On polynomial and rational quadratic differential forms

K. Takaba * **H.L. Trentelman** ** **J.C. Willems** ***

* *Department of Applied Mathematics and Physics Graduate School of Informatics Kyoto University, Kyoto 606-8501, Japan (e-mail: takaba@amp.i.kyoto-u.ac.jp).*
** *Mathematics Department, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands (e-mail: h.l.trenteman@math.rug.nl)*
*** *ESAT-SISTA, K.U. Leuven, Kasteelpark Arenberg 10, Leuven B-3001, Belgium (e-mail: Jan.Willems@esat.kuleuven.be)*

Abstract: In system and control theory, we often encounter the situation of investigating some functionals which represent Lyapunov functions, energy storage, performance measures, e.t.c. In particular, if the system is linear time-invariant, the functionals can be quadratic forms of system variables and their derivatives. Such a quadratic form is called a *quadratic differential form (QDF)* which was introduced in the context of the behavioral system theory by Willems and Trentelman (1998). The contribution of this keynote paper is twofold. Firstly, this paper illustrates the basic features of QDF's in terms of polynomial matrices. We also discuss the dissipativity and Lyapunov stability on the basis of QDF's. The second contribution is to present a new and more general formulation of QDF's in terms of rational functions rather than polynomials. A QDF defined by rational functions is called a *rational QDF*. Unlike polynomial QDF's, a rational QDF defines a set of values of a quadratic functional. It is shown that the several features of polynomial QDF's (nonnegativity, average nonnegativity, e.t.c.) can be generalized to the case of rational QDF's.

Keywords: behavior, linear systems, polynomial methods, quadratic differential form, rational representation

1. INTRODUCTION

In analysis and synthesis of dynamical systems, we often encounter the situation of investigating some functionals which represent Lyapunov functions, energy supply, energy storage, performance measures, and so forth. In particular, if the system under consideration is linear time-invariant, the functionals can be quadratic forms of the system variables and their derivatives. Such a quadratic form is called a *quadratic differential form (QDF)* which was introduced in the context of the behavioral system theory by Willems and Trentelman (1998).

In the behavioral framework, the QDF's have been playing a crucial role in many aspects of system and control theory: Lyapunov stability (Willems and Trentelman 1998, Peeters and Rapisarda 2001, Kojima and Takaba 2005), dissipation theory (Willems and Trentelman 1998, Willems and Trentelman 2002, Kaneko and Fujii 2000, 2003), linear quadratic optimal control (Willems 1993, Willems and Valcher 2005), $\mathcal{H}^\infty$ control (Trentelman and Willems 1999, Willems and Trentelman 2002, Belur and Trentelman2002) and stability analysis of uncertain or nonlinear interconnections (Pendharkar Pillai 2007, Takaba 2005, Willems and Takaba 2007).

This keynote paper discusses the basic features of QDF's and their generalization.

We first introduce the definition of a QDF in terms of polynomial matrices in Section 2. Note that a QDF has one-to-one correspondence with a two-variable polynomial matrix, whose indeterminates represent the differentiations of the variables. Hence, in terms of polynomials, we will illustrate the basic features of QDF's including nonnegativity, average nonnegativity, half-line nonnegativity and so forth, which are closely related to the dissipation theory and Lyapunov stability.

In Section 3, as a generalization of the 'polynomial' QDF's described above, we will present a new formulation of a QDF in terms of rational functions, and will examine the basic features of such a type of QDF's. We call such a QDF defined by rational functions a *rational QDF*. Note that the need for such rational QDF's arises, for example, in the stability analysis of interconnected or feedback systems via rational multipliers or integral quadratic constraints (e.g. Megretski and Rantzer 1997, Iwasaki and Hara 1998). It will be shown that the several features of polynomial QDF's such as nonnegativity and average nonnegativity can be generalized to the case of rational QDF's.

Some basics on a linear differential system and its representation by a rational function in the behavioral framework are given in Appendix A.

Notations:

$\mathbb{R}$: the field of real numbers
$\mathbb{C}$: the field of complex numbers
$\mathrm{i}\mathbb{R} := \{\lambda \in \mathbb{C} \mid \lambda = \mathrm{i}\omega,\ \omega \in \mathbb{R}\}$
$\mathbb{R}[\xi]$: the ring of polynomials
$\mathbb{R}[\zeta, \eta]$: the ring of two-variable polynomials
$\mathbb{R}(\xi)$: the ring of rational functions
$\mathbb{R}(\zeta, \eta)$: the ring of two-variable rational functions
$\mathbb{R}^{\mathtt{p}\times\mathtt{q}}$: $\mathtt{p} \times \mathtt{q}$ real matrices
$\mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$: $\mathtt{p} \times \mathtt{q}$ polynomial matrices
$\mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\zeta, \eta]$: $\mathtt{p} \times \mathtt{q}$ two-variable polynomial matrices
$\mathbb{R}^{\mathtt{p}\times\mathtt{q}}(\xi)$: $\mathtt{p} \times \mathtt{q}$ rational matrices
$\mathbb{R}^{\mathtt{p}\times\mathtt{q}}(\zeta, \eta)$: $\mathtt{p} \times \mathtt{q}$ two-variable rational matrices
$\mathbb{R}_{\mathrm{s}}^{\mathtt{p}\times\mathtt{p}}$: $\mathtt{p} \times \mathtt{p}$ symmetric real matrices $\mathbb{R}_{\mathrm{s}}^{\mathtt{p}\times\mathtt{p}}[\zeta, \eta]$: $\mathtt{p} \times \mathtt{p}$ symmetric two-variable polynomial matrices
$\mathbb{R}_{\mathrm{s}}^{\mathtt{p}\times\mathtt{p}}(\zeta, \eta)$: $\mathtt{p} \times \mathtt{p}$ symmetric two-variable rational matrices
$\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$: infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$
$\mathfrak{D}$: compact support functions

Note that we will often use "$\bullet$" to denote irrelevant dimensions of a vector or a matrix.

2. QUADRATIC DIFFERENTIAL FORMS IN THE POLYNOMIAL SETTING

We here give the original definition of a QDF in terms of polynomial matrices, and review the basic features of QDF's from Willems and Trentelman (1998), Rapisarda and Willems (2004), Kaneko and Fujii (2004).

2.1 Definitions and basic calculus

A quadratic differential form (QDF) Q_Φ is defined as a quadratic form of $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ and its derivatives. Namely,

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$Q_\Phi(w)(t) = \sum_{i=0}^{k}\sum_{j=0}^{k} \left(\frac{d^i w}{dt^i}\right)^\top \Phi_{ij} \left(\frac{d^j w}{dt^j}\right)$$

where $\Phi_{ij} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and $\Phi_{ji}^\top = \Phi_{ij}$. $(i = 0, 1, \ldots, k)$. Throughout this paper, we will assume that all variables are $\mathfrak{C}^\infty$ functions of time t for simplicity of discussion. Also, we often drop the time dependence for ease of notation.

We can associate Q_Φ with a symmetric two-variable polynomial matrix

$$\Phi(\zeta, \eta) = \sum_{i=0}^{k}\sum_{j=0}^{k} \zeta^i \eta^j \Phi_{ij} \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta].$$

By "symmetric", we mean $\Phi(\zeta, \eta) = \Phi^\top(\eta, \zeta)$. Notice that the indeterminates ζ and η correspond to the differentiations on $w^\top$ and w, respectively.

Furthermore, we define the *coefficient matrix of* $\Phi(\zeta, \eta)$ as

$$\tilde{\Phi} := \begin{pmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0k} \\ \Phi_{10} & \Phi_{11} & \cdots & \Phi_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{k0} & \Phi_{k1} & \cdots & \Phi_{kk} \end{pmatrix} \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}(k+1)\times\mathtt{w}(k+1)}$$

It is clearly seen that there is one-to-one correspondence among Q_Φ, $\Phi(\zeta, \eta)$ and $\tilde{\Phi}$.

We consider the factorization of $\Phi \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ in the form of

$$\Phi(\zeta, \eta) = M^\top(\zeta)\Sigma M(\eta), \quad M \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi],$$
$$\Sigma = \mathrm{diag}(I_{\mathtt{r}_+}, -I_{\mathtt{r}_-}) \in \mathbb{R}^{\bullet\times\bullet}.$$

Note that a constant matrix in the form of $\mathrm{diag}(I_{\mathtt{r}_+}, -I_{\mathtt{r}_-})$ is called a *signature matrix*. If $M(\xi)$ has minimal row number among such factorizations of $\Phi(\zeta, \eta)$, then we say that the factorization is a *symmetric canonical factorization* or simply a *canonical factorization*. The canonical factorization of $\Phi(\zeta, \eta)$ can be obtained by the following procedure. Let $\mathtt{v} = \mathrm{rank}\tilde{\Phi}$, and factor Φ as $\tilde{\Phi} = \tilde{M}^\top \Sigma \tilde{M}$, $\tilde{M} \in \mathbb{R}^{\mathtt{v}\times\mathtt{w}}$, $\Sigma \in \mathbb{R}_{\mathrm{s}}^{\mathtt{v}\times\mathtt{v}}$, where Σ is a signature matrix. Then, a canonical factorization of $\Phi(\zeta, \eta)$ is obtained by

$$\Phi(\zeta, \eta) = M^\top(\zeta)\Sigma M(\eta), \quad M(\xi) = \tilde{M} \begin{pmatrix} I_{\mathtt{w}} \\ \xi I_{\mathtt{w}} \\ \vdots \\ \xi^k I_{\mathtt{w}} \end{pmatrix}$$

Of course, the canonical factorization of a two-variable polynomial matrix is not unique. If we obtain two different canonical factorizations

$$\Phi(\zeta, \eta) = M_1^\top(\zeta)\Sigma M_1(\eta) = M_2^\top(\zeta)\Sigma M_2(\eta),$$

then there exists a nonsingular matrix $U \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}$ such that

$$\Sigma = U^\top \Sigma U, \quad M_1(\xi) = U M_2(\xi).$$

Therefore, the canonical factorization is unique modulo a pseudo-unitary transformation U (p. 1709 in Willems and Trentelman 1998).

Example 1. We consider the QDF

$$Q_\Phi(w) = \dot{w}_1^2 + 2w_1\dot{w}_2,$$

where "$\dot{}$" denotes time differentiation. Since

$$Q_\Phi(w) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}^\top \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix}^\top \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}^\top \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} + \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix},$$

this QDF is induced by the polynomial matrix

$$\Phi(\zeta, \eta) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \zeta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \eta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \zeta\eta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \zeta\eta & \eta \\ \zeta & 0 \end{pmatrix}$$

Moreover, the coefficient matrix $\tilde{\Phi}$ is

$$\tilde{\Phi} = \left(\begin{array}{cc:cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

Since $\mathrm{rank}\tilde{\Phi} = 3$, the canonical factorization of $\Phi(\zeta, \eta)$ is given by

$$M(\xi) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\xi \\ -\xi & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\xi \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\diamond$

One of the important properties of QDF's is the fact that the derivative of a QDF is also a QDF. Namely, there holds

$$\frac{d}{dt}Q_\Phi(w)(t) = Q_\Psi(w)(t) \ \ \forall t \in \mathbb{R},\ \forall w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$$
$$\Leftrightarrow\ (\zeta + \eta)\Phi(\zeta, \eta) = \Psi(\zeta, \eta)$$

We hereafter use the notation

$$\overset{\bullet}{\Phi}(\zeta, \eta) := (\zeta + \eta)\Phi(\zeta, \eta).$$

Example 1 (continued). Again, consider $Q_\Phi(w) = \dot{w}_1^2 + 2w_1\dot{w}_2$. Direct calculation shows that

$$\frac{d}{dt}Q_\Phi(w) = 2\dot{w}_1\ddot{w}_1 + 2\dot{w}_1\dot{w}_2 + 2w_1\ddot{w}_2.$$

is induced by

$$\overset{\bullet}{\Phi}(\zeta,\eta) = (\zeta+\eta)\begin{pmatrix}\zeta\eta & \eta\\ \zeta & 0\end{pmatrix} = \begin{pmatrix}\zeta^2\eta+\zeta\eta^2 & \zeta\eta+\eta^2\\ \zeta\eta+\zeta^2 & 0\end{pmatrix}.$$

◇

2.2 Nonnegativity

Definition 1. Let $\Phi \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be given. A QDF Q_Φ, or $\Phi(\zeta,\eta)$, is said to be *nonnegative*, denoted by $\Phi \geq 0$, if

$$Q_\Phi(w)(t) \geq 0 \ \forall t \in \mathbb{R},\ \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}).$$

Furthermore, Q_Φ (or $\Phi(\zeta,\eta)$) is called *positive*, denoted by $\Phi > 0$, if it is nonnegative and $Q_\Phi(w)(t) = 0\ \forall t$ implies $w(t) = 0\ \forall t$. The nonpositivity and negativity of a QDF are also defined in the same way.

Proposition 1. Let $\Phi \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be given.

(i) The following are equivalent.
  (a) $\Phi \geq 0$ (nonnegative).
  (b) There exists $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that
  $$\Phi(\zeta,\eta) = D^\top(\zeta)D(\eta). \tag{1}$$
  (c) $\tilde{\Phi} \geq 0$.

(ii) The following are equivalent.
  (a) $\Phi > 0$ (positive).
  (b) There exists $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that (1) holds, and $D(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.

Note also that $\tilde{\Phi} > 0$ implies the positivity of Q_Φ, but the converse is not true.

Example 2. Consider the QDF

$$Q_\Phi(w) = \dot{w}_1^2 + w_1^2 + w_2^2$$

which is induced by

$$\Phi(\zeta,\eta) = \begin{pmatrix}1+\zeta\eta & 0\\ 0 & 1\end{pmatrix}.$$

It is obvious that Q_Φ is positive. We have

$$\tilde{\Phi} = \left(\begin{array}{cc:cc} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hdashline 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$$

This $\tilde{\Phi}$ is nonnegative definite, but not positive definite. Now, $\Phi(\zeta,\eta)$ is factored as $\Phi(\zeta,\eta) = D^\top(\zeta)D(\eta)$ with

$$D(\xi) = \begin{pmatrix}1 & 0\\ 0 & 1\\ \xi & 0\end{pmatrix}.$$

Actually, this $D(\xi)$ is a canonical factor of $\Phi(\zeta,\eta)$, and $D(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. ◇

2.3 Average nonnegativity and dissipation inequality

We often need to consider integrals of QDF's, in several applications such as LQ optimal control, $\mathcal{H}^\infty$ control, etc. In particular, we are interested in boundedness or nonnegativity of such integrals.

Definition 2. Let $\Phi \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be given.
(i) A QDF Q_Φ, or $\Phi(\zeta,\eta)$, is said to be *average nonnegative*, denoted by $\int Q_\Phi \geq 0$, if

$$\int_{-\infty}^{\infty} Q_\Phi(w)(t)dt \geq 0 \ \ \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \cap \mathfrak{D} \tag{2}$$

(ii) A QDF Q_Φ, or $\Phi(\zeta,\eta)$, is said to be *half-line nonnegative*, denoted $\int^t Q_\Phi \geq 0$, if

$$\int_{-\infty}^{0} Q_\Phi(w)(t)dt \geq 0 \ \ \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \cap \mathfrak{D} \tag{3}$$

The next proposition characterizes necessary and sufficient conditions for the average nonnegativity of Q_Φ.

Proposition 2. The following are equivalent for $\Phi \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$.

(i) $\int Q_\Phi \geq 0$ (average nonnegative).
(ii) $\Phi(\bar{\lambda},\lambda) \geq 0 \ \ \forall \lambda \in \mathrm{i}\mathbb{R}$
(iii) There exists $\Psi \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that
$$\frac{d}{dt}Q_\Psi(w)(t) \leq Q_\Phi(w)(t)\ \forall t \in \mathbb{R},\ \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \tag{4}$$
(iv) There exist $\Psi \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that
$$(\zeta+\eta)\Psi(\zeta,\eta) + F^\top(\zeta)F(\eta) = \Phi(\zeta,\eta) \tag{5}$$

The inequality (4) is closely related to the dissipativity of a linear differential system. In fact, the inequality (4) is referred to as a *dissipation inequality*, in which Q_Φ and Q_Ψ are called a *supply rate* and a *storage function*, respectively. The reason for these names is explained as follows.

Let a controllable system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{v}}$ be given by the image representation $v = \mathrm{im}M(\frac{d}{dt})w$, $M \in \mathbb{R}^{\mathtt{v}\times\mathtt{w}}[\xi]$, namely $\mathfrak{B} = \{v \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{v}})|\ \exists w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})\ \text{s.t.}\ v = M(\frac{d}{dt})w\}$. See Appendix A for the details about linear differential systems. Consider the quadratic form $s(t) = v^\top(t)\Sigma v(t)$, $\Sigma \in \mathbb{R}_{\rm s}^{\mathtt{v}\times\mathtt{v}}$. Using the image representation, $s(t)$ can be expressed as a QDF of the latent variable w as

$$s = Q_\Phi(w) = [M(\frac{d}{dt})w]^\top\Sigma[M(\frac{d}{dt})w] \tag{6}$$

where $\Phi(\zeta,\eta) = M^\top(\zeta)\Sigma M(\eta)$. If we view $s(t) = Q_\Phi(w)(t)$ as the rate of energy supply into the system $\mathfrak{B}$ and $Q_\Psi(w)(t)$ as the energy stored in $\mathfrak{B}$, then (4) claims that the rate of increase of the stored energy $Q_\Psi(\ell)$ never exceeds the rate of energy supply Q_Φ, namely, the system $\mathfrak{B}$ dissipates the energy.

If Q_Φ is average nonnegative, a storage function can be obtained by the following procedure. First, we perform the polynomial spectral factorization

$$\Phi(-\xi,\xi) = F^\top(-\xi)F(\xi). \tag{7}$$

This factorization is possible whenever the condition (ii) in the above proposition is satisfied. Then, one of the storage functions for Q_Φ is induced by

$$\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - F^\top(\zeta)F(\eta)}{\zeta+\eta}.$$

Clearly, the resulting $\Phi(\zeta,\eta)$ and $F(\xi)$ satisfy (5).

It is seen from (5) that there holds

$$\frac{d}{dt}Q_\Psi(w)(t) + Q_\Delta(w)(t) = Q_\Phi(w)(t) \tag{8}$$
$$\forall t \in \mathbb{R},\ \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$$

where we have defined $\Delta(\zeta,\eta) = F^\top(\zeta)F(\eta)$, and Q_Δ is called a *dissipation rate*. Note that $\Delta \geq 0$ holds by its definition and Proposition 1 (i). It is also immediate from (8) that

$$\int_{-\infty}^{\infty} Q_\Delta(w)(t)dt = \int_{-\infty}^{\infty} Q_\Phi(w)(t)dt, \ \ \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \cap \mathfrak{D} \tag{9}$$

Hence, as another characterization of the average nonnegativity (dissipativity) of Q_Φ, we conclude that $\int Q_\Phi \geq 0$ is equivalent to the existence of a dissipation rate Q_Δ satisfying $\Delta \geq 0$ and (9).

As we have seen, a storage function or a dissipation rate is obtained by performing the spectral factorization of (7). This claims that the storage functions for Q_Φ are characterized by the choice of the spectral factor $F(\xi)$. We introduce the Hurwitz and anti-Hurwitz spectral factorizations as

$$\Phi(-\xi,\xi) = H^\top(-\xi)H(\xi), \quad H(\xi): \text{ Hurwitz}, \tag{10}$$

$$\Phi(-\xi,\xi) = A^\top(-\xi)A(\xi), \quad A(\xi): \text{ anti-Hurwitz}. \tag{11}$$

Then, the minimal and maximal storage functions are characterized in the next proposition.

Proposition 3. Assume that Q_Φ is average nonnegative for a given $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}_{\mathrm{s}}[\zeta,\eta]$. Then, there exist Ψ_-, $\Psi_+ \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}_{\mathrm{s}}[\zeta,\eta]$ such that

$$\Psi_- \leq \Psi \leq \Psi_+ \tag{12}$$

holds for any $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}_{\mathrm{s}}[\zeta,\eta]$ which induces a storage function for Q_Φ. If $\Phi(\bar{\lambda},\lambda) > 0$ for all $\lambda \in \mathrm{i}\mathbb{R}$, such $\Psi_-(\zeta,\eta)$ and $\Psi_+(\zeta,\eta)$ are given by

$$\Psi_-(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - H^\top(\zeta)H(\eta)}{\zeta+\eta}, \tag{13}$$

$$\Psi_+(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - A^\top(\zeta)A(\eta)}{\zeta+\eta}, \tag{14}$$

where $H(\xi)$ and $A(\xi)$ are the Hurwitz and anti-Hurwitz spectral factors define in (10),(11).

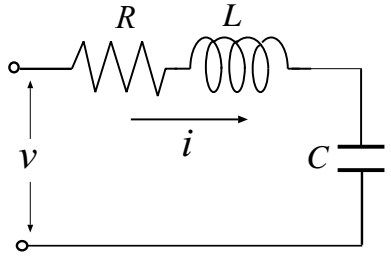


Fig. 1. RLC electrical network (Example 3)

Example 3. We consider a simple electrical network consisting of a resistor R, an inductor L and a capacitor C (Fig. 1). Let $V(t)$ and $I(t)$ be the port voltage and current at time t, respectively. We also define $q(t)$ as the electric charge in the capacitor. Using fundamental laws of electrical elements and circuits, the dynamics of this system is described by

$$\frac{dV}{dt} = \frac{1}{C}I + R\frac{dI}{dt} + L\frac{d^2I}{dt^2} \tag{15}$$

Then, an image representation of this system is given by

$$\begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} R\frac{d}{dt} + L\frac{d^2}{dt^2} + \frac{1}{C} \\ \frac{d}{dt} \end{pmatrix} q,$$

where the electric charge q serves as a latent variable.

As well known, the power (supply rate) into the electrical network is the product $V \cdot I$, which is equivalently described by the QDF

$$VI = Q_\Phi(q),$$

$$\Phi(\zeta,\eta) = \frac{1}{2}\begin{pmatrix} \frac{1}{C} + R\zeta + L\zeta^2 \\ \zeta \end{pmatrix}^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{C} + R\eta + L\eta^2 \\ \eta \end{pmatrix}$$

Define

$$Q_\Psi(q) = \frac{1}{2C}q^2 + \frac{L}{2}(\frac{dq}{dt})^2 = \frac{1}{2C}q^2 + \frac{L}{2}I^2$$

where $\Psi(\zeta,\eta) = \frac{1}{2C} + \frac{1}{2}L\zeta\eta$. Then, $Q_\Psi(q)$ represents the energy stored in the system, namely the sum of the energies stored in the capacitor and the inductor. This $Q_\Psi(q)$ satisfies the dissipation inequality, because

$$\begin{aligned} \frac{d}{dt}Q_\Psi(q) &= \frac{1}{C}QI + LI\frac{dI}{dt} \\ &= (\frac{1}{C}q + L\frac{dI}{dt})I \\ &= (V - RI)I \\ &= Q_\Phi(q) - RI^2 \end{aligned}$$

Hence, we conclude that this system is dissipative with respect to the supply rate $Q_\Phi(q) = VI$, with the dissipation rate $Q_\Delta(q) = RI^2$ and the storage function $Q_\Psi(q)$. In fact, the direct calculation proves that

$$(\zeta+\eta)\Psi(\zeta,\eta) + \Delta(\zeta,\eta) = \Phi(\zeta,\eta), \quad \Delta(\zeta,\eta) = R\zeta\eta \tag{16}$$

◇

Remark 1. If $R = 0$ holds, then we have $\frac{d}{dt}Q_\Psi(q)(t) = Q_\Phi(q)(t)$ for all $t \in \mathbb{R}$ and $q \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$. This implies that the system preserves the supplied energy without dissipating it. Hence such a system is called *lossless*.

A nonnegative storage function often plays an important role in many issues arising in system and control theory such as stability analyses. Note that Example 3 is an example of the system which admits a nonnegative storage function.

The next proposition relates the half-line positivity in Definition 2 (ii) and the existence of a nonnegative storage function.

Proposition 4. Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}_{\mathrm{s}}[\zeta,\eta]$ be given. The following are equivalent.

(i) Q_Φ is average nonnegative and admits a nonnegative storage function.
(ii) $\int^t Q_\Phi \geq 0$ (half-line nonnegative).

Remark 2. The frequency domain condition for the half-line nonnegativity is more complicated than that for the average nonnegativity. In fact, Q_Φ is half-line nonnegative if and only if

- $\Phi(\bar{\lambda},\lambda) \geq 0 \ \ \forall \lambda \in \mathrm{i}\mathbb{R}$.
- The *Pick matrix* T_Φ associated with $\Phi(\zeta,\eta)$ is nonnegative definite.

The Pick matrix is an Hermitian matrix constructed in terms of the stable zeros of the spectral density matrix $\Phi(-\xi,\xi)$. For the detail, see Section 9 in Willems and Trentelman (1998).

2.4 QDF along behavior, and Lyapunov stability

In the analysis and control of linear dynamical systems, we often need to study QDF's whose variables are produced from the systems. More precisely, we consider the properties of properties of $Q_\Phi(w)$, when w belongs to a certain linear differential behavior $\mathfrak{B} \subset \mathfrak{L}^{\mathtt{w}}$.

Definition 3. Let the behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be given. Consider two different QDF's Q_{Φ_1} and Q_{Φ_2} induced by Φ_1, $\Phi_2 \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. We say that Q_{Φ_1} and Q_{Φ_2} are equivalent over $\mathfrak{B}$, denoted by $\Phi_1 \overset{\mathfrak{B}}{=} \Phi_2$, if

$$Q_{\Phi_1}(w)(t) = Q_{\Phi_2}(w)(t) \ \ \forall t \in \mathbb{R},\ \forall w \in \mathfrak{B}.$$

Let $\mathfrak{B}$ be represented by the image representation $w = M(\frac{d}{dt})\ell$, $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{l}}[\xi]$. Then, we easily that

$$Q_\Phi(w)(t) = Q_\Psi(\ell)(t) \ \ \forall t \in \mathbb{R},$$

where we have defined $\Psi(\zeta,\eta) = M^\top(\zeta)\Phi(\zeta,\eta)M(\eta)$. Since ℓ is free in the image representation, the trajectory of $Q_\Phi(w)(t)$ over $\mathfrak{B}$ is completely characterized by $Q_\Psi(\ell)$. We thus obtain the following result.

Proposition 5. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be represented by the image representation $w = M(\frac{d}{dt})\ell$, $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{l}}[\xi]$. Then, $\Phi_1 \overset{\mathfrak{B}}{=} \Phi_2$ if and only if

$$M^\top(\zeta)\Phi_1(\zeta,\eta)M(\eta) = M^\top(\zeta)\Phi_2(\zeta,\eta)M(\eta).$$

Next, assume that $\mathfrak{B}$ is represented by the kernel representation

Proposition 6. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be represented by the kernel representation $R(\frac{d}{dt})w = 0$, $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$. Then, the following are equivalent.

(i) $\Phi_1 \overset{\mathfrak{B}}{=} \Phi_2$.
(ii) There exists $X \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\zeta,\eta]$ such that

$$\Phi_1(\zeta,\eta) = \Phi_2(\zeta,\eta) + X^\top(\eta,\zeta)R(\eta) + R^\top(\zeta)X(\zeta,\eta)$$

The next proposition characterizes the nonnegativity and positivity of a QDF along the behavior.

Proposition 7. Let $\Phi \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be given. Assume that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be represented by the kernel representation $R(\frac{d}{dt})w = 0$, $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$.

(i) $\Phi \overset{\mathfrak{B}}{\geq} 0$ holds if and only if there exists $X \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\zeta,\eta]$ and $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ satisfying

$$\Phi(\zeta,\eta) = X^\top(\eta,\zeta)R(\eta) + R^\top(\zeta)X(\zeta,\eta) + D^\top(\zeta)D(\eta) \tag{17}$$

(ii) $\Phi \overset{\mathfrak{B}}{>} 0$ holds if and only if there exist $X \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\zeta,\eta]$ and $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ satisfying (17), with $R(\xi)$ and $D(\xi)$ being right coprime, i.e. $\begin{pmatrix} R(\lambda) \\ D(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$.

The above result plays an important role in the Lyapunov stability analysis of a linear differential system. Recall that a linear differential behavior $\mathfrak{B}$ is said to be *asymptotically stable* if

$$w(t) \to 0 \ (\text{as } t \to +\infty) \ \ \forall w \in \mathfrak{B}.$$

Proposition 8. Let an autonomous behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be given by the kernel representation $R(\frac{d}{dt})w = 0$, $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$. Then, the following are equivalent.

(i) $\mathfrak{B}$ is asymptotically stable.
(ii) There exists $\Psi \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that

$$\Psi \overset{\mathfrak{B}}{\geq} 0 \ \& \ \overset{\bullet}{\Psi} \overset{\mathfrak{B}}{<} 0$$

(iii) There exist $\Psi \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, $X, Y \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\zeta,\eta]$, $D, E \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that

$$\Psi(\zeta,\eta) = X^\top(\eta,\zeta)R(\eta) + R^\top(\zeta)X(\zeta,\eta) + D^\top(\zeta)D(\eta) \tag{18}$$

$$(\zeta+\eta)\Psi(\zeta,\eta) = Y^\top(\eta,\zeta)R(\eta) + R^\top(\zeta)Y(\zeta,\eta) - E^\top(\zeta)E(\eta), \tag{19}$$

where $R(\xi)$ and $E(\xi)$ are right coprime.

The condition (ii) in the above theorem is equivalent to the existence of a QDF Q_Ψ satisfying

$$Q_\Psi(w)(t) \geq 0 \ \& \frac{d}{dt}Q_\Psi(w)(t) < 0 \ \forall t \in \mathbb{R},\ \forall w \in \mathfrak{B}.$$

If we regard $Q_\Psi(w)(t)$ as the energy stored in the system at time t, these inequalities mean that the energy monotonically decays to its steady-state value. For this reason, the QDF Q_Ψ is called a *Lyapunov function* of $\mathfrak{B}$. Note also that the equivalence (ii)⇔(iii) immediately follows from Proposition 7.

Example 4. Consider an unforced mechanical system described by the following equation of motion

$$m\ddot{x} + d\dot{x} + kx = 0,$$

where m, d and k represent the mass, damping coefficient and spring constant, respectively. Namely,

$$\mathfrak{B} = \left\{ x \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) |\ R(\tfrac{d}{dt})x = 0 \right\}, \quad R(\xi) = m\xi^2 + d\xi + k$$

Then, the QDF

$$Q_\Psi(x) = \frac{1}{2}(kx^2 + m\dot{x}^2), \quad \Psi(\zeta,\eta) = \frac{1}{2}(k + m\zeta\eta).$$

is a Lyapunov function that proves the asymptotic stability of $\mathfrak{B}$. In fact, (18) and (19) are satisfied for

$$X(\zeta,\eta), \ \ Y(\zeta,\eta) = \frac{1}{2}\eta, \ \ D(\xi) = \begin{pmatrix} 1 \\ \xi \end{pmatrix}, \ \ E(\xi) = \sqrt{d}\xi.$$

In addition,

$$\begin{pmatrix} R(\lambda) \\ E(\lambda) \end{pmatrix} = \begin{pmatrix} m\lambda^2 + d\lambda + k \\ \sqrt{d}\lambda \end{pmatrix}$$

has full column rank for all $\lambda \in \mathbb{C}$. Hence, the asymptotic stability of $\mathfrak{B}$ is proven by Proposition 8 (i)⇔(iii).

As easily seen from this example, the main advantage of the QDF-based stability condition is that the asymptotic stability can be checked by using the artless high-order differential equation without converting it into a state-space form. ◇

3. RATIONAL QUADRATIC DIFFERENTIAL FORMS

The rest of this paper is devoted to more general QDF's defined by rational functions, which we call *rational QDF's*.

Quadratic constraints on rational transfer functions such as bounded realness, positive realness, integral quadratic constraints (IQCs) and quadratic separators (see e.g. Megretski and Rantzer 1997, Iwasaki and Hara 1998) have been playing a crucial role in the analyses of linear dynamical systems (especially, stability analysis of feedback systems). Also, by using state-space representations, the well-

known Kalman-Yakubovich-Popov (KYP) lemma clarified the connection between such constraints and dissipativity.

The purpose of this section is to introduce the notion of rational QDF's as the foundation for studying the above quadratic constraints on rational functions from viewpoint of the behavioral approach.

We now formulate a rational QDF induced by a symmetric two-variable *rational* matrix $\Phi \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}(\zeta,\eta)$ rather than a two-variable polynomial matrix. Similarly to the polynomial case, we mean $\Phi(\zeta,\eta) = \Phi^\top(\eta,\zeta)$ by the term "symmetric".

Definition 4. A symmetric two-variable rational matrix $\Phi \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}(\zeta,\eta)$ is said to be *factorizable* if there exist a rational matrix $G \in \mathbb{R}^{\bullet\times\mathtt{w}}(\xi)$ and a signature matrix $\Sigma \in \mathbb{R}_{\rm s}^{\bullet\times\bullet}$ such that

$$\Phi(\zeta,\eta) = G^\top(\zeta)\Sigma G(\eta), \quad . \tag{20}$$

It is straightforward to show that $\Phi(\zeta,\eta)$ is factorizable iff the least common multiple of the denominators of all entries of $\Phi(\zeta,\eta)$ is factored as $\varphi(\zeta)\varphi(\eta)$, namely, $\Phi(\zeta,\eta)$ is expressed as

$$\Phi(\zeta,\eta) = \frac{\Pi(\zeta,\eta)}{\varphi(\zeta)\varphi(\eta)}, \quad \Pi \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta], \ \varphi \in \mathbb{R}[\xi]. \tag{21}$$

Of course, the factorization in (20) is not unique. We say that the factorization $\Phi(\zeta,\eta) = G^\top(\zeta)\Sigma G(\eta)$ is *canonical*, if $\mathtt{row}(G) \le \mathtt{row}(G')$ for any other $G' \in \mathbb{R}^{\bullet\times\mathtt{w}}(\xi)$ and $\Sigma' = \mathrm{diag}(I_{\mathtt{r}_+}, -I_{\mathtt{r}_-}) \in \mathbb{R}_{\rm s}^{\bullet\times\bullet}$ satisfying $\Phi(\zeta,\eta) = {G'}^\top(\zeta)\Sigma' G'(\eta)$, where $\mathtt{row}(\,\cdot\,)$ denotes the number of rows of a (rational) matrix.

We hereafter make the following assumption.

Assumption 1. The two-variable rational matrix $\Phi \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}(\zeta,\eta)$ is symmetric, factorizable and admits a canonical factorization in the form of (20) with $\mathtt{v} := \mathtt{row}(G)$.

We now at the position to define a rational QDF.

Definition 5. Under Assumption 1, *the rational QDF with respect to* Φ is the set defined by

$$Q_\Phi^{\mathbf{r}}(w) := \left\{ s \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \;\middle|\; \exists v \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{v}}) \text{ s.t. } s = v^\top \Sigma v \ \&\ v = G(\frac{d}{dt})w \right\}.$$

Note that we put the superscript "$\mathbf{r}$" in order to distinguish the rational QDF from polynomial QDF's.

The rational QDF is well-defined in the sense that it is uniquely defined regardless of the choice of the canonical factorization. The reason of this uniqueness is as follows. Suppose that $\Phi \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}(\zeta,\eta)$ is expressed as in (21). Moreover, assume that $\Phi(\zeta,\eta)$ admits two different canonical factorizations as

$$\Phi(\zeta,\eta) = G_1^\top(\zeta)\Sigma G_1(\eta) = G_2^\top(\zeta)\Sigma G_2(\eta).$$

By the definition of the canonical factorization, we must have $\mathtt{row}(G_1) = \mathtt{row}(G_2) =: \mathtt{v}$. Then, it is obvious that $\varphi(\xi)G_i(\xi)$ $(i=1,2)$ is a polynomial matrix in $\mathbb{R}^{\mathtt{v}\times\mathtt{w}}[\xi]$ and induces a canonical factorization of the polynomial matrix $\Pi(\zeta,\eta)$ in the polynomial matrix sense. (Otherwise, it can be shown that $G_i(\xi)$ is not a canonical factor of $\Phi(\zeta,\eta)$.) As discussed in Sub-section 2.1, there exists a nonsingular constant matrix U such that $\varphi(\xi)G_1(\xi) = U\varphi(\xi)G_2(\xi)$ and $U^\top\Sigma U = \Sigma$. Hence, we have $G_1(\xi) = UG_2(\xi)$. As a result, we see that the rational QDF's induced from two different canonical factors are identical, because there holds

$$\begin{aligned}
&\left\{ s \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \mid \exists v_1 \text{ s.t. } s = v_1^\top\Sigma v_1 \ \&\ v_1 = G_1(\tfrac{d}{dt})w \right\} \\
&= \left\{ s \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \mid \exists v_1 \text{ s.t. } s = v_1^\top\Sigma v_1 \ \&\ v_1 = UG_2(\tfrac{d}{dt})w \right\} \\
&= \left\{ s \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \mid \exists v_2 \text{ s.t. } s = v_2^\top U^\top\Sigma U v_2 \right. \\
&\qquad\qquad \left. \&\ v_2 = G_2(\tfrac{d}{dt})w \right\} \\
&= \left\{ s \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \mid \exists v_2 \text{ s.t. } s = v_2^\top\Sigma v_2 \ \&\ v_2 = G_2(\tfrac{d}{dt})w \right\},
\end{aligned}$$

where the most left- and right-hand sides are the rational QDF's induced by $G_1(\xi)$ and $G_2(\xi)$, respectively.

Example 5. We again consider the electrical network in Fig. 1. We see from (15) that a rational representation of this system is given by

$$\begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 1 \\ H(\frac{d}{dt}) \end{pmatrix} V, \quad H(\xi) = \frac{\xi}{\frac{1}{C} + R\xi + L\xi^2} \tag{22}$$

As we have seen earlier, the supply rate is given by

$$s = VI = (V \ \ I) \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}. \tag{23}$$

This expression of s is associated with the rational matrix

$$\Phi(\zeta,\eta) = \frac{1}{2}\,(1 \ \ H(\zeta)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ H(\eta) \end{pmatrix} = \frac{1}{2}\,(H(\zeta) + H(\eta)).$$

One of the canonical factorization of $\Phi(\zeta,\eta)$ is given by

$$G(\xi) = \frac{1}{2}\begin{pmatrix} 1 + H(\xi) \\ 1 - H(\xi) \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Therefore, the supply rate s can be expressed as the rational QDF with

$$s = v^\top \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v, \quad v = \frac{1}{2}\begin{pmatrix} V + I \\ V - I \end{pmatrix}, \ w = V$$

◇

Proposition 9. For two symmetric rational matrices Φ_1, $\Phi_2 \in \mathbb{R}_{\rm s}^{\mathtt{w}\times\mathtt{w}}(\zeta,\eta)$, we have

$$Q_{\Phi_1+\Phi_2}^{\mathbf{r}}(w) \subseteq Q_{\Phi_1}^{\mathbf{r}}(w) + Q_{\Phi_2}^{\mathbf{r}}(w) \quad \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}).$$

Proof. We introduce the canonical factorizations of Φ_1, Φ_2 and $\Phi_1 + \Phi_2$ as

$$\begin{aligned}
\Phi_1(\zeta,\eta) &= G_1^\top(\zeta)\Sigma_1 G_1(\eta), \\
\Phi_2(\zeta,\eta) &= G_2^\top(\zeta)\Sigma_2 G_2(\eta), \\
\Phi_1(\zeta,\eta) + \Phi_2(\zeta,\eta) &= G^\top(\zeta)\Sigma G_(\eta)
\end{aligned}$$

Since the rows of $G(\xi)$ are linearly independent over $\mathbb{R}$, there exists a nonsingular square matrix K such that

$$\begin{pmatrix} G_1(\xi) \\ G_2(\xi) \end{pmatrix} = K \begin{pmatrix} G(\xi) \\ 0 \end{pmatrix}, \quad (I \ 0)\, K^\top \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} K \begin{pmatrix} I \\ 0 \end{pmatrix} = \Sigma. \tag{24}$$

Assume that $s \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ belongs to $Q_{\Phi_1+\Phi_2}^{\mathbf{r}}(w)$. There exists a $v \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)$ such that $s = v^\top\Sigma v$ and $v = G(\frac{d}{dt})w$. Define

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = K \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

It then follows from (24) that

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} G_1(\frac{d}{dt}) \\ G_2(\frac{d}{dt}) \end{pmatrix} w, \quad v^\top\Sigma v = v_1^\top\Sigma_1 v_1 + v_2^\top\Sigma_2 v_2$$

This implies that $v_1^\top \Sigma_1 v_1 \in Q^{\mathbf{r}}_{\Phi_1}(w)$, $v_2^\top \Sigma_2 v_2 \in Q^{\mathbf{r}}_{\Phi_2}(w)$, and hence $v \in Q^{\mathbf{r}}_{\Phi_1}(w) + Q^{\mathbf{r}}_{\Phi_2}(w)$. That is, $Q^{\mathbf{r}}_{\Phi_1+\Phi_2}(w) \subseteq Q^{\mathbf{r}}_{\Phi_1}(w) + Q^{\mathbf{r}}_{\Phi_2}(w)$ holds. ■

Let the right coprime factorization of $G(\xi)$ in (20) over $\mathbb{R}[\xi]$ be given by

$$G(\xi) = N(\xi)D^{-1}(\xi), \quad N \in \mathbb{R}^{\mathbf{v}\times\mathbf{w}}[\xi],\ D \in \mathbb{R}^{\mathbf{w}\times\mathbf{w}}[\xi]. \quad (25)$$

Then, we see from Appendix A that the pair $(v, w) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{v}+\mathbf{w}})$ is a solution of $v = G(\frac{d}{dt})w$ iff there exists $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}})$ satisfying

$$v = N(\frac{d}{dt})\ell \ \ \& \ \ w = D(\frac{d}{dt})\ell \quad (26)$$

Since $s = v^\top \Sigma v$ is rewritten as $s = [N(\frac{d}{dt})\ell]^\top \Sigma [N(\frac{d}{dt})\ell] = Q_{N^\top \Sigma N}(\ell)$, we have

$$Q^{\mathbf{r}}_\Phi(w) = \left\{ Q_{N^\top\Sigma N}(\ell) \,\middle|\, w = D(\frac{d}{dt})\ell,\ \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}}) \right\}. \quad (27)$$

Therefore, the rational QDF $Q^{\mathbf{r}}_\Phi$ inherits the properties (e.g. nonnegativity, average nonnegativity, etc) of the polynomial QDF $Q_{N^\top\Sigma N}$ as discussed below.

In the same way as the polynomial case, we introduce the notation

$$\overset{\bullet}{\Phi}(\zeta, \eta) = (\zeta + \eta)\Phi(\zeta, \eta).$$

Proposition 10. Let a symmetric rational matrix $\Phi \in \mathbb{R}^{\mathbf{w}\times\mathbf{w}}_{\mathrm{s}}(\zeta, \eta)$ be given. Then, under Assumption 1, we obtain

$$Q^{\mathbf{r}}_{\overset{\bullet}{\Phi}}(w) = \left\{ r \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}) \,\middle|\, \exists s \in Q^{\mathbf{r}}_\Phi(w) \text{ s.t. } r = \frac{d}{dt}s \right\} \quad (28)$$

Proof: Let a right coprime factorization be given by (25). We also define

$$\Theta(\zeta, \eta) = D^\top(\zeta)\Phi(\zeta, \eta)D(\eta) = N^\top(\zeta)\Sigma N(\eta).$$

Then, $\Theta(\zeta, \eta)$ is a symmetric polynomial matrix in $\mathbb{R}^{\mathbf{w}\times\mathbf{w}}[\zeta, \eta]$. Moreover, let $M \in \mathbb{R}^{\mathbf{v}\times\mathbf{w}}[\xi]$ and $\Sigma' \in \mathbb{R}^{\mathbf{v}\times\mathbf{v}}$ induce a canonical factorization

$$(\zeta + \eta)\Theta(\zeta, \eta) = M^\top(\zeta)\Sigma' M(\eta).$$

Then, there holds

$$\frac{d}{dt}\left\{ (N(\frac{d}{dt})\ell)^\top \Sigma N(\frac{d}{dt})\ell \right\} = \left(M(\frac{d}{dt})\ell \right)^\top \Sigma' \left(M(\frac{d}{dt})\ell \right) \quad \forall \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}}) \quad (29)$$

Note that, by the definition of the canonical factorization, it can be shown that the pair (M, D) is right coprime. We also see that $H(\xi) := M(\xi)D^{-1}(\xi)$ is a canonical factor of $\overset{\bullet}{\Phi}(\zeta, \eta)$. As a result, we get

$$\begin{aligned} Q^{\mathbf{r}}_{\overset{\bullet}{\Phi}}(w) &= \left\{ r \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}) \,\middle|\, \exists z \text{ s.t. } r = z^\top \Sigma' z \ \& \ z = H(\tfrac{d}{dt})w \right\} \\ &= \left\{ Q_{M^\top\Sigma' M}(\ell) \,\middle|\, w = D(\tfrac{d}{dt})\ell,\ \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}}) \right\} \\ &= \left\{ \tfrac{d}{dt} Q_{N^\top\Sigma' N}(\ell) \,\middle|\, w = D(\tfrac{d}{dt})\ell,\ \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}}) \right\} \\ &= \left\{ \tfrac{d}{dt}s \,\middle|\, s = Q_{N^\top\Sigma' N}(\ell) \ \& \ w = D(\tfrac{d}{dt})\ell \ \& \ \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}}) \right\} \\ &= \left\{ \tfrac{d}{dt}s \,\middle|\, s \in Q^{\mathbf{r}}_\Phi(w) \right\} \end{aligned}$$

This completes the proof. ■

3.1 Nonnegativity of rational QDF

Definition 6. Under Assumption 1, $Q^{\mathbf{r}}_\Phi$ is said to be *nonnegative*, denoted by $\Phi \geq 0$, if

$$[\![\ s(t) \geq 0\ \forall t \in \mathbb{R}\ \forall s \in Q^{\mathbf{r}}_\Phi(w)\ ]\!]\ \ \forall w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}})$$

The nonpositivity of $Q^{\mathbf{r}}_\Phi$ is also defined in the same way.

Proposition 11. Under Assumption 1,

$$\Phi \geq 0 \ \Leftrightarrow\ \exists K \in \mathbb{R}^{\bullet\times\mathbf{w}}(\xi) \text{ s.t. } \Phi(\zeta, \eta) = K^\top(\zeta)K(\eta) \quad (30)$$

Proof:

($\Leftarrow$) Obvious because we can assume without loss of generality that $\Phi(\zeta, \eta) = K^\top(\zeta)K(\eta)$ is a canonical factorization. Even if this not the case, we can easily show that the canonical factorization is given in the form of $\Phi(\zeta, \eta) = G^\top(\zeta)G(\eta)$, $\Sigma = I_{\mathbf{v}}$.

($\Rightarrow$) Under Assumption 1, we again introduce a right coprime factorization of (25). It then follows from (27) and Definition 6 that

$$\begin{aligned} \Phi \geq 0 \ \Leftrightarrow s(t) = Q_{N^\top\Sigma N}(\ell)(t) \geq 0\ \ \forall t \in \mathbb{R}, \\ \forall \ell \text{ s.t. } w = D(\tfrac{d}{dt})\ell,\ \forall w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}}) \end{aligned}$$

As shown Proposition A.2 (i), we see that $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}}) \Leftrightarrow \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}})$ for the solutions of $w = D(\frac{d}{dt})\ell$. Therefore, we have

$$\begin{aligned} \Phi \geq 0 &\Leftrightarrow Q_{N^\top\Sigma N}(\ell)(t) \geq 0\ \ \forall t \in \mathbb{R},\ \forall \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}}) \\ &\Leftrightarrow N^\top(\zeta)\Sigma N(\eta) \geq 0 \ \ \text{(in the sense of } \textit{polynomial} \text{ QDF's)} \\ &\Leftrightarrow\ \exists F \in \mathbb{R}^{\bullet\times\mathbf{w}}[\xi] \text{ s.t. } N^\top(\zeta)\Sigma N(\eta) = F^\top(\zeta)F(\eta) \\ &\qquad \text{(by Proposition 1 (i))} \end{aligned}$$

As a result, by defining $K(\xi) = F(\xi)D^{-1}(\xi)$, we obtain $\Phi(\zeta, \eta) = K^\top(\zeta)K(\eta)$. This completes the proof. ■

Also, the inequality between two rational QDF's is defined as follows.

$$\begin{aligned} &\Phi_1 - \Phi_2 \geq 0 \\ &:\Leftrightarrow s(t) \geq 0\, \forall t \in \mathbb{R}, \forall s \in Q^{\mathbf{r}}_{\Phi_1-\Phi_2}(w), \forall w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}}) \end{aligned}$$

Note that $\Phi_1 - \Phi_2 \geq 0$ does *not* imply $s_1(t) \geq s_2(t), \forall t \in \mathbb{R}$, $\forall s_i \in Q^{\mathbf{r}}_{\Phi_i}(w)$ $(i = 1, 2)$, $\forall w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}})$.

3.2 Average nonnegativity and dissipation inequality

In order to define the average nonnegativity, we introduce the set $\hat{Q}^{\mathbf{r}}_\Phi(w)$ as

$$\begin{aligned} \widehat{Q}^{\mathbf{r}}_\Phi(w) = \big\{ s \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}) \,\big|\, \exists v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{v}}) \cap \mathfrak{D} \text{ s.t.} \\ s = v^\top \Sigma v \ \& \ v = G(\tfrac{d}{dt})w \big\}. \end{aligned}$$

Clearly, $\hat{Q}^{\mathbf{r}}_\Phi(w)$ is a subset of $Q^{\mathbf{r}}_\Phi(w)$, and any $s \in \hat{Q}^{\mathbf{r}}_\Phi(w)$ has compact support, guaranteeing the existence of the integral $\int_{-\infty}^{\infty} s(t)dt$. .

Definition 7. Under Assumption 1, $Q^{\mathbf{r}}_\Phi$ is called *average nonnegative*, denoted by $\int Q^{\mathbf{r}}_\Phi \geq 0$, if there holds

$$\int_{-\infty}^{\infty} s(t)dt \geq 0,\ \forall s \in \hat{Q}^{\mathbf{r}}_\Phi(w),\ \forall w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{w}}) \cap \mathfrak{D}. \quad (31)$$

The average nonpositivity is also defined in the same way.

It may be noted that the integral inequality in (31) can be interpreted as a kind of integral quadratic constraints (Megretski and Rantzer 1997) in the time domain setting.

Lemma 1. Under Assumption 1, $Q^{\mathbf{r}}_\Phi$ is average nonnegative if and only if there exist a factorizable $\Psi \in \mathbb{R}^{\mathbf{w}\times\mathbf{w}}_{\mathrm{s}}(\zeta, \eta)$ and an $F \in \mathbb{R}^{\bullet\times\mathbf{w}}(\xi)$ such that

$$(\zeta + \eta)\Psi(\zeta, \eta) + F^\top(\zeta)F(\eta) = \Phi(\zeta, \eta) \quad (32)$$

Proof:

($\Rightarrow$) In view of (26) and Proposition A.2 (ii), $(v,\ w)$ has compact support iff so does ℓ. It thus follows that

$$\begin{aligned}
&\int Q_\Phi^{\mathbf{r}} \geq 0 \\
&\Leftrightarrow \int_{-\infty}^{\infty} v^\top(t)\Sigma v(t)dt \geq 0 \ \forall v \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{v}}) \cap \mathfrak{D} \text{ s.t. } v = G(\tfrac{d}{dt})w, \\
&\qquad\qquad \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \\
&\Leftrightarrow \int_{-\infty}^{\infty} v^\top(t)\Sigma v(t)dt \geq 0,\ v = N(\tfrac{d}{dt})\ell \ \forall \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \cap \mathfrak{D} \\
&\Leftrightarrow \int Q_{N^\top(\zeta)\Sigma N(\eta)} \geq 0 \quad \text{(in the sense of } \textit{polynomial} \text{ QDF's)} \\
&\Leftrightarrow \exists\, \hat\Psi \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta],\ \hat F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi] \ \text{ such that} \\
&\qquad (\zeta+\eta)\hat\Psi(\zeta,\eta) + \hat F^\top(\zeta)\hat F(\eta) = N^\top(\zeta)\Sigma N(\eta) \quad (33) \\
&\qquad\qquad \text{(by Proposition 2 (i)}\Leftrightarrow\text{(iv))}
\end{aligned}$$

Pre- and post-multiplying (33) by $D^{-\top}(\zeta)$ and $D^{-1}(\eta)$, respectively, we obtain (32) with

$$\begin{aligned}
\Psi(\zeta,\eta) &:= D^{-\top}(\zeta)\hat\Psi(\zeta,\eta)D^{-1}(\eta), \\
F(\xi) &:= \hat F(\xi)D^{-1}(\xi).
\end{aligned}$$

It is clear from this definition that $\Psi(\zeta,\eta)$ is factorizable.

($\Leftarrow$) Since $\Phi(\zeta,\eta) = G^\top(\zeta)\Sigma G(\eta)$ and $G(\xi) = N(\xi)D^{-1}(\xi)$, it follows from (32) that

$$\begin{aligned}
(\zeta+\eta)D^\top(\zeta)\Psi(\zeta,\eta)D(\eta) + (F(\zeta)D(\zeta))^\top F(\eta)D(\eta) \\
= N^\top(\zeta)\Sigma N(\eta) \quad (34)
\end{aligned}$$

To prove the sufficiency, we first show that we can always find a solution of (32) such that both $D^\top(\zeta)\Psi(\zeta,\eta)D(\eta)$ and $F(\eta)D(\eta)$ are polynomial matrices. Suppose that this is not the case. It follows from (34) that

$$N^\top(-\xi)\Sigma N(\xi) = (F(-\xi)D(-\xi))^\top F(\xi)D(\xi)$$

Since the left-hand side is a polynomial matrix, $F(\xi)D(\xi)$ must be factored as

$$F(\xi)D(\xi) = U(\xi)W(\xi),$$

where $W(\xi)$ is a polynomial matrix, and $U(\xi)$ is a unitary rational matrix, namely $U^\top(-\xi)U(\xi) = I$. Then, we redefine $F(\xi)$ and $\Psi(\zeta,\eta)$ as

$$\begin{aligned}
F(\xi) &\leftarrow W(\xi)D^{-1}(\xi) \\
\Psi(\zeta,\eta) &\leftarrow \Psi(\zeta,\eta) + \tfrac{1}{\zeta+\eta}\left\{F^\top(\zeta)F(\eta) - D^{-\top}(\zeta)W^\top(\zeta)W(\eta)D^{-1}(\eta)\right\}
\end{aligned}$$

Note that this new $\Psi(\zeta,\eta)$ is still factorizable because $F^\top(\zeta)F(\eta) - D^{-\top}(\zeta)W^\top(\zeta)W(\eta)D^{-1}(\eta)$ has $\zeta+\eta$ as a factor of its numerator. Furthermore, since $(\zeta+\eta)D^\top(\zeta)\Psi(\zeta,\eta)D(\eta)$ is a polynomial matrix from (34), we conclude $D^\top(\zeta)\Psi(\zeta,\eta)D(\eta)$ is a polynomial matrix. Thus, we hereafter assume that both $D^\top(\zeta)\Psi(\zeta,\eta)D(\eta)$ and $F(\eta)D(\eta)$ are polynomial matrices.

Next, we define

$$\Xi(\zeta,\eta) = D^\top(\zeta)\Psi(\zeta,\eta)D(\eta) \quad (35)$$

$$\Delta(\zeta,\eta) = (F(\zeta)D(\zeta))^\top F(\eta)D(\eta) \quad (36)$$

Since $\Xi(\zeta,\eta)$ and $\Delta(\zeta,\eta)$ are polynomial matrices. the polynomial QDF's Q_Ξ and Q_Δ satisfies

$$Q_{N^\top\Sigma N}(\ell) = \frac{d}{dt}Q_\Xi(\ell) + Q_\Delta(\ell) \quad (37)$$

Hence, for every $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \cap \mathfrak{D}$, integrating (37) from $-\infty$ to ∞ yields

$$\begin{aligned}
\int_{-\infty}^{\infty} Q_{N^\top\Sigma N}(\ell)dt &= \int_{-\infty}^{\infty} Q_\Delta(\ell)dt + Q_\Xi(\ell)(\infty) - Q_\Xi(\ell)(-\infty) \\
&= \int_{-\infty}^{\infty} Q_\Delta(\ell)dt \\
&\geq 0 \quad (38)
\end{aligned}$$

In the last inequality, we have used the fact that $\Delta \geq 0$. By Proposition A.2, for the solution of $v = G(\frac{d}{dt})w$, v and w have compact support iff $\ell \in \mathfrak{D}$. Hence, (38) implies that $Q_\Phi^{\mathbf{r}}$ is average nonnegative. ■

Remark 3. We give a remark regarding to Q_Ψ and Q_Δ introduced in the proof of sufficiency. Since $Q_\Delta(\ell)(t) \geq 0$ holds for all $t \in \mathbb{R}$ and for all $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, (37) reduces to a version of *dissipation inequality*

$$\begin{aligned}
Q_{N^\top\Sigma N}(\ell)(t) \geq \frac{d}{dt}Q_\Xi(\ell)(t) \ \ \forall t \in \mathbb{R}, \\
\forall \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \text{ s.t. } w = D(\frac{d}{dt})\ell
\end{aligned}$$

In this case, Q_Ξ and Q_Δ respectively serve as a *storage function* and a *dissipation rate* for the QDF Q_Φ represented by a rational matrix in $\mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}(\zeta,\eta)$.

The next lemma provides a frequency domain condition for the average nonnegativity.

Lemma 2. Under Assumption 1, there exist a $\Psi \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}(\zeta,\eta)$ and a $F \in \mathbb{R}^{\bullet\times\mathtt{w}}(\xi)$ satisfying (32) if and only if

$$\Phi(\bar\lambda,\lambda) \geq 0 \ \ \forall \lambda \in \mathrm{i}\mathbb{R}\backslash\{\text{poles of } G(\xi)\} \quad (39)$$

Proof: ($\Rightarrow$) It readily follows from (32) that

$$\Phi(\bar\lambda,\lambda) = (F(\lambda))^* F(\lambda) \geq 0 \ \forall \lambda \in \mathrm{i}\mathbb{R}\backslash\{\text{poles of } G(\xi)\}$$

($\Leftarrow$) Let $G(\xi) = N(\xi)D^{-1}(\xi)$ be a right coprime factorization over $\mathbb{R}[\xi]$. Since $\det D(\lambda) \neq 0$ holds for all $\lambda \in \mathrm{i}\mathbb{R}\backslash\{\text{poles of } G(\xi)\}$, and since $N(\lambda)$ is continuous in $\lambda \in \mathbb{C}$, we obtain

$$\begin{aligned}
&\Phi(\bar\lambda,\lambda) \geq 0 \ \forall \lambda \in \mathrm{i}\mathbb{R}\backslash\{\text{poles of } G(\xi)\} \\
&\Leftrightarrow\ N^\top(\bar\lambda)\Sigma N(\lambda) \geq 0 \ \forall \lambda \in \mathrm{i}\mathbb{R}
\end{aligned}$$

The latter condition is equivalent to the existence of polynomial matrices $\hat\Psi \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and $\hat F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that

$$(\zeta+\eta)\hat\Psi(\zeta,\eta) + \hat F^\top(\zeta)\hat F(\eta) = N^\top(\zeta)\Sigma N(\eta)$$

(see Propositions 2 (ii)$\Leftrightarrow$(iv)). Thus, we obtain (32) by defining $\Psi(\zeta,\eta) = D^{-\top}(\zeta)\hat\Psi(\zeta,\eta)D^{-1}(\eta)$ and $F(\xi) = \hat F(\xi)D^{-1}(\xi)$, and by pre- and post-multiplying the above equation by $D^{-\top}(\xi)$ and $D^{-1}(\xi)$, respectively. ■

Summarizing the discussions in Lemmas 1 and 2, we obtain the following theorem.

Theorem 1. Under Assumption 1, the following statements are equivalent.

(i) $\int Q_\Phi^{\mathbf{r}} \geq 0$ (average nonnegative).

(ii) There exists a symmetric and factorizable rational matrix $\Psi \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}(\zeta,\eta)$ such that

$$\overset{\bullet}{\Psi} - \Phi \leq 0 \quad (40)$$

(iii) $\Phi(\bar\lambda,\lambda) \geq 0$ holds for all $\lambda \in \mathrm{i}\mathbb{R}\backslash\{\text{poles of } G(\xi)\}$.

(iv) There exists a symmetric and factorizable rational matrix $\Psi \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}(\zeta,\eta)$ and a rational matrix $F \in \mathbb{R}^{\bullet\times\mathtt{w}}(\xi)$ such that

$$\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta) + F^\top(\zeta)F(\eta) \qquad (41)$$

Moreover, among Ψ's satisfying (40) or (41), there exists one for which $D^\top(\zeta)\Psi(\zeta,\eta)D(\eta)$ is a polynomial matrix for the right coprime factors (N,D) of $G(\xi)$.

Proof: It is easily seen from Proposition 11 that $\overset{\bullet}{\Psi} - \Phi \leq 0$ holds iff there exists $F \in \mathbb{R}^{\bullet\times \mathtt{w}}(\xi)$ satisfying

$$\Phi(\zeta,\eta) - (\zeta+\eta)\Psi(\zeta,\eta) = F^\top(\zeta)F(\eta).$$

Therefore, (i)$\Leftrightarrow$(ii) and (ii)$\Leftrightarrow$(iii) immediately follow from Lemmas 1 and 2, respectively. The existence of $\Psi(\zeta,\eta)$ for which $\overset{\bullet}{\Psi} - \Phi \leq 0$ and $D^\top(\zeta)\Psi(\zeta,\eta)D(\eta)$ is a polynomial matrix is also clear from the proof of the sufficiency of Lemma 1. ■

Remark 4. The equivalence between (iii) and (iv), namely Lemma 2, claims that the para-Hermitian rational matrix $\Phi(-\xi,\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}(\xi)$ admits a spectral factorization

$$\Phi(-\xi,\xi) = F^\top(-\xi)F(\xi), \quad F \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}(\xi),$$

if and only if the condition (iii) is satisfied (This condition was proved by Youla (1961)). Hence, the above result provides an behavioral interpretation to Youla's spectral factorizability condition.

Example 5 (continued). Since

$$\Phi(-\xi,\xi) = \frac{(-R\xi^2}{(\frac{1}{C} - R\xi + L\xi^2)(\frac{1}{C} + R\xi + L\xi^2)},$$

we have

$$\Phi(\bar{\lambda},\lambda) = R\left|\frac{\lambda}{\frac{1}{C} + R\lambda + L\lambda^2}\right|^2 \geq 0 \quad \forall \lambda \in \mathrm{i}\mathbb{R}.$$

Hence, the spectral factor of $\Phi(-\xi,\xi)$ exists, and is given by

$$F(\xi) = \frac{\sqrt{R}\xi}{\frac{1}{C} + R\xi + L\xi^2}.$$

It follows that

$$\begin{aligned}\Psi(\zeta,\eta) &= \frac{\Phi(\zeta,\eta) - F(\zeta)F(\eta)}{\zeta+\eta} \\ &= \frac{\frac{1}{2C} + \frac{1}{2}L\zeta\eta}{(\frac{1}{C} + R\eta + L\eta^2)(\frac{1}{C} + R\zeta + L\zeta^2)}\end{aligned}$$

Since the right coprime factors of $G(\xi)$ are given by

$$\begin{aligned}D(\xi) &= \frac{1}{C} + R\xi + L\xi^2, \\ N(\xi) &= \frac{1}{2}\begin{pmatrix} D(\xi)+\xi \\ D(\xi)-\xi \end{pmatrix} = \begin{pmatrix} \frac{1}{C} + (R+1)\xi + L\xi^2 \\ \frac{1}{C} + (R-1)\xi + L\xi^2 \end{pmatrix},\end{aligned}$$

the polynomial matrices in (35),(36) are given by

$$\begin{aligned}\Xi(\zeta,\eta) &= D(\zeta)\Psi(\zeta,\eta)D(\eta) = \frac{1}{2C} + \frac{1}{2}L\zeta\eta, \\ \Delta(\zeta,\eta) &= D(\zeta)F(\zeta)F(\eta)D(\eta) = R\zeta\eta.\end{aligned}$$

In addition, the observable (polynomial) image representation of $v = G(\frac{d}{dt})w$ is given by

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(V+I) \\ \frac{1}{2}(V-I) \\ \hline V \end{pmatrix} = \begin{pmatrix} N(\frac{d}{dt}) \\ \hline D(\frac{d}{dt}) \end{pmatrix} q,$$

Note that, in this case, the latent variable is exactly the electrical charge q. Hence, $\Xi(\zeta,\eta)$ and $\Delta(\zeta,\eta)$ recovers the storage function and the dissipation rate in Example 3 as follows.

$$Q_\Xi(q) = \frac{1}{2C}q^2 + \frac{1}{2}LI^2, \quad Q_\Delta(q) = RI^2, \quad I = \frac{dq}{dt}.$$

◇

Next, we consider the notion of half-line nonnegativity.

Definition 8. Under Assumption 1, $Q^{\mathtt{r}}_\Phi$ is called *half-line nonnegative*, denoted by $\int^t Q^{\mathtt{r}}_\Phi \geq 0$, if there holds

$$\int_{-\infty}^{\infty} s(t)dt \geq 0 \quad \forall s \in \hat{Q}^{\mathtt{r}}_\Phi(w), \ \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \cap \mathfrak{D} \quad (42)$$

The half-line nonpositivity is also defined in the same way.

Theorem 2. Under Assumption 1, the following statements are equivalent.

(i) $\displaystyle\int^t Q^{\mathtt{r}}_\Phi \geq 0$ (half-line nonnegative).

(ii) There exists a symmetric and factorizable rational matrix $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}_{\mathrm{s}}(\zeta,\eta)$ such that

$$\overset{\bullet}{\Psi} - \Phi \leq 0 \ \ \& \ \ \Psi \geq 0 \qquad (43)$$

(iii) There exists a symmetric and factorizable $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}_{\mathrm{s}}(\zeta,\eta)$ and a rational matrices $F, K \in \mathbb{R}^{\bullet\times\mathtt{w}}(\xi)$ satisfying (41) and $\Psi(\zeta,\eta) = K^\top(\zeta)K(\eta)$.

Moreover, among Ψ's satisfying (41) or (43), there exists one for which $D^\top(\zeta)\Psi(\zeta,\eta)D(\eta)$ is a polynomial matrix for the right coprime factors (N,D) of $G(\xi)$.

We omit the proof, since the theorem can be proved almost in the same way as Theorem 1. In particular, it turns out from (27) and Propositions A.1, A.2 that the half-line nonnegativity of the rational QDF is equivalently translated to that of a polynomial QDF:

$$\int^t Q^{\mathtt{r}}_\Phi(w) \geq 0 \ \Leftrightarrow \ \int^t Q_{N^\top(\zeta)\Sigma N(\eta)} \geq 0$$

Therefore, we have the same observation as Remark 3 that, for $\Psi(\zeta,\eta)$ satisfying the conditions in Theorem **??**, the polynomial QDF $Q_\Xi(\ell)$, $\Xi(\zeta,\eta) = D^\top(\zeta)\Psi(\zeta,\eta)D(\eta)$ serves as a "nonnegative" storage function for the system defined by $v = G(\frac{d}{dt})w$.

4. CONCLUDING REMARKS

I this paper, we first illustrated the basic features of a QDF including its definition, calculus, nonnegativity, average nonnegativity, e.t.c. We also clarified the relation between QDF's and dissipativity and Lyapunov stability of a linear differential system. Although we have concentrated only on the QDF's are defined by ordinary differential equations in the continuous-time, it should be noted that there have been a number of research works about the discrete-time QDF's (Kaneko and Fujii 2000, 2003, Kojima and Takaba 2005), and the QDF's defined by partial differential equations or multivariable polynomials (Pillai and Shankar 1999, Pillai and Willems 2002, Napp Avelli and Trentelman 2007).

Secondly, we presented a new and more general formulation of a QDF in terms of rational functions rather than polynomials. It turned out that the rational QDF defines a set unlike the polynomial QDF. We also showed that the

notions of nonnegativity, average nonnegativity and half-line nonnegativity can be generalized for the rational QDF, and derived their necessary and sufficient conditions. It remains as future topics to study the rational QDF along a given behavior and its applications to the Lyapunov stability and the dissipation theory. It should be noted that the rational QDF has many important applications in system and control theory: for example, the dissipation theory for linear systems defined by rational representations, the stability analysis of interconnected systems with rational multipliers, and so forth.

The first author would like to acknowledge that this work is supported by JSPS Grant-in-Aid for Scientific Research (C) No. 18560431.

REFERENCES

M.N. Belur and H.L. Trentelman, Algorithmic issues in the synthesis of dissipative systems, *Mathematical and Computer Modelling of Dynamical Systems*, vol. 8, no. 4, pp. 407–428, 2002.

M.N. Belur and H.L. Trentelman, The strict dissipativity synthesis problem and the rank of the coupling QDF, *Syst. Contr. Letters*, vol. 51, pp. 247–258, 2004.

T. Iwasaki and S. Hara, Well-posedness of feedback systems: insights into exact robustness analysis and approximate computations, *IEEE Trans. Automat. Contr.*, vol. 43, no. 5, pp. 619–630, 1998.

O. Kaneko and T. Fujii, Discrete-time average positivity and spectral factorization in a behavioral framework, *Syst. Contr. Letters*, vol. 39, pp. 31–44, 2000.

O. Kaneko and T. Fujii, When is a storage function a state function in discrete time?, *SIAM J. Contr. Optimiz.*, vol. 42, no. 4, pp. 1374–1394, 2003.

O. Kanekoa nd T. Fujii, The behavioral approach to dissipation theory of dynamical systems: Based on QDF's, *Syst. Contr. Inform.*, vol. 48, no. 5, 2004 (in Japanese).

C. Kojima and K. Takaba, A generalized Lyapunov stability theorem for discrete-time systems based on quadratic difference forms, *Proc. 44th IEEE Conf. Decis. Contr., and Eur. Contr. Conf. 2005*, pp. 2911–2916, 2005. *CDC-ECC2005*, 2005.

A. Megretski and A. Rantzer, System analysis via integral quadratic constraints, *IEEE Trans. Automat. Contr.*, vol. 42, no. 6, pp. 819-830, 1997.

D. Napp Avelli and H.L. Trentelman, Algorithms for multidimensional spectral factorization and sum of squares, *Linear Algebra and its Applications*, to appear, 2007.

R. Peeters and P. Rapisarda, A two-variable approach to solve the polynomial Lyapunov equation, *Syst. Contr. Letters* vol. 42, pp. 117–126, 2001.

I. Pendharkar and H. Pillai, Systems with sector bound nonlinearities: A behavioral approach, *Syst. Contr. Letters*, to appear, 2007.

H. Pillai and S. Shankar, A behavioural approach to control of distributed systems, *SIAM J. Contr. Optimiz.*, vol. 37, pp. 388–408, 1999.

H. Pillai and J.C. Willems, Lossless and dissipative distributed systems, *SIAM J. Contr. Optimiz.*, vol. 40, pp. 1406–1430, 2002.

J.W. Polderman and J.C. Willems, *Introduction to Mathematical Systems Theory: A Behavioral Approach*, Springer-Verlag, 1998.

P. Rapisarda and J.C. Willems, An introduction to quadratic differential forms, *Proc. of 16th Symp. Mathematical Theory of Networks and Systems (MTNS2004)*, Leuven, 2004.

K. Takaba, Robust stability analysis of uncertain interconnection in the behavioral framework, *Proc. of 16th IFAC World Congress*, Prague, 2005.

H.L. Trentelman and J.C. Willems, H_∞ control in a behavioral context: The full information case, *IEEE Trans. Automat. Contr.*, vol. 44, no. 3, pp. 521–536, 1999.

H.L. Trentelman and J.C. Willems, Dissipative linear differential systems and the state space H^∞ control problem, *Int. J. of Robust and Nonlinear Control*, vol. 10, pp. 1039–1057, 2000.

J.C, Willems, LQ-control: a behavioral approach, *Proc. of 32nd IEEE Conf. on Decis. and Contr.*, San Antonio, pp. 3664–3668, 1993.

J.C. Willems and H.L. Trentelman, On quadratic differential forms, *SIAM J. Contr. Optimiz.*, vol. 36, pp. 1703–1749, 1998.

J.C. Willems and H.L. Trentelman, Synthesis of dissipative systems using quadratic differential forms: Parts I and II, *IEEE Trans. Automat. Contr.*, vol. 47, pp. 53–86, 2002.

J.C. Willems and M.E. Valcher, Linear-quadratic control and quadratic differential forms, *Proc. of 16th IFAC World Congress*, Prague, Paper code: Mo-A15-TO/1, 2005.

J.C. Willems and K. Takaba, Dissipativity and stability of interconnections, *Int. J. of Robust and Nonlinear Control*, vol. 17, pp. 563–586, 2007.

J.C. Willems and Y. Yamamoto, Behaviors defined by rational functions, *Linear Algebra and its Applications*, vol. 425, pp. 226–24, 2007.

D.C. Youla, On the factorization of rational matrices, *IRE Trans. Inform. Theory*, vol. 7, pp. 172-189, 1961.

Appendix A. LINEAR DIFFERENTIAL SYSTEMS

Representation by polynomial matrices

In the behavioral approach, a dynamical system is characterized by the triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is the time axis and $\mathbb{W}$ is the signal space where the system variables take their values at each time instant. The set $\mathfrak{B}$ is the *behavior*, namely, the set of all possible trajectories which meet the dynamic laws of the system. Throughout this paper, we will identify a dynamical system with its behavior for ease of notation.

We are mainly interested in a *linear differential system*, that is described by a a differential-algebraic equation with constant coefficients. Typically, such an equation is given by

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_L \frac{d^L}{dt^L} w = 0,$$

or in short hand notation,

$$R\left(\frac{d}{dt}\right) w = 0,$$

where $R(\xi) = R_0 + R_1\xi + \cdots + R_L\xi^L \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$. We call the variable $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ *a manifest variable*. Then, the behavior is defined by

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \;\middle|\; R\left(\frac{d}{dt}\right) w = 0 \right\}.$$

In short, we denote the behavior as $\mathfrak{B} = \ker R(\frac{d}{dt})$. For the obvious reason, the above representation is called *a kernel representation*. We define $\mathfrak{L}^{\mathtt{w}}$ as the set of such linear time-invariant differential behaviors with $\mathtt{w}$ variables. For simplicity of discussion, we assume that the behaviors are defined in the class of $\mathfrak{C}^\infty$-functions.

Recall that there are more than one polynomial matrices which induce kernel representations of $\mathfrak{B}$. A polynomial matrix $R(\xi)$ satisfying $\mathfrak{B} = \ker R(d/dt)$ is said to be *minimal* if the number of rows of $R(\xi)$ is less than or equal to that of any other polynomial matrix which induces a kernel representation of $\mathfrak{B}$.

A system $\mathfrak{B}$ is called *controllable* if, for any w_1, $w_2 \in \mathfrak{B}$, there exist a $w \in \mathfrak{B}$ and a positive constant T such that $w(t) = w_1(t)$ $(t \leq 0)$ and $w(t) = w_2(t-T)$ $(t \geq T)$. The family of controllable linear time-invariant differential systems is denoted by $\mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. When a kernel representation of $\mathfrak{B}$ is induce by $R(\xi)$, $\mathfrak{B}$ is controllable iff $\mathrm{rank}(R(\lambda))$ is constant for all $\lambda \in \mathbb{C}$. If $R(\xi)$ induces a minimal kernel representation of a controllable system, then $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$.

If $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, there exists a polynomial matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ such that $R(\xi)M(\xi) = 0$. In this case, $\mathfrak{B}$ can be rewritten as

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \;\middle|\; \exists \ell \text{ s.t. } w = M\left(\frac{d}{dt}\right)\ell \right\},$$

or in short $\mathfrak{B} = \mathrm{im} M(\frac{d}{dt})$, where $\ell : \mathbb{R} \to \mathbb{R}^m$ is an auxiliary variable called *a latent variable*. The representation $w = M(\frac{d}{dt})\ell$ is called *an image representation*. The image representation is said to be *observable* if $M(\frac{d}{dt})\ell = 0$ implies $\ell = 0$. The behavior $\mathfrak{B} = \mathrm{im}\, M(\frac{d}{dt})$ is observable if and only if $M(\lambda)$ has full column rank for any $\lambda \in \mathbb{C}$.

Suppose that $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ induces a minimal kernel representation of $\mathfrak{B} \subset \mathfrak{L}^{\mathtt{w}}$. Then, there exists a nonsingular permutation matrix Π such that

$$R(\xi)\Pi^{-1} = (Q(\xi) \; -P(\xi)), \; \det P \neq 0,$$
$$\Pi w = \begin{pmatrix} u \\ y \end{pmatrix}, \; u : \mathbb{R} \to \mathbb{R}^{\mathtt{m}}, \; y : \mathbb{R} \to \mathbb{R}^{\mathtt{p}}, \; \mathtt{p} + \mathtt{m} = \mathtt{w}.$$

Then, u and y serve as the input and output of $\mathfrak{B}$, respectively, and the transfer function from u to y is defined by

$$G(\xi) = P(\xi)^{-1}Q(\xi)$$

For the obvious reason, the above partition is called *the input/output (I/O) partition of* $\mathfrak{B}$. It should be noted that the choice of inputs and outputs is not unique, and is *not* given *a priori*. The dimensions of u and y (namely, m and p) are invariant for any choice of inputs and outputs and for any representation of $\mathfrak{B}$. We refer to these dimensions as *input and output cardinalities of* $\mathfrak{B}$, and denote them by $\mathtt{m}(\mathfrak{B})$ and $\mathtt{p}(\mathfrak{B})$, respectively. It should also be noted that, the system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is autonomous if and only if $\mathtt{m}(\mathfrak{B}) = 0$ and $\mathtt{p}(\mathfrak{B}) = \mathtt{w}$.

A system $\mathfrak{B}$ is said to be *asymptotically stable* if $w(t) \to 0$ $(t \to \infty)$ holds for all $w \in \mathfrak{B}$. Clearly, $\mathfrak{B}$ must be autonomous in order to be asymptotically stable. The behavior $\mathfrak{B} = \ker R(\frac{d}{dt})$ is asymptotically stable if and only if $R(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}_+$. In the case where $R(\xi)$ is square, $\mathfrak{B}$ is asymptotically stable iff $R(\xi)$ is Hurwitz, namely $\det R(\xi) = 0$ has all roots in $\mathrm{Re}\,\xi < 0$.

Representation by rational matrices

We next consider the linear system represented by

$$v = G(\frac{d}{dt})w, \quad G \in \mathbb{R}^{\mathtt{v}\times\mathtt{w}}(\xi) \tag{A.1}$$

For more various rational function representations such as $G(\frac{d}{dt})w = 0$ and $G(\frac{d}{dt})w = H(\frac{d}{dt})\ell$, the readers are recommended to refer to Willems and Yamamoto (2007).

We introduce left and right coprime factorizations of $G(\xi)$ over $\mathbb{R}[\xi]$ as

$$G(\xi) = X^{-1}(\xi)Y(\xi) = N(\xi)D^{-1}(\xi) \tag{A.2}$$
$$X \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}[\xi], \; Y \in \mathbb{R}^{\mathtt{v}\times\mathtt{w}}[\xi],$$
$$N \in \mathbb{R}^{\mathtt{v}\times\mathtt{w}}[\xi], \; D \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$$

Then, along the line of Willems and Yamamoto (2007), the solution of (A.1) is defined as follows.

Definition 9.

$$\left[\!\!\left[(v,w) : \; \mathbb{R} \to \mathbb{R}^{\mathtt{v}+\mathtt{w}} \text{ is a solution of } v = G(\frac{d}{dt})w \right]\!\!\right]$$
$$:\Leftrightarrow \; \left[\!\!\left[(v,w) \text{ satisfies } X(\frac{d}{dt})v = Y(\frac{d}{dt})w \right]\!\!\right]$$

Proposition A.1. Let $G \in \mathbb{R}^{\mathtt{v}\times\mathtt{w}}(\xi)$ be given. Then,

$$[\![(v,w) \text{ satisfies (A.1).}]\!]$$
$$\Leftrightarrow \; \left[\!\!\left[\exists \ell \text{ s.t. } v = N(\frac{d}{dt})\ell, \; w = D(\frac{d}{dt})\ell \right]\!\!\right]$$

Proof: Recall that $X(\frac{d}{dt})v = Y(\frac{d}{dt})w$ is a kernel representation of the behavior of (v,w) in terms of a "polynomial" matrix $(X(\xi) \;\; Y(\xi))$. Hence, the proposition is obvious from the definition of the coprime factors and the standard results on kernel and image (polynomial) representations (see e.g. Section 6.6 in Polderman and Willems (1998)). ■

Proposition A.2. Consider the solution to the differential equation

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} N(\frac{d}{dt}) \\ D(\frac{d}{dt}) \end{pmatrix} \ell \tag{A.3}$$

associated with the rational representation $v = G(\frac{d}{dt})w$.

(i) $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \Leftrightarrow \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$
(ii) $(v,w) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}+\mathtt{w}})$ has compact support iff $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ has compact support.

Proof: (i) Immediate.

(ii) By the right coprimeness, $\begin{pmatrix} N(\lambda) \\ D(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$. In other words, ℓ is observable from (v,w). The observability proves the statement in (ii). ■

Consider the situation where w is arbitrarily given and v is determined from (A.1). Since $\det X \neq 0$ and $\det D \neq 0$, w is free in (A.1). Hence, (A.1) defines a linear differential behavior $\mathfrak{B}_v \in \mathfrak{L}^{\mathtt{v}}$.

$$\mathfrak{B}_v = \left\{ v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}}) | \; \exists w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \text{ s.t. } v = G(\frac{d}{dt})w \right\}$$

Indeed, the above propositions claim that $\mathfrak{B}_v$ is represented by an image representation as

$$\mathfrak{B}_v = \left\{ v = N(\tfrac{d}{dt})\ell \,|\, \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \right\}.$$

Note that, in general, $G(\frac{d}{dt})$ does *not* define a map from w to v. This is because there are a number of ℓ's satisfying $w = D(\frac{d}{dt})\ell$ for a fixed w. Of course, if $G(\xi)$ is a polynomial matrix, then $G(\frac{d}{dt})$ defines a differential map.

The above propositions claim that, though $G(\frac{d}{dt})$ is not a map, $G(\frac{d}{dt})\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$ is a linear differential behavior which is characterized as the image space of the differential operator $N(\frac{d}{dt})$.