^{(k+1)\top} \cdots \times_4 \mathbf{x}_4^{(k+1)\top}$, based on which we compute the remaining two matrices $\mathcal{A} \times_1 \mathbf{x}_1^{(k+1)\top} \cdots \times_4 \mathbf{x}_4^{(k+1)\top} \times_7 \mathbf{x}_7^{(k)\top} \times_8 \mathbf{x}_8^{(k)\top}$ and $\mathcal{A} \times_1 \mathbf{x}_1^{(k+1)\top} \cdots \times_6 \mathbf{x}_6^{(k+1)\top}$. By doing this, for dense tensors, the computational complexity can be reduced from $4 \cdot n^8$ (the vanilla way) to $2 \cdot n^8 + 4 \cdot n^4$.

5.2 Solving (1.1)

The tensors $\mathcal{A}$ in this experiment are given by $\mathcal{A} = \sum_{i=1}^{n} \mathbf{a}_{1,i} \otimes \cdots \otimes a_{d,i}$, where every vector $\mathbf{a}_{k,i}$, $1 \leq k \leq d$, $1 \leq i \leq n$ is of dimension n, randomly generated by the standard normal distribution; $d \in \{3, 4, 7, 8\}$. For each specified order and dimension, we generate 50 instances. We also generate sparse tensors, where "SP" denotes the sparsity. Considering the implementation of HOPM, since it can be seen as an ALS, we use the function "cpd_als" in Tensorlab [52] to avoid implementing it by ourselves. The parameter ϵ of BSVMM variant (3.21) is set to 0.01.

The objective value and the computational time of each method are reported in Table 1, all of which are averaged. We first claim that we have observed that all the methods converge to a singular vector tuple. Considering the objective value, we first observe that BSVMM and its variant generate the same value; that might be because we choose a small ϵ for the variant. Comparing with HOPM, BSVMM and its variant obtain larger values than HOPM in an average sense, which might be because BSVMM and its variant can escape from some stationary points, as explained in Section 1.1, while HOPM might get stuck on them. Considering the computational time, when the order is small and the dimension is large, BSVMM and its variant perform not as good as HOPM; this is due to that when the dimension is high, BSVMM and its variant has more expensive subproblems to solve. However, the table shows that the computational time of BSVMM and its variant is still acceptable. When the order is high and the dimension is not very high, BSVMM and its variant consume less time, as in this case, computing the subproblems is not very expensive, and we use a hierarchical way to prepare the matrices. In summary, BSVMM can be an alternative method to HOPM, and, considering both convergence properties and empirical performance, BSVMM variant (3.21) with a small ϵ can be a better choice than BSVMM.

Table 1: Reports of BSVMM, BSVMM variant (3.21) and HOPM on solving (1.1).

		BSVMM		BSVMM variant (3.21)		HOPM	
d,n	SP (%)	Value	Time (S.)	Value	Time (S.)	Value	Time (S.)
3,200	0	2936.0	0.30	2936.0	0.30	**2977.9**	0.23
3,200	70	**923.5**	0.32	**923.5**	0.31	922.8	0.22
3,200	90	**325.8**	0.47	**325.8**	0.47	323.4	0.28
3,300	0	**5452.0**	0.83	**5452.0**	0.83	5392.2	0.74
3,300	70	**1671.3**	0.93	**1671.3**	0.92	1626.1	0.77
3,300	90	**578.9**	1.29	**578.9**	1.29	574.7	0.91
4,50	0	2782.5	0.17	2782.5	0.16	**2892.4**	0.22
4,50	70	**849.3**	0.17	**849.3**	0.17	848.6	0.23
4,50	90	194.9	0.09	194.9	0.09	**202.9**	0.09
4,80	0	**6899.2**	1.00	**6899.2**	1.01	6864.3	1.65
4,80	70	**2089.8**	1.05	**2089.8**	1.05	2024.2	1.78
4,80	90	685.3	1.18	685.3	1.18	**696.2**	1.84
7,10	0	**2871.1**	0.30	**2871.1**	0.29	2808.2	0.53
7,10	70	**1193.2**	0.52	**1193.2**	0.51	1188.0	0.67
7,10	90	**476.2**	0.70	**476.2**	0.70	457.5	0.95
7,12	0	6605.9	0.95	6605.9	0.94	**6630.7**	1.75
7,12	70	**2309.1**	1.68	**2309.1**	1.66	2118.2	2.41
7,12	90	**884.9**	2.34	**884.9**	2.32	881.0	3.21
8,7	0	**2516.1**	0.17	**2516.1**	0.16	2374.9	0.35
8,7	70	1058.2	0.36	1055.2	0.36	**1063.5**	0.60
8,7	90	333.0	0.72	333.0	0.73	**331.8**	0.82
8,9	0	7133.7	1.32	7133.7	1.29	**7146.0**	2.51
8,9	70	**3005.5**	2.12	**3005.5**	2.10	2741.2	3.82
8,9	90	1010.0	3.39	1010.0	3.37	**1027.4**	4.72

5.3 Solving (4.32)

The tensors are generated as follows: First they are generated as in the previous experiment; then they are symmetrized; finally the tensors are normalized to have unit Frobenius norm, so that the shift parameter for EMM (variant) and SS-HOPM can be easily and reasonably chosen.

EMM (variant) is compared with SS-HOPM [30]. The parameter for EMM (variant) ($\alpha d/2+\epsilon$ in Algorithm 4.41) is chosen from $\{0.05, 0.1, 0.5, 1\}$, while that for SS-HOPM is chosen from the same parameter set. We also compare EMM with the adaptive shifted power method (GEAP) [31], which can be regarded as a self-adaptive SS-HOPM when specified to problem (4.32). Both SS-HOPM and GEAP are available from Tensor Toolbox [5].

The results, including objective value, computational time and iterations (Iter for short) are reported in Table 2. Almost in all cases, all the methods converge to a Z-eigenpair/l^2-eigenpair. Considering the objective value, EMM ($\alpha = 0.1$ and $\alpha = 0.5$) yields better results in most cases in an average sense, followed by SS-HOPM ($\alpha = 0.05$) and EMM ($\alpha = 0.05$). Considering the computational time, for EMM, there is an increasing trend as the shift parameter gets larger. This is because that as discussed in Remark 4.1, EMM can be seen as a PGM. Considering both efficiency and effectiveness, from the table, $\alpha = 0.1$ may be the best choice for EMM. We notice that the comparisons of computational time between EMM and GEAP/SS-HOPM is unfair, because we find that the codes of GEAP/SS-HOPM requires first checking that whether the input tensor is symmetric, which is very time consuming when the order of the tensor is high. Instead, if considering the iterations, in some cases, EMM needs more iterations for convergence, which indicates that EMM might not be as efficient as GEAP/SS-HOPM. In this regard, choosing the shift parameters adaptively, as those in GEAP, might help to accelerate the convergence of EMM. Nevertheless, from the table, it seems that the computational time of EMM is still acceptable. In summary, EMM can be an alternative method for solving (4.32).

Table 2: Reports of EMM, SS-HOPM and GEAP on solving (4.32).

	GEAP			SS-HOPM ($\alpha = 0.05$)			EMM ($\alpha = 0.05$)			SS-HOPM ($\alpha = 0.1$)			EMM ($\alpha = 0.1$)			SS-HOPM ($\alpha = 0.5$)			EMM ($\alpha = 0.5$)			SS-HOPM ($\alpha = 1$)			EMM ($\alpha = 1$)		
d,n	Value	Time	Iter	Value	Time	Iter	Value	Time	Iter	Value	Time	Iter	Value	Time	Iter	Value	Time	Iter	Value	Time	Iter	Value	Time	Iter	Value	Time	Iter
4,10	2.96E-01	0.14	27.5	**2.98E-01**	0.20	63.6	2.92E-01	0.01	19.7	2.86E-01	0.05	16.4	2.92E-01	0.01	24.6	2.92E-01	0.13	43.4	2.92E-01	0.02	60.7	2.86E-01	0.21	65.8	2.86E-01	0.03	101.3
4,50	7.21E-02	1.44	15.9	7.21E-02	1.33	16.5	7.21E-02	0.97	21.8	7.11E-02	1.41	23.6	**7.38E-02**	1.48	33.4	7.07E-02	2.09	81.6	**7.38E-02**	5.18	116.8	5.76E-02	2.58	125.1	**7.38E-02**	9.37	211.5
4,60	5.85E-02	2.62	14.5	5.84E-02	2.48	17.4	**6.22E-02**	1.99	24.8	5.83E-02	2.69	26.6	**6.22E-02**	3.12	26.6	5.17E-02	3.90	83.9	**6.22E-02**	10.91	136.3	1.90E-02	2.97	39.8	5.84E-02	18.64	229.0
4,80	**4.83E-02**	10.92	15.4	4.79E-02	10.95	21.7	4.78E-02	8.44	30.0	4.65E-02	11.55	34.2	4.78E-02	13.64	48.1	9.11E-03	10.88	22.6	4.78E-02	48.46	173.7	6.90E-05	9.18	1.0	4.78E-02	89.45	313.4
6,5	1.60E-01	0.15	15.4	1.80E-01	1.04	308.8	1.94E-01	0.13	356.0	1.59E-01	0.45	113.0	**1.98E-01**	0.06	163.8	1.60E-01	0.17	29.1	**1.98E-01**	0.02	54.8	1.60E-01	0.24	47.7	**1.98E-01**	0.04	92.7
5,10	1.76E-01	0.15	18.7	**2.59E-01**	0.10	10.9	1.02E-01	0.08	112.1	2.01E-01	0.11	16.6	2.31E-01	0.02	16.8	1.75E-01	0.21	48.7	2.31E-01	0.04	46.9	1.47E-01	0.28	73.5	2.31E-01	0.06	80.8
5,20	**1.56E-01**	2.93	13.6	**1.56E-01**	2.85	11.7	1.22E-01	0.32	14.5	**1.56E-01**	2.88	17.1	1.22E-01	0.47	21.3	**1.56E-01**	3.24	57.7	1.22E-01	1.53	69.9	1.22E-01	3.29	64.7	1.22E-01	2.76	125.5
5,30	6.09E-02	22.92	17.7	6.10E-02	22.43	19.1	**7.10E-02**	3.30	19.2	5.91E-02	22.86	26.4	**7.10E-02**	4.86	28.9	4.18E-02	24.53	68.5	**7.10E-02**	16.54	96.8	1.51E-02	23.68	41.1	**7.10E-02**	29.41	172.8
6,10	1.69E-01	5.79	19.1	1.69E-01	5.77	13.9	**1.76E-01**	1.35	170.5	1.69E-01	5.74	18.8	**1.76E-01**	0.29	34.9	1.30E-01	5.89	48.1	**1.76E-01**	0.83	103.2	1.03E-01	6.01	59.5	1.69E-01	1.04	129.7
6,20	**1.00E-01**	328.30	13.2	**1.00E-01**	326.01	15.4	**1.00E-01**	8.01	18.4	9.16E-02	323.66	21.8	**1.00E-01**	12.81	29.2	3.69E-02	323.57	16.6	**1.00E-01**	47.64	110.0	3.68E-02	326.76	29.7	**1.00E-01**	90.40	199.5
7,6	7.91E-02	14.84	16.7	1.07E-01	15.00	11.6	1.07E-01	0.17	44.3	7.15E-02	15.04	10.6	**1.67E-01**	0.11	27.5	6.45E-02	15.28	31.3	**1.67E-01**	0.31	78.5	6.27E-02	15.13	43.1	1.07E-01	0.40	110.9
7,8	1.31E-01	95.51	15.9	1.19E-01	93.50	13.5	1.37E-01	0.36	18.4	1.31E-01	92.92	20.1	**1.61E-01**	0.44	23.0	9.96E-02	92.71	26.4	**1.61E-01**	1.54	78.5	9.93E-02	92.71	44.4	1.54E-01	2.54	133.1
8,5	3.32E-02	99.31	10.7	4.59E-02	97.69	112.9	2.29E-02	0.46	140.2	1.70E-02	98.53	114.8	6.39E-02	0.26	97.6	3.09E-02	99.00	29.8	**6.51E-02**	0.37	136.7	2.88E-02	99.01	30.9	6.15E-02	0.53	165.4
8,6	1.62E-01	546.27	13.3	6.58E-02	528.48	108.8	1.61E-01	2.72	171.7	8.29E-02	526.54	109.9	**1.63E-01**	2.21	137.2	8.43E-02	532.27	25.8	1.62E-01	1.69	91.1	7.57E-02	559.45	35.2	9.44E-02	2.57	136.0

5.4 Polyadic decomposition

Let $\tilde{\mathcal{A}} = \mathcal{A} + \mathcal{E}$, where $\mathcal{A} = \sum_{i=1}^{n} \mu_i \otimes \cdots \otimes \mu_i$ is a symmetric and orthogonally decomposable tensor and $\mathcal{E}$ is a perturbation. Given $\tilde{\mathcal{A}}$, the goal is to (approximately) find $\mathcal{A}$. In [2], a robust power method was proposed to achieve this goal. In order to obtain the approximated rank-1 components $\overline{\mu}_i \otimes \cdots \otimes \overline{\mu}_i$, this method iteratively solves subproblems as (4.32), where the data tensors are the deflated tensors $\mathcal{A} - \sum_{i=1}^{k} \overline{\mu}_i \otimes \cdots \otimes \overline{\mu}_i$, $k = 0, 1, \ldots, n-1$. Please refer to [2, Algorithm 1] for details.

In [2], the robust power method uses the tensor power method to solve subproblems (4.32) with deflated tensors. Here, we use EMM to solve the subproblems instead. That is, given $\tilde{\mathcal{A}}$, we apply the robust power method to find $\mathcal{A}$, where the tensor power iteration (Lines 4–5 of [2, Algorithm 1]) is replaced by EMM.

In this experiment, we randomly generate tensors $\tilde{\mathcal{A}} = \mathcal{A} + \epsilon \cdot \mathcal{E}$, where $\mathcal{A} = \sum_{i=1}^{n} \lambda_i \mu_i \otimes \cdots \otimes \mu_i$, $\lambda_i > 0$, $\mu_i \in \mathbb{R}^n$ and $\{\mu_1, \ldots, \mu_n\}$ is an orthonormal basis; $\mathcal{E}$ is also a symmetric tensor with $\|\mathcal{E}\|_F = 1$. The minimal λ_i, $\lambda_{\min} = \min_{1 \le i \le n}\{\lambda_i\}$, according to [2, Theorem 5.1], should satisfy $\lambda_{\min} \ge \epsilon \cdot n / C_1$ where C_1 is a constant. Here we set $C_1 = 10$. Let $\overline{\mathcal{A}} = \sum_{i=1}^{n} \overline{\lambda}_i \overline{\mu}_i \otimes \cdots \otimes \overline{\mu}_i$ be the tensor recovered by the robust power method with EMM. Denote $\Delta_{\mathcal{A}} := \mathcal{A} - \overline{\mathcal{A}}$, $\Delta_{\mu_i} := \mu_i - \overline{\mu_i}$ and $\Delta_{\lambda_i} := \lambda_i - \overline{\lambda_i}$. Then $\|\Delta_{\mathcal{A}}\|_F$, $\|\Delta_{\mu_i}\|$ and $\max_{1 \le i \le n} |\Delta_{\lambda_i}|$ are used to evaluate the approximation errors of the algorithm.

The approximation errors of the approximation are reported in Table 3. The tensor order varies from 3 to 6, the dimension is either 8 or 10, and the magnitude of the perturbation is 0.1 or 0.5. The table shows that in all cases, the approximation errors are small, namely, the robust power method using EMM as subroutines can recover the true tensors well. In particular, one can check that the errors are significantly lower than the theoretical upper bound proposed in [2, Theorem 5.1]. Summarizing, EMM can be useful for decomposing a tensor into (approximating a tensor by) a sum of several rank-1 terms.

Table 3: Approximation errors of polyadic decomposition solved by [2, Algorithm 1] with EMM ; the last column shows the theoretical upper bound of $\|\Delta_{\mathcal{A}}\|_F$ (derived from [2, Theorem 5.1]).

d,n	ϵ	$\|\Delta_{\mu_1}\|$	$\|\Delta_{\mu_2}\|$	$\|\Delta_{\mu_3}\|$	$\|\Delta_{\mu_4}\|$	$\|\Delta_{\mu_5}\|$	$\|\Delta_{\mu_6}\|$	$\|\Delta_{\mu_7}\|$	$\|\Delta_{\mu_8}\|$	$\|\Delta_{\mu_9}\|$	$\|\Delta_{\mu_{10}}\|$	$\max_i\{\|\Delta_{\lambda_i}\|\}$	$\|\Delta_{\mathcal{A}}\|_F$	Up.Bd.
3,8	0.1	4.0E-05	9.3E-05	3.0E-04	4.7E-04	4.5E-04	4.7E-04	4.1E-04	4.3E-04	–	–	1.2E-02	6.5E-02	5.5
3,8	0.5	1.3E-04	1.0E-04	1.8E-04	1.5E-04	3.0E-04	3.1E-04	3.2E-04	4.9E-04	–	–	6.6E-02	3.7E-01	22.5
4,8	0.1	4.9E-04	1.8E-03	2.4E-03	3.3E-03	3.9E-03	4.7E-03	2.3E-03	4.2E-03	–	–	1.0E-02	5.0E-02	5.5
4,8	0.5	2.8E-04	6.7E-04	1.4E-03	1.1E-03	1.1E-03	1.8E-03	1.7E-03	2.0E-03	–	–	5.3E-02	2.1E-01	22.5
6,8	0.1	9.3E-07	7.3E-06	5.9E-06	9.0E-06	5.4E-06	9.6E-06	8.1E-06	5.7E-06	–	–	2.8E-03	1.8E-02	5.5
6,8	0.5	7.1E-05	5.6E-05	2.0E-04	1.1E-04	3.2E-04	2.9E-04	3.0E-04	4.4E-04	–	–	2.2E-02	9.1E-02	22.5
3,10	0.1	2.0E-05	6.1E-05	1.2E-04	7.8E-05	1.3E-04	1.5E-04	1.7E-04	1.8E-04	1.4E-04	2.4E-04	1.3E-02	6.2E-02	5.5
3,10	0.5	4.2E-04	5.9E-04	1.2E-03	8.8E-04	1.6E-03	2.2E-03	2.7E-03	2.7E-03	1.8E-03	2.7E-03	6.4E-02	3.7E-01	22.5
4,10	0.1	5.3E-03	6.9E-03	6.1E-03	7.7E-03	8.3E-03	1.1E-02	6.4E-03	9.3E-03	1.2E-02	1.0E-02	8.2E-03	4.0E-02	5.5
4,10	0.5	1.2E-03	5.4E-03	6.4E-03	1.0E-02	1.1E-02	1.2E-02	9.7E-03	1.4E-02	1.1E-02	1.8E-02	3.0E-02	1.8E-01	22.5
6,10	0.1	5.7E-05	2.9E-05	4.1E-05	7.4E-05	5.0E-05	1.0E-04	9.4E-05	1.2E-04	1.4E-04	8.7E-05	2.0E-03	1.4E-02	5.5
6,10	0.5	3.7E-05	8.8E-05	1.5E-04	1.4E-04	2.3E-04	2.0E-04	3.2E-04	2.7E-04	2.4E-04	3.4E-04	2.0E-02	7.9E-02	22.5

6 Concluding Remarks

This paper studied some convergence results of BSVMM. The convergence rate of BSVMM was shown to be $O(1/K)$ in the ergodic sense. The subsequence convergence was established when one of the limit points admits some regularities; if all the limit points satisfy the regularities, then the whole sequence is proved to converge to a partial maximizer. To gain better convergence results, variants of BSVMM were proposed. Along the line of BSVMM, an eigenvalue maximization method for symmetric rank-1 approximation to symmetric tensor was investigated. Preliminary experiments were given to show the effectiveness of the methods.

Further open problems can be studied in future work: for example, for convergence of BSVMM, can the

related assumptions be relaxed? Does Assumption 3.1 hold for a generic tensor? What is the local convergence rate of BSVMM? How to accelerate the convergence of EMM (for example, we can consider an adaptive version of EMM, similar to [31])?

Acknowledgement

This work was supported by ERC AdG A-DATADRIVE-B (290923), EU's Seventh Framework Programme (FP7/2007-2013) / ERC AdG BIOTENSORS (339804), GOA/10/09 MaNet, CoE PFV/10/002 (OPTEC), BIL12/11T, C1 project C16/15/059-nD, FWO G.0377.12, G.088114N, G.0830.14N, iMinds SBO 2014; IUAP P7/19 (DYSCO). The second author was supported by the NSFC Grant 11401428. This paper reflects only the authors' views and the Union is not liable for any use that may be made of the contained information.

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A An Abstract Convergence Result

Let $\{\mathbf{x}^{(k)}\}$ be a sequence generated by a descent algorithm $\mathtt{A}$ for solving a problem $\min_{\mathbf{x}} F(\mathbf{x})$ which might be nonsmooth and nonconvex. It has been proved that, provided that $F(\cdot)$ is a lower semicontinuous (l.s.c.) and Kurdyka-Łojasiewicz (KL) function, the whole sequence generated by $\mathtt{A}$ (including a variety of algorithms such as proximal gradient, projected gradient, forward-backward splitting, (proximal) coordinate descent) globally converges [4, 8, 54]. In particular, it was refined in [4, Theorem 2.9] that if $\mathtt{A}$ satisfies the three assumptions specified in Assumption A.1, then the whole sequence globally converges. We recall this theorem here, and place the definitions of KL property/function and semi-algebraic sets/functions into the next section.

Theorem A.1 (Abstract convergence, c.f. [4])**.** *Let $F(\cdot)$ be a proper l.s.c. function, and $\{\mathbf{x}^{(k)}\}$ be a sequence satisfying Assumption A.1 for solving $\min_{\mathbf{x}} F(\mathbf{x})$. If $F(\cdot)$ has the KL property at the limit point $\mathbf{x}^*$ specified in Item 3 of Assumption A.1, then $\{\mathbf{x}^{(k)}\} \to \mathbf{x}^*$, and $\mathbf{x}^*$ is a critical point of $F(\cdot)$. Moreover, $\sum_{k=0}^{+\infty} \|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| < +\infty$.*

Remark A.1. *A l.s.c. function has the KL property at any noncritical point; see e.g. [4, Remark 2.5], while a KL function satisfies the KL property at each point of* $\mathrm{dom}\partial F$ *(Definition B.1).*

The assumptions required in the above theorem are listed in the following.

Assumption A.1. *1. There is a constant $c > 0$ such that*

$$F(\mathbf{x}^{(k)}) - F(\mathbf{x}^{(k+1)}) \geq \frac{c}{2}\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|^2.$$

2. There is a $\mathbf{w}^{(k+1)} \in \partial F(\mathbf{x}^{(k+1)})$ and a constant $b > 0$ such that

$$\|\mathbf{w}^{(k+1)}\| \leq b\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|.$$

3. There is a subsequence $\{\mathbf{x}^{(k)}\}_{k\in\kappa} \subseteq \{\mathbf{x}^{(k+1)}\}$ and $\mathbf{x}^*$ such that

$$\mathbf{x}^{(k)} \to \mathbf{x}^* \text{ and } F(\mathbf{x}^{(k)}) \to F(\mathbf{x}^*), \qquad k(\in \kappa) \to \infty.$$

B Kurdyka-Łojasiewicz Property/Function and Semi-Algebraic Set/-Function

Definition B.1. (KL property/function, c.f. [4, Definition 2.4]) *Let $F(\cdot)$ be an extended real-valued function, and let $\mathbf{x}^* \in \mathrm{dom}\partial F$, where $\mathrm{dom}\partial F := \{\mathbf{x} \mid \partial F(\mathbf{x}) \neq \emptyset\}$. If there exist $\eta \in (0, +\infty]$, a neighborhood U of $\mathbf{x}^*$, and a continuous concave function $\varphi : [0, \eta) \to \mathbb{R}_+$, such that 1) $\varphi(0) = 0$, 2) φ is C^1 on $(0, \eta)$, 3) for*

all $s \in (0, \eta)$, $\varphi'(s) > 0$, *and 4) for all* $\mathbf{x} \in U \cap \{\mathbf{x} \mid F(\mathbf{x}^*) < F(\mathbf{x}) < F(\mathbf{x}^*) + \eta\}$, *the Kurdyka-Łojasiewicz inequality holds*

$$\varphi'(F(\mathbf{x}) - F(\mathbf{x}^*))\mathrm{dist}(0, \partial F(\mathbf{x})) \geq 1,$$

then $F(\cdot)$ *has the KL property at* $\mathbf{x}^*$, *where* $\mathrm{dist}(0, \partial F(\mathbf{x}))$ *denotes the distance from* 0 *to the set* $\partial F(\mathbf{x})$.

$F(\cdot)$ *is said to be a KL function if it is proper, l.s.c., and satisfies the KL property at each point of* $\mathrm{dom}\partial F$.

Commonly, it is not easy to check that whether a nonsmooth function is a KL function from the definition. Nevertheless, KL functions are ubiquitous in real-world applications, including real subanalytic functions and semi-algebraic functions; see, e.g. [8]. Definitions and properties concerning the semi-algebraic sets and functions are introduced in the following.

Definition B.2 (Semi-algebraic set and function, see e.g., [7])**.** *A set* $C \subset \mathbb{R}^n$ *is called semi-algebraic, if there exists a finite number of real polynomials* $p_{ij}, q_{ij} : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq s, 1 \leq j \leq t$ *such that*

$$C = \bigcup_{i=1}^{s} \bigcap_{j=1}^{t} \{\mathbf{x} \in \mathbb{R}^n \mid p_{ij}(\mathbf{x}) = 0,\ q_{ij}(\mathbf{x}) > 0\}.$$

An extended real-valued function F *is said to be semi-algebraic if its graph*

$$\mathrm{graph}\, F := \big\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} \mid F(\mathbf{x}) = t\big\}$$

is a semi-algebraic set.

Operations that keep the semi-algebraic property of sets include: finite unions, finite intersections, complementation and Cartesian products. On the other hand, the Tarski-Seidenberg theorem, which is stated in the following, can also be used to identify semi-algebraic sets.

Theorem B.1 (Tarski-Seidenberg theorem, c.f. [14])**.** *Let* C *be a semi-algebraic set of* $\mathbb{R}^n$. *Then the image* $\pi(C)$ *is a semi-algebraic set if*

1. $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ *is the projection on the first* $(n-1)$ *coordinates.*
2. $\pi : \mathbb{R}^n \to \mathbb{R}^m$ *is a polynomial mapping.*

Some common examples of semi-algebraic functions are listed in the following, which are provided in [8].

1. Real polynomial functions.
2. Indicator functions of semi-algebraic sets.
3. Finite sums, product and composition of semi-algebraic functions.

C Auxiliary Results

In order that Theorem A.1 works through this paper, we need to show that $G(\cdot)$ defined in Section 2.2 is a KL function.

Proposition C.1. $-G(\cdot)$ *is a l.s.c. and KL function.*

Proof. Recall that $-G(\boldsymbol{X}) = -F(\boldsymbol{X}) + \sum_{i=1}^{d/2} I_{C_{2i-1,2i}}(X_{2i-1,2i})$. Here

$$C_{2i-1,2i} = \{X \mid \|X\|_F = 1, \mathrm{rank}(X) = 1\} = \{X \mid \|X\|_F = 1, \mathrm{rank}(X) \leq 1\},$$

which is the intersection of two closed sets and hence is closed, and so $I_{C_{2i-1,2i}}(\cdot)$ is l.s.c. because it is the indicator function of a closed set. Therefore $-G(\cdot)$ is l.s.c..

To show that $-G(\cdot)$ is a KL function, according to the discussions in the previous section, we only need to show that it is semi-algebraic. Again by the discussions in the previous section, it suffices to prove that each $C_{2i-1,2i}$ is a semi-algebraic set. It is clear that $C_{2i-1,2i}$ is the image of the polynomial mapping $\pi(\mathbf{u},\mathbf{v}) = \mathbf{u}\mathbf{v}^\top$ with $\mathbf{u} \in \{\mathbf{x} \in \mathbb{R}^{n_{2i-1}} \mid \|\mathbf{x}\| = 1\}$ and $\mathbf{v} \in \{\mathbf{y} \in \mathbb{R}^{n_{2i}} \mid \|\mathbf{y}\| = 1\}$, which are two semi-algebraic sets. As a result, the semi-algebraic property of $C_{2i-1,2i}$ follows from the Tarski-Seidenberg theorem (Theorem B.1) and finally, $-G(\cdot)$ is a KL function. The proof has been completed. □

Lemma C.1. *For any even number d and $\boldsymbol{X}, \boldsymbol{Y} \in C_{1,2} \times \cdots \times C_{2d-1,2d}$, there holds*

$$\|X_{1,2} \otimes \cdots \otimes X_{d-1,d} - Y_{1,2} \otimes \cdots \otimes Y_{d-1,d}\|_F \le d/2 \|\boldsymbol{X} - \boldsymbol{Y}\|_F.$$

Proof. We apply the induction method on d. When $d = 2$, the above inequality is true. Suppose it holds when $d = 2m$. When $d = 2m + 2$,

$$\begin{aligned}
&\|X_{1,2} \otimes \cdots \otimes X_{2m+1,2m+2} - Y_{1,2} \otimes \cdots \otimes Y_{2m+1,2m+2}\|_F \\
&\le \|X_{1,2} \otimes \cdots \otimes X_{2m+1,2m+2} - X_{1,2} \otimes \cdots \otimes X_{2m-1,2m} \otimes Y_{2m+1,2m+2}\|_F + \\
&\qquad\qquad \|X_{1,2} \otimes \cdots \otimes X_{2m-1,2m} \otimes Y_{2m+1,2m+2} - Y_{1,2} \otimes \cdots \otimes Y_{2m+1,2m+2}\|_F \\
&\le \|X_{2m+1,2m+2} - Y_{2m+1,2m+2}\|_F + m\|(X_{1,2}, \ldots, X_{2m-1,2m}, 0) - (Y_{1,2}, \ldots, Y_{2m-1,2m}, 0)\|_F \\
&\le (m+1)\|\boldsymbol{X} - \boldsymbol{Y}\|_F,
\end{aligned}$$

where the second inequality follows from the induction and that $\|X \otimes Y\|_F = 1$ when $X, Y \in C_{2i-1,2i}$ for any i. □

Lemma C.2. *There is a constant $c > 0$, such that for any $X, Y \in C = \{X \in \mathbb{S}^{n\times n} \mid \|X\|_F = 1, \mathrm{rank}(X) = 1\}$, there holds $\|X \otimes X \otimes \cdots \otimes X - Y \otimes Y \otimes \cdots \otimes Y\|_F \le c\|X - Y\|_F$.*

Proof. This is a consequence of Lemma C.1. □

D Discussions on the Convergence Issues of the Overlapping Updates

We have mentioned that BSVMM can be updated in the following manner:

$$(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)}, \mathbf{x}_3^{(k)}) \to (\mathbf{x}_1^{(k+0.5)}, \mathbf{x}_2^{(k+0.5)}, \mathbf{x}_3^{(k)}) \to (\mathbf{x}_1^{(k+0.5)}, \mathbf{x}_2^{(k+1)}, \mathbf{x}_3^{(k+0.5)}) \to (\mathbf{x}_1^{(k+1)}, \mathbf{x}_2^{(k+1)}, \mathbf{x}_3^{(k+1)}).$$

To see its convergence, let us assume that $\lim_{k(\in\kappa)\to\infty} \boldsymbol{x}^{(k)} = \boldsymbol{x}^*$ where $\boldsymbol{x}^*$ is a limit point. Similar to Assumption 3.1, we assume that $\mathcal{A} \times_1 \mathbf{x}^{*\top}$, $\mathcal{A} \times_2 \mathbf{x}_2^{*\top}$, and $\mathcal{A} \times_3 \mathbf{x}_3^{*\top}$ have simple leading singular values. Following the proofs of Lemma 3.1 and Theorem 3.2, when $k(\in \kappa)$ is sufficiently large, there holds

$$F(\mathbf{x}_1^{(k+0.5)}, \mathbf{x}_2^{(k+0.5)}, \mathbf{x}_3^{(k)}) \ge F(\boldsymbol{x}^{(k)}) + \frac{c}{2}\|\mathbf{x}_1^{(k+0.5)}\mathbf{x}_2^{(k+0.5)\top} - \mathbf{x}_1^{(k)}\mathbf{x}_2^{(k)\top}\|_F^2,$$

which implies that when $k(\in \kappa) \to \infty$,

$$\|\mathbf{x}_1^{(k+0.5)}\mathbf{x}_2^{(k+0.5)\top} - \mathbf{x}_1^{(k)}\mathbf{x}_2^{(k)\top}\|_F^2 = 2 - 2\langle \mathbf{x}_1^{(k+0.5)}, \mathbf{x}_1^{(k)}\rangle\langle \mathbf{x}_2^{(k+0.5)}, \mathbf{x}_2^{(k)}\rangle \to 0,$$

and so $\lim_{k(\in\kappa)\to\infty} \mathbf{x}_1^{(k+0.5)} = \mathbf{x}_1^*$ and $\lim_{k(\in\kappa)\to\infty} \mathbf{x}_2^{(k+0.5)} = \mathbf{x}_2^*$. Similarly, we have

$$\begin{aligned}
F(\mathbf{x}_1^{(k+0.5)}, \mathbf{x}_2^{(k+1)}, \mathbf{x}_3^{(k+0.5)}) &\ge F(\mathbf{x}_1^{(k+0.5)}, \mathbf{x}_2^{(k+0.5)}, \mathbf{x}_3^{(k)}) + \frac{c}{2}\|\mathbf{x}_2^{(k+1)}\mathbf{x}_3^{(k+0.5)\top} - \mathbf{x}_2^{(k+0.5)}\mathbf{x}_3^{(k)}\|_F^2, \\
F(\boldsymbol{x}^{(k+1)}) &\ge F(\mathbf{x}_1^{(k+0.5)}, \mathbf{x}_2^{(k+1)}, \mathbf{x}_3^{(k+0.5)}) + \frac{c}{2}\|\mathbf{x}_1^{(k+1)}\mathbf{x}_3^{(k+1)\top} - \mathbf{x}_1^{(k+0.5)}\mathbf{x}_3^{(k+0.5)\top}\|_F^2.
\end{aligned}$$

The above discussion finally yields

$$\lim_{k(\in\kappa)\to\infty} \boldsymbol{x}^{(k+1)} = \lim_{k(\in\kappa)\to\infty} \boldsymbol{x}^{(k+0.5)} = \boldsymbol{x}^*.$$

Using these relations, we can show that $\boldsymbol{x}^*$ is a singular vector tuple (in fact, by the structure of the updating rule, one can verify that $(\mathbf{x}_1^*, \mathbf{x}_2^*) = \arg\max_{\|\mathbf{x}_1\|=\|\mathbf{x}_2\|=1} \mathbf{x}_1^\top \left(\mathcal{A} \times_3 \mathbf{x}_3^{*\top}\right) \mathbf{x}_2$, $(\mathbf{x}_2^*,\mathbf{x}_3^*) = \operatorname{argmax}_{\|\mathbf{x}_2\|=\|\mathbf{x}_3\|=1} \mathbf{x}_2^\top \left(\mathcal{A} \times_1 \mathbf{x}_1^{*\top}\right) \mathbf{x}_3$ and $(\mathbf{x}_1^*,\mathbf{x}_3^*) = \operatorname{argmax}_{\|\mathbf{x}_1\|=\|\mathbf{x}_3\|=1} \mathbf{x}_1^\top \left(\mathcal{A} \times_2 \mathbf{x}_2^{*\top}\right) \mathbf{x}_3$). Furthermore, its global convergence can be also proved (similar to the proofs of Theorems 3.3 and 3.4). For higher order cases, similar convergence results can be derived.