Fourth-order Cumulant Tensor Based Identification of Underdetermined MIMO FIR Channels

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Abstract

Blind MIMO FIR identification based on cumulants have a long history. Most of the existing methods can only handle overdetermined MIMO FIR channels, that is the involved matrices are assumed to have full column rank. However, by exploiting the tensorial structure of the cumulants it is possible to identify underdetermined MIMO FIR channels. The main contribution of this paper are new uniqueness results for blind fourth-order cumulant based identification of underdetermined MIMO FIR channels. Second, we also present subspace methods for the actual computation of the MIMO FIR channels.

Keywords: cumulants, tensor, canonical polyadic decomposition, parallel factor (PARAFAC), canonical decomposition (CANDECOMP), block-Toeplitz matrices, MIMO, blind identification.

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1 Introduction

Blind Multiple-Input-Multiple-Output (MIMO) Finite Impulse Response (FIR) identification based on cumulants dates back to [5], [2], [17]. However, the mentioned papers only partially exploited the structure of the problem. Consequently the methods presented in the mentioned papers can only find overdetermined MIMO FIR channels up to a nonsingular ambiguity matrix. More recently, matrix based subspace methods for identifying MIMO FIR channels have been proposed in [22], [8]. Again they are only capable of finding an overdetermined MIMO FIR system up to a nonsingular ambiguity matrix. In order to remove the nonsingular ambiguity matrix a second computational stage is necessary. Depending on the problem the ambiguity matrix can be found by means of for instance ICA-based [3], CM-based [19] or FA-based [20] methods.

Only a little attention has been given to the underdetermined MIMO FIR identification problem. Common to the few existing results is that they recognizing and exploiting the tensorial structure of the problem. Based on eigenanalysis, then one of the first uniqueness results for blind underdetermined MIMO FIR identification was presented in [16]. Moreover, based on Kruskal's uniqueness theorem [14], [11] it was shown in [21] that under some approximations it is possible to generically identify some underdetermined MIMO FIR systems based on Kth order cross-polyspectra. Based on subspace analysis we will in this paper show that more relaxed identifiability conditions can be obtained than those achieved by eigenanalysis or Kruskal's theorem.

In [16] no practical numerical method was provided while in [21] the unknown channel parameters was numerically found by means of the iterative Alternating Least Squares (ALS) method which can be sensitive to local minima and slow convergence. In this paper we propose subspace

methods which convert the problem into a Simultaneous Matrix Diagonalization (SMD) problem involving a congruence transform. In the absence of noise a SMD problem can be solved by means of a generalized Singular Value Decomposition (SVD).

To summarize, the main contribution of this paper are new uniqueness results for fourth-order cumulant based identification of underdetermined MIMO FIR channels. Second, we propose subspace methods for the actual computation of the MIMO FIR channels.

The rest of the introduction will present the notation of the paper followed by a brief discussion of the Canonical Polyadic Decompostion (CPD). Next, in section 2 we discuss a cumulant based approach for blind identification of MIMO FIR channels. Section 3 provides new cumulant tensor based identification results for MIMO FIR channels while section 4 present new subspace methods for the actual computation of the MIMO FIR parameters. Finally, in section 5 some simulation results are reported. We end the paper with a conclusion in section 6.

1.1 Notations

Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively. Let $\otimes$ denote the Kronecker product

$$\mathbf{A} \otimes \mathbf{B} \triangleq \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

and let $\odot$ denote the Khatri-Rao product

$$\mathbf{A} \odot \mathbf{B} \triangleq \begin{bmatrix} \mathbf{a}_1 \otimes \mathbf{b}_1 & \mathbf{a}_2 \otimes \mathbf{b}_2 & \dots \end{bmatrix},$$

where $\mathbf{a}_r$ and $\mathbf{b}_r$ denotes the rth column vector of $\mathbf{A}$ and $\mathbf{B}$, respectively. Let $\text{blockdiag}\left(\mathbf{A}^{(1)}, \dots, \mathbf{A}^{(R)}\right)$ denote the block-diagonal matrix holding the matrices $\{\mathbf{A}^{(r)}\}$ on its block-diagonal. Moreover, let $\circ$ denote the outer product of three vectors $\mathbf{a}^{(n)} \in \mathbb{C}^{I_n}$ such that $\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \mathbf{a}^{(3)} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ satisfies

$$\left(\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \mathbf{a}^{(3)}\right)_{i_1,i_2,i_3} = a^{(1)}_{i_1} a^{(2)}_{i_2} a^{(3)}_{i_3}.$$

The identity matrix, anti-identity matrix and the all-ones vector are denoted by are denoted by $\mathbf{I}_R \in \mathbb{C}^{R\times R}$, $\mathbf{J}_R \in \mathbb{C}^{R\times R}$ and $\mathbf{1}_R \in \mathbb{C}^R$, respectively, i.e.,

$$\mathbf{I}_R = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{C}^{R\times R}, \quad \mathbf{J}_R = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & 1 & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{R\times R}, \quad \mathbf{1}_R = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{C}^R.$$

The all-zero vector and matrix are denoted by $\mathbf{0}_R \in \mathbb{C}^R$ and $\mathbf{0}_{R,S} \in \mathbb{C}^{R\times S}$, respectively. Furthermore, let $(\cdot)^T$, $(\cdot)^*$, $(\cdot)^H$ and $\mathrm{Col}\,(\cdot)$ denote the transpose, conjugate, conjugate-transpose, column space of a matrix, respectively. Moreover, let $(\cdot)^\perp$ and $\dim(\cdot)$ denote the complementary subspace and dimension of a subspace, respectively.

Given $\mathbf{A} \in \mathbb{C}^{I\times J}$, then $\mathrm{Vec}\,(\mathbf{A}) \in \mathbb{C}^{IJ}$ will denote the column vector defined by $(\mathrm{Vec}\,(\mathbf{A}))_{i+(j-1)I} = (\mathbf{A})_{ij}$. Given $\mathbf{a} \in \mathbb{C}^{IJ}$, then the reverse operation is $\mathrm{Unvec}\,(\mathbf{a}) = \mathbf{A} \in \mathbb{C}^{I\times J}$ such that $(\mathbf{a})_{i+(j-1)I} = (\mathbf{A})_{ij}$. Let $D_k(\mathbf{A}) \in \mathbb{C}^{J\times J}$ denote the diagonal matrix holding row k of $\mathbf{A} \in \mathbb{C}^{I\times J}$ on its diagonal.

Matlab index notation will be used to extract submatrices from a given matrix. For example $\mathbf{A}(1:k,:)$ denotes the submatrix of $\mathbf{A}$ consisting of the rows from 1 to k of $\mathbf{A}$. Let

$$\mathbf{T} = \left[\mathbf{T}^{(1)T}, \ldots, \mathbf{T}^{(N)T}\right]^T \in \mathbb{C}^{MN\times P},$$

where $\mathbf{T}^{(n)} \in \mathbb{C}^{M\times P}$, then

$$\begin{aligned} \overline{\mathbf{T}} &= \left[\mathbf{T}^{(2)T}, \ldots, \mathbf{T}^{(N)T}\right]^T \in \mathbb{C}^{M(N-1)\times P} \\ \underline{\mathbf{T}} &= \left[\mathbf{T}^{(1)T}, \ldots, \mathbf{T}^{(N-1)T}\right]^T \in \mathbb{C}^{M(N-1)\times P}. \end{aligned}$$

The k-rank of a matrix $\mathbf{A}$ is denoted by $\mathrm{k}(\mathbf{A})$. It is equal to the largest integer $\mathrm{k}(\mathbf{A})$ such that every subset of $\mathrm{k}(\mathbf{A})$ columns of $\mathbf{A}$ is linearly independent.

1.2 Canonical Polyadic Decomposition (CPD)

A rank-1 tensor $\mathcal{X} \in \mathbb{C}^{I_1\times I_2\times I_3}$ is equal to the outer product of some non-zero vectors $\mathbf{a}^{(n)} \in \mathbb{C}^{I_n}$, $n \in [1,3]$, such that $\mathcal{X}_{i_1 i_2 i_3} = \prod_{n=1}^3 a^{(n)}_{i_n}$. The rank of a tensor $\mathcal{X}$ is equal to the minimal number of rank-1 tensors that yield $\mathcal{X}$ in a linear combination. Assume that the rank of $\mathcal{X}$ is R, then it can be written as

$$\mathcal{X} = \sum_{r=1}^R \mathbf{a}_r^{(1)} \circ \mathbf{a}_r^{(2)} \circ \mathbf{a}_r^{(3)}. \tag{1}$$

The decomposition (1) will in this paper be referred to as the CPD of $\mathcal{X}$. Let us stack the vectors $\{\mathbf{a}_r^{(n)}\}$ into the matrices

$$\mathbf{A}^{(n)} \;=\; \left[\; \mathbf{a}_1^{(n)}, \;\; \cdots \;\; , \mathbf{a}_R^{(n)} \;\right] \in \mathbb{C}^{I_n \times R}, \quad n \in [1,3]. \tag{2}$$

The matrices $\mathbf{A}^{(n)}$ in (2) will be referred to as the matrix factors of the tensor $\mathcal{X}$ in (1). We say that the CPD of $\mathcal{X}$ is partially symmetric if $\mathbf{A}^{(m)} = \mathbf{A}^{(n)}$ for some $m \neq n$. Similarly, we say that the CPD of $\mathcal{X}$ is partially Hermitian symmetric if $\mathbf{A}^{(m)*} = \mathbf{A}^{(n)}$ for some $m \neq n$.

The following matrix representation of a CPD of $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ will be used in this paper. Let $\mathbf{X}^{(i_1 \cdot\cdot)} \in \mathbb{C}^{I_2 \times I_3}$ denote the matrix such that $\mathbf{X}^{(i_1 \cdot\cdot)}_{i_2 i_3} = \mathcal{X}_{i_1 i_2 i_3}$, then

$$\mathbf{X}^{(i_1 \cdot\cdot)} \;=\; \mathbf{A}^{(2)} D_{i_1}\left(\mathbf{A}^{(1)}\right) \mathbf{A}^{(3)T}$$

and

$$\mathbb{C}^{I_1 I_2 \times I_3} \ni \mathbf{X}_{(1)} \triangleq \begin{bmatrix} \mathbf{X}^{(1 \cdot\cdot)} \\ \vdots \\ \mathbf{X}^{(I_1 \cdot\cdot)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(2)} D_1\left(\mathbf{A}^{(1)}\right) \\ \vdots \\ \mathbf{A}^{(2)} D_{I_1}\left(\mathbf{A}^{(1)}\right) \end{bmatrix} \mathbf{A}^{(3)T} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}\right) \mathbf{A}^{(3)T}.$$

The CPD of $\mathcal{X}$ in (1) is said to be essentially unique if all triplets $\left(\overline{\mathbf{A}}^{(1)}, \overline{\mathbf{A}}^{(2)}, \overline{\mathbf{A}}^{(3)}\right)$ satisfying relation (1) are related via

$$\overline{\mathbf{A}}^{(n)} = \mathbf{A}^{(n)} \Pi \Delta_n \,, \quad \forall n \in [1,3],$$

where Π is a permutation matrix and Δ_n are diagonal matrix satisfying $\Delta_1 \Delta_2 \Delta_3 = \mathbf{I}_R$. A general and commonly used identifiability result for a CPD, known as Kruskal's sufficiency condition, is stated in the following theorem.

Theorem 1.1. *Let $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ be a tensor with rank R, then the CPD of $\mathcal{X}$ is essentially unique if*

$$\sum_{n=1}^{3} k\left(A^{(n)}\right) \geq 2R + 2 \,. \tag{3}$$

Proof. The original proof can be found in [7] and more accessible proofs can be found in [11], [14]. Furthermore, a shortened proof has recently been obtained in [10]. □

However, the condition (3) is not necessary for essential uniqueness of the CPD [15]. For instance, if one of the matrix factors of the CPD has full column rank, then the following generic uniqueness result was obtained in [4]:

Theorem 1.2. *Let $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ be a tensor of rank R, then the CPD of $\mathcal{X}$ is essentially unique in the generic case if*

$$2R(R-1) \leq I_1(I_1-1)I_2(I_2-1) \quad \textit{and} \quad R \leq I_3. \tag{4}$$

2 Fourth-order Cumulant Tensor for MIMO FIR Channels

Assume that R transmitted signals are passing through a MIMO FIR channel

$$y_m(n) = \sum_{r=1}^{R} \sum_{l_r=0}^{L_r} h_m^{(r)}(l_r) s^{(r)}(n-l_r) + v(n),$$

where $y_m(n)$ denotes the output of the mth receive antenna at the nth sample period, $\{s^{(r)}(n)\}$ are the unknown transmitted signals, $\{h_m^{(r)}(n)\}$ are unknown channel coefficients, $v(n)$ is an Additive White Gaussian Noise (AWGN) term and L_r denotes the maximal filter order for the channel between the receive antennas and the rth transmitted signal. Assume that the transmitted signals $\{s^{(r)}(n)\}$ and the AWGN term $v(n)$ are mutually statistically independent from each other and without loss of generality we also assume that $L_1 \geq \cdots \geq L_R$. Based on the observed data sequence $\{y_n\}$ the blind identification problem amounts to estimate the channel coefficients $\{h_m^{(r)}(n)\}$ up to the inherent permutation and scaling ambiguities.

As in [8], [1] we work with the spatio-temporal cumulants

$$\begin{aligned} C_{m_1 m_2 m_3 m_4}(l_1, l_2, l_3) &\triangleq \operatorname{Cum}\left[y_{m_1}^*(n) y_{m_2}(n+l_1) y_{m_3}^*(n+l_2) y_{m_4}(n+l_3) \right] \\ &= \sum_{r=1}^{R} \gamma_r \sum_{l=0}^{L_1} h_{m_1}^{(r)*}(l) h_{m_2}^{(r)}(l+l_1) h_{m_3}^{(r)*}(l+l_2) h_{m_4}^{(r)}(l+l_3)\,, \end{aligned} \tag{5}$$

where $\gamma_r = \operatorname{Cum}\left[s^{(r)*}(n) s^{(r)}(n) s^{(r)*}(n) s^{(r)}(n) \right]$, $m_1, m_2, m_3, m_4 \in [1, M]$ and $l_1, l_2, l_3 \in [-L_1, L_1]$ and $h_{m_n}^{(r)}(l) = 0$, $\forall l > L_r$. Let us store the channel coefficients into

the parameter matrices with

$$\mathbf{P}^{(r)} = \left[\mathbf{p}_0^{(r)}, \ldots, \mathbf{p}_{L_r}^{(r)}\right] = \begin{bmatrix} h_1^{(r)}(0) & \cdots & h_1^{(r)}(L_r) \\ \vdots & \ddots & \vdots \\ h_M^{(r)}(0) & \cdots & h_M^{(r)}(L_r) \end{bmatrix} \in \mathbb{C}^{M\times(L_r+1)}. \tag{6}$$

The goal is now to find the parameter matrices $\mathbf{P}^{(r)}$ from the cumulants (5). Let us store the cumulants (5) into the tensor $\mathcal{T} \in \mathbb{C}^{M\times M(2L_1+1)\times M(2L_1+1)\times M(2L_1+1)}$ such that

$$\mathcal{T}_{m_1,(l_1+L_1)M+m_2,(l_2+L_1)M+m_3,(l_3+L_1)M+m_4} = C_{m_1m_2m_3m_4}(l_1, l_2, l_3),$$

then $\mathcal{T}$ admits the factorization

$$\mathcal{T} = \sum_{r=1}^{(\sum_{r=1}^{R} L_r+R)} \mathbf{a}_r^* \circ \mathbf{h}_r \circ \mathbf{h}_r^* \circ \mathbf{h}_r\,, \tag{7}$$

where $\mathbf{a}_r$ denotes the rth column vector of

$$\mathbf{A} = \left[\gamma_1\mathbf{P}^{(1)}, \ldots, \gamma_R\mathbf{P}^{(R)}\right] \in \mathbb{C}^{M\times(\sum_{r=1}^{R} L_r+R)}$$

and $\mathbf{h}_r$ denotes the rth column vector of

$$\mathbf{H} = \left[\mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(R)}\right] \in \mathbb{C}^{M(2L_1+1)\times(\sum_{r=1}^{R} L_r+R)} \tag{8}$$

with banded block-Hankel matrices of the form

$$\mathbf{H}^{(r)} = \begin{bmatrix} \mathbf{0}_{M(L_1-L_r)} & \cdots & \mathbf{0}_{M(L_1-L_r)} & \mathbf{0}_{M(L_1-L_r)} \\ \mathbf{0}_M & \cdots & \mathbf{0}_M & \mathbf{p}_0^{(r)} \\ \vdots & \iddots & \iddots & \vdots \\ \mathbf{0}_M & \mathbf{p}_0^{(r)} & \cdots & \mathbf{p}_{L_r-1}^{(r)} \\ \mathbf{p}_0^{(r)} & \mathbf{p}_1^{(r)} & \cdots & \mathbf{p}_{L_r}^{(r)} \\ \vdots & \vdots & \iddots & \mathbf{0}_M \\ \mathbf{p}_{L_r-1}^{(r)} & \mathbf{p}_{L_r}^{(r)} & \iddots & \vdots \\ \mathbf{p}_{L_r}^{(r)} & \mathbf{0}_M & \cdots & \mathbf{0}_M \\ \mathbf{0}_{M(L_1-L_r)} & \mathbf{0}_{M(L_1-L_r)} & \cdots & \mathbf{0}_{M(L_1-L_r)} \end{bmatrix} = \sum_{l=0}^{L_r} \mathbf{E}_l \otimes \mathbf{p}_l^{(r)} \in \mathbb{C}^{M(2L_1+1)\times(L_r+1)}, \tag{9}$$

where $\mathbf{E}_l \in \mathbb{C}^{(2L_1+1)\times(L_r+1)}$ are basis matrices. More precisely, the basis matrices

$$(\mathbf{E}_l)_{ij} = \begin{cases} 1, & i = L_r + l - j + 2 \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

will be used. Let $\mathbf{T}^{(i_1,i_2)} \in \mathbb{C}^{M(2L_1+1)\times M(2L_1+1)}$ be constructed such that $\mathbf{T}^{(i_1,i_2)}_{i_4 i_3} = \mathcal{T}_{i_1 i_2 i_3 i_4}$, then

$$\mathbf{T}^{(i_1,i_2)} = \mathbf{H} D_{i_1}(\mathbf{A}^*) D_{i_2}(\mathbf{H}) \mathbf{H}^H, \quad \forall i_1 \in [1,M], \forall i_2 \in [1, M(2L_1+1)]. \tag{11}$$

We will also apply the matrices

$$\mathbf{X}^{(i_1,i_2)} = \mathbf{T}^{(i_1,i_2)H} = \mathbf{H} D_{i_1}(\mathbf{A}) D_{i_2}(\mathbf{H}^*) \mathbf{H}^H, \quad \forall i_1 \in [1,M], \forall i_2 \in [1, M(2L_1+1)]. \tag{12}$$

Stack the matrices $\{\mathbf{T}^{(i_1,i_2)}\}$ and $\{\mathbf{X}^{(i_1,i_2)}\}$ into the matrix

$$\begin{aligned} \mathbf{Y} &\triangleq \Big[\mathrm{Vec}\left(\mathbf{T}^{(1,1)}\right), \mathrm{Vec}\left(\mathbf{T}^{(2,1)}\right), \ldots, \mathrm{Vec}\left(\mathbf{T}^{(M(2L_1+1),M)}\right), \\ &\quad \mathrm{Vec}\left(\mathbf{X}^{(1,1)}\right), \mathrm{Vec}\left(\mathbf{X}^{(2,1)}\right), \ldots, \mathrm{Vec}\left(\mathbf{X}^{(M(2L_1+1),M)}\right)\Big] \\ &= (\mathbf{H}^* \odot \mathbf{H}) \mathbf{D}^T \in \mathbb{C}^{M^2(2L_1+1)^2 \times 2M^2(2L_1+1)}, \end{aligned} \tag{13}$$

where

$$\mathbf{D} = \begin{bmatrix} \mathbf{A}^* \odot \mathbf{H} \\ \mathbf{A} \odot \mathbf{H}^* \end{bmatrix} \in \mathbb{C}^{2M^2(2L_1+1)\times(\sum_{r=1}^{R} L_r + R)}. \tag{14}$$

Based on the matrix representation $\mathbf{Y}$ in (13) of the cumulant tensor $\mathcal{T}$ in (7) we will in the following section present new uniqueness results for blind identification of underdetermined MIMO FIR channels.

3 Identifiability of MIMO FIR Channels

In this section we study the identifiability of MIMO FIR channels based on fourth-order cumulants. Subsection 3.1 provides a deterministic uniqueness condition for fourth-order cumulant based MIMO FIR identification while subsection 3.2 presents a generic version of it.

3.1 Deterministic Uniqueness Result for MIMO FIR Identication

For simplicity, let us first derive a uniqueness condition for the case when $L_r = L$, $\forall r \in [1,R]$ and it will be stated in proposition 3.2 which in turn is based on the following lemma.

Lemma 3.1. *Let $\boldsymbol{H}$ be a block-Hankel matrix of the form* (8)*, then*

$$\boldsymbol{h}_l^{(r)} \otimes \boldsymbol{h}_l^{(r)} = \left(\Pi_{M(2L+1)} \otimes \Pi_{M(2L+1)}\right)\left(\boldsymbol{h}_{l-1}^{(r)} \otimes \boldsymbol{h}_{l-1}^{(r)}\right),$$

where

$$\Pi_{M(2L+1)} \;=\; \begin{bmatrix} \boldsymbol{0}_{M2L,M} & \boldsymbol{I}_{M2L} \\ \boldsymbol{I}_M & \boldsymbol{0}_{M,M2L} \end{bmatrix} \in \mathbb{C}^{M(2L+1)\times M(2L+1)}. \tag{15}$$

Proof. The lemma is a direct consequence of the mixed product property of Kronecker products, i.e., we have $\mathbf{h}_l^{(r)} = \Pi_{M(2L+1)}\mathbf{h}_{l-1}^{(r)}$ and hence

$$\mathbf{h}_l^{(r)} \otimes \mathbf{h}_l^{(r)} = \left(\Pi_{M(2L+1)} \otimes \Pi_{M(2L+1)}\right)\left(\mathbf{h}_{l-1}^{(r)} \otimes \mathbf{h}_{l-1}^{(r)}\right).$$

□

Proposition 3.2. *Consider the cumulant tensor $\mathcal{T} \in \mathbb{C}^{M\times M(2L+1)\times M(2L+1)\times M(2L+1)}$ in* (7) *with $L_r = L$, $\forall r \in [1, R]$ and matrix representation $\boldsymbol{Y} = (\boldsymbol{H}^* \odot \boldsymbol{H})\,\boldsymbol{D}^T$, where $\boldsymbol{H} \in \mathbb{C}^{M(2L+1)\times R(L+1)}$ is of the form* (8) *and $\boldsymbol{D} \in \mathbb{C}^{2M(2L+1)\times R(L+1)}$ is of the form* (14)*. Assume that the matrices $\boldsymbol{D}$, $(\boldsymbol{H}^* \odot \boldsymbol{H}) \in \mathbb{C}^{M^2(2L+1)^2\times R(L+1)}$,*

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{P}D_1\left(\boldsymbol{P}^*\right) \\ \vdots \\ \boldsymbol{P}D_M\left(\boldsymbol{P}^*\right) \\ \boldsymbol{P}_0D_{M+1}\left(\boldsymbol{P}^*\right) \\ \vdots \\ \boldsymbol{P}_0D_{M(L+1)}\left(\boldsymbol{P}^*\right) \end{bmatrix} \in \mathbb{C}^{M^2(2L+1)\times R} \tag{16}$$

and

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{P}_LD_1\left(\boldsymbol{P}^*\right) \\ \vdots \\ \boldsymbol{P}_LD_{ML}\left(\boldsymbol{P}^*\right) \\ \boldsymbol{P}D_{ML+1}\left(\boldsymbol{P}^*\right) \\ \vdots \\ \boldsymbol{P}D_{M(L+1)}\left(\boldsymbol{P}^*\right) \end{bmatrix} \in \mathbb{C}^{M^2(2L+1)\times R} \tag{17}$$

have full column rank, where

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{p}_0^{(1)} & \cdots & \boldsymbol{p}_0^{(R)} \\ \vdots & \ddots & \vdots \\ \boldsymbol{p}_L^{(1)} & \cdots & \boldsymbol{p}_L^{(R)} \end{bmatrix} \in \mathbb{C}^{M(L+1)\times R} \tag{18}$$

and

$$\boldsymbol{P}_0 = \left[\begin{array}{ccc} \boldsymbol{p}_0^{(1)} & \cdots & \boldsymbol{p}_0^{(R)} \end{array} \right] \in \mathbb{C}^{M \times R} \tag{19}$$

$$\boldsymbol{P}_L = \left[\begin{array}{ccc} \boldsymbol{p}_L^{(1)} & \cdots & \boldsymbol{p}_L^{(R)} \end{array} \right] \in \mathbb{C}^{M \times R}. \tag{20}$$

Assume also that the CPD $\boldsymbol{Z} = (\boldsymbol{P}^* \odot \boldsymbol{P})\,\boldsymbol{A}$ *is essentially unique, where* $\boldsymbol{A} \in \mathbb{C}^{R \times R}$ *is a nonsingular matrix. Let* $\widehat{\boldsymbol{H}} \in \mathbb{C}^{M(2L+1) \times R(L+1)}$ *be another block-Hankel matrix with the same structure as* $\boldsymbol{H}$*, then* $Col\left(\widehat{\boldsymbol{H}}^* \odot \widehat{\boldsymbol{H}}\right) \subseteq Col\left(\boldsymbol{H}^* \odot \boldsymbol{H}\right) = Col\left(\boldsymbol{Y}\right)$ *if and only if* $\left(\widehat{\boldsymbol{H}}^* \odot \widehat{\boldsymbol{H}}\right)\Pi_{\boldsymbol{H}} = \left(\boldsymbol{H}^* \odot \boldsymbol{H}\right)\Pi_{\boldsymbol{H}}\left(\boldsymbol{I}_{L+1} \otimes \Pi\Delta\right)$*, where* $\Pi_{\boldsymbol{H}} \in \mathbb{C}^{R(L+1) \times R(L+1)}$ *is the permutation matrix of the form* (21)*,* $\Pi \in \mathbb{C}^{R \times R}$ *is a permutation matrix and* $\Delta \in \mathbb{C}^{R \times R}$ *is a diagonal matrix.*

Proof. Let us introduce the permutation matrix

$$\Pi_{\mathbf{H}} = \left[\Pi^{(0)}, \ldots, \Pi^{(L)}\right] \in \mathbb{C}^{R(L+1) \times R(L+1)}, \tag{21}$$

where

$$\Pi^{(i)} = \mathbf{I}_R \otimes \mathbf{e}_{i+1}^{(L+1)} = \left[\begin{array}{c|c|c|c} \begin{pmatrix} \mathbf{0}_i \\ 1 \\ \mathbf{0}_{L-i} \end{pmatrix} & \mathbf{0}_{L+1} & \cdots & \mathbf{0}_{L+1} \\ \hline \mathbf{0}_{L+1} & \begin{pmatrix} \mathbf{0}_i \\ 1 \\ \mathbf{0}_{L-i} \end{pmatrix} & \cdots & \mathbf{0}_{L+1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{0}_{L+1} & \mathbf{0}_{L+1} & \cdots & \begin{pmatrix} \mathbf{0}_i \\ 1 \\ \mathbf{0}_{L-i} \end{pmatrix} \end{array}\right] \in \mathbb{C}^{R(L+1) \times (L+1)},$$

and $\mathbf{e}_{i+1}^{(L+1)} \in \mathbb{C}^{L+1}$ is unit vector with unit entry at position $i+1$ and zero elsewhere, then

$$\mathbf{Z} = \left[\mathbf{Z}^{(0)}, \ldots, \mathbf{Z}^{(L)}\right] = \left(\mathbf{H}^* \odot \mathbf{H}\right)\Pi_{\mathbf{H}}, \tag{22}$$

where

$$\mathbf{Z}^{(0)} = \left[\begin{array}{c} \mathbf{0}_{ML,R} \\ \mathbf{P}^* \end{array} \right] \odot \left[\begin{array}{c} \mathbf{0}_{ML,R} \\ \mathbf{P} \end{array} \right] \in \mathbb{C}^{M^2(2L+1)^2 \times R}.$$

Due to lemma 3.1 we get

$$\mathbf{Z}^{(l)} = \left[\begin{array}{c} \mathbf{0}_{M(L+1-l),R} \\ \mathbf{P}^* \\ \mathbf{0}_{M(l-1),R} \end{array} \right] \odot \left[\begin{array}{c} \mathbf{0}_{M(L+1-l),R} \\ \mathbf{P} \\ \mathbf{0}_{M(l-1),R} \end{array} \right] = \left(\Pi_{M(2L+1)}^{l} \otimes \Pi_{M(2L+1)}^{l}\right)\mathbf{Z}^{(0)}, \tag{23}$$

where $\Pi_{M(2L+1)}$ is the permutation matrix (15). Assume that $\left(\widehat{\mathbf{H}}^* \odot \widehat{\mathbf{H}}\right)\Pi_{\mathbf{H}} = (\mathbf{H}^* \odot \mathbf{H})\,\Pi_{\mathbf{H}}\,(\mathbf{I}_{L+1} \otimes \Pi\Delta)$, then obviously $\mathrm{Col}\left(\widehat{\mathbf{H}}^* \odot \widehat{\mathbf{H}}\right) \subseteq \mathrm{Col}\,(\mathbf{H}^* \odot \mathbf{H}) = \mathrm{Col}\,(\mathbf{Y})$. Conversely, assume that $\mathrm{Col}\left(\widehat{\mathbf{H}}^* \odot \widehat{\mathbf{H}}\right) \subseteq \mathrm{Col}\,(\mathbf{H}^* \odot \mathbf{H}) = \mathrm{Col}\,(\mathbf{Y})$, then there exists a nonsingular matrix $\mathbf{M} \in \mathbb{C}^{R(L+1)\times R(L+1)}$ such that

$$\widehat{\mathbf{Z}} = \left(\widehat{\mathbf{H}}^* \odot \widehat{\mathbf{H}}\right)\Pi_{\mathbf{H}} = (\mathbf{H}^* \odot \mathbf{H})\,\Pi_{\mathbf{H}}\mathbf{M} = \mathbf{Z}\mathbf{M}. \tag{24}$$

In the first part of the proof we will capitalize on the block-banded structure of $\mathbf{Z}$ and $\widehat{\mathbf{Z}}$. Partition $\mathbf{M}$ as follows

$$\mathbf{M} \;=\; \begin{bmatrix} \mathbf{M}^{(1,1)} & \cdots & \mathbf{M}^{(1,L+1)} \\ \vdots & \ddots & \vdots \\ \mathbf{M}^{(L+1,1)} & \cdots & \mathbf{M}^{(L+1,L+1)} \end{bmatrix}, \quad \mathbf{M}^{(p,q)} \in \mathbb{C}^{R\times R}, \quad \forall p,q \in [1, L+1].$$

Let $q = M(2L+1)$, $p = M(L+1)$ and let us introduce the row-indexing vector

$$\mathbf{r}_1^{(l)} = \left[\begin{array}{l} [(l-1)Mq + M(l-1) + 1 : (l-1)Mq + M(l-1) + p]^T \\ [(l-1)Mq + M(l-1) + 1 + q : (l-1)Mq + M(l-1) + p + q]^T \\ \vdots \\ [(l-1)Mq + M(l-1) + 1 + q(M-1) : (l-1)Mq + M(l-1) + p + q(M-1)]^T \\ \hline [(l-1)Mq + M(l-1) + 1 + qM : (l-1)Mq + M(l-1) + M + qM]^T \\ [(l-1)Mq + M(l-1) + 1 + q(M+1) : (l-1)Mq + M(l-1) + M + q(M+1)]^T \\ \vdots \\ [(l-1)Mq + M(l-1) + 1 + q(p-1) : (l-1)Mq + M(l-1) + M + q(p-1)]^T \end{array}\right] \in \mathbb{C}^{Mq\times R},$$

then from (23) and (24) we observe that

$$\mathbf{0}_{M^2(2L+1),R} = \widehat{\mathbf{Z}}\left(r_1^{(1)}, (n-1)R : nR\right) = \mathbf{S}\mathbf{M}^{(L+1,n)}, \quad \forall n < L+1\,, \tag{25}$$

where $\mathbf{S}$ is of the form (16). Since it is assumed that $\mathbf{S}$ has full column rank we can conclude from (25) that $\mathbf{M}^{(L+1,n)} = \mathbf{0}_{R,R}$, $\forall n < L+1$. Consider now

$$\mathbf{0}_{M^2(2L+1),R} = \widehat{\mathbf{Z}}\left(r_1^{(2)}, (n-1)R : nR\right) = \mathbf{S}\mathbf{M}^{(L,n)}, \quad \forall n < L\,. \tag{26}$$

Again, relation (26) implies that $\mathbf{M}^{(L,n)} = \mathbf{0}_{R,R}$, $\forall n < L$. By repeating this procedure we can conclude that $\mathbf{M}^{(m,n)} = \mathbf{0}_{R,R}$ $\forall n < m$, i.e., $\mathbf{M}$ is lower block-triangular.

Next, let $q = M(2L+1)$, $p = M(L+1)$, $v = 2L+2$, $w = L+1$ and let us introduce the row-indexing vector

$$\mathbf{r}_2^{(l)} = \begin{bmatrix} [w-l)Mq + M(v-l) - M + 1 : (w-l) * Mq + M(v-l)]^T \\ [w-l)Mq + M(v-l) - M + 1 + q : (w-l) * Mq + M(v-l) + q]^T \\ \vdots \\ [w-l)Mq + M(v-l) - M + 1 + q(ML-1) : (w-l)Mq + M(v-l) + q(ML-1)]^T \\ [(2L+1-l)Mq + (M-1)(v-l) : (2L+1-l)Mq + M(v-l)]^T \\ [(2L+1-l)Mq + (M-1)(v-l) + q : (2L+1-l)Mq + M(v-l) + q]^T \\ \vdots \\ [(2L+1-l)Mq + (M-1)(v-l) + q(p-1) : (2L+1-l)Mq + M(v-l) + q(p-1)]^T \end{bmatrix} \in \mathbb{C}^{Mq \times R},$$

then from (23) and (24) we observe that

$$\mathbf{0}_{M^2(2L+1),R} = \widehat{\mathbf{Z}}\left(r_2^{(1)}, (n-1)R : nR\right) = \mathbf{T}\mathbf{M}^{(1,n)}, \quad \forall n > 1\,, \tag{27}$$

where $\mathbf{T}$ is of the form (17). Since we assume that $\mathbf{T}$ has full column rank we can conclude from (27) that $\mathbf{M}^{(1,n)} = \mathbf{0}_{R,R}$, $\forall n > 1$. Consider now

$$\mathbf{0}_{M^2(2L+1),R} = \widehat{\mathbf{Z}}\left(r_2^{(2)}, (n-1)R : nR\right) = \mathbf{T}\mathbf{M}^{(2,n)}, \quad \forall n > 2\,. \tag{28}$$

Again, relation (28) implies that $\mathbf{M}^{(2,n)} = \mathbf{0}_{R,R}$, $\forall n > 2$. By repeating this procedure we can conclude that $\mathbf{M}^{(m,n)} = \mathbf{0}_{R,R}$ $\forall n > m$, i.e., $\mathbf{M}$ is also upper block-triangular. Thus, $\mathbf{M}$ is equal to a block-diagonal matrix. We note in passing that if $\mathbf{S}$ and $\mathbf{T}$ have full column rank, then this also implies that $\mathbf{H}^* \odot \mathbf{H}$ has full column rank.

Second, due to the block-Hankel structure $(\mathbf{H}^* \odot \mathbf{H})\,\Pi_{\mathbf{H}}$, the matrix $\mathbf{M}$ further simplifies to $\mathbf{M} = \mathbf{I}_{L+1} \otimes \mathbf{A}$, where $\mathbf{A} \in \mathbb{C}^{R \times R}$ is a nonsingular matrix.

In the third and final part of the proof we exploit the CPD structure of the problem. Note that

$$\widehat{\mathbf{Z}}\,(:, 1:R) = \mathbf{Z}^{(0)}\mathbf{A} = \begin{bmatrix} \mathbf{0}_{L,R} \\ \mathbf{P}^* \end{bmatrix} \odot \begin{bmatrix} \mathbf{0}_{L,R} \\ \mathbf{P} \end{bmatrix} \mathbf{A}.$$

Since we assume that the CPD $(\mathbf{P}^* \odot \mathbf{P})\,\mathbf{A}$ is essentially unique this implies that the ambiguity matrix $\mathbf{A}$ reduces to $\mathbf{A} = \Pi\Delta$, where $\Pi \in \mathbb{C}^{R \times R}$ is a permutation matrix and $\Delta \in \mathbb{C}^{R \times R}$ is a diagonal matrix. □

A direct consequence of proposition 3.2 is the following corollary.

Corollary 3.3. *Consider the cumulant tensor* $\mathcal{T} \in \mathbb{C}^{M \times M(2L+1) \times M(2L+1) \times M(2L+1)}$ *in* (7) *with* $L_r = L$, $\forall r \in [1, R]$ *and matrix representation* $\boldsymbol{Y} = (\boldsymbol{H}^* \odot \boldsymbol{H})\, \boldsymbol{D}^T$, *where* $\boldsymbol{H} = \left[\boldsymbol{H}^{(1)}, \ldots, \boldsymbol{H}^{(R)}\right] \in \mathbb{C}^{M(2L+1) \times R(L+1)}$ *is of the form* (8) *and* $\boldsymbol{D} \in \mathbb{C}^{2M(2L+1) \times R(L+1)}$ *is of the form* (14). *Assume that the matrices* $\boldsymbol{D}$, $\boldsymbol{S} \in \mathbb{C}^{M^2(2L+1) \times R}$ *of the form* (16) *and* $\boldsymbol{T} \in \mathbb{C}^{M^2(2L+1) \times R}$ *of the form* (17) *have full column rank. Note that this also implies that* $(\boldsymbol{H}^* \odot \boldsymbol{H}) \in \mathbb{C}^{M^2(2L+1)^2 \times R(L+1)}$ *has full column rank. Let* $\boldsymbol{S} = \left[\boldsymbol{S}^{(1)}, \ldots, \boldsymbol{S}^{(R)}\right] \in \mathbb{C}^{M^2(2L+1)^2 \times R(L+1)}$ *be another matrix with the same block-Hankel structure as* $(\boldsymbol{H}^* \odot \boldsymbol{H})$, *that is*

$$\boldsymbol{S}^{(r)} = \sum_{m=1}^{M(L+1)} \sum_{n=1}^{M(L+1)} (\boldsymbol{E}_m \odot \boldsymbol{E}_n)\, s^{(r)}_{m+(n-1)(L+1)},$$

where $\boldsymbol{E}_m \in \mathbb{C}^{(2L+1) \times (L+1)}$ *are basis matrices of the form* (10) *and* $\boldsymbol{s}^{(r)} \in \mathbb{C}^{M^2(L+1)^2}$. *Then* $Col\,(\boldsymbol{S}) \subseteq Col\,(\boldsymbol{H}^* \odot \boldsymbol{H}) = Col\,(\boldsymbol{Y})$ *if and only if* $(\boldsymbol{H}^* \odot \boldsymbol{H})\, \Pi_{\boldsymbol{H}} = \boldsymbol{S} \Pi_{\boldsymbol{H}}\, (\boldsymbol{I}_{L+1} \otimes \boldsymbol{A})$, *where* $\boldsymbol{A} \in \mathbb{C}^{R \times R}$ *is a nonsingular matrix and* $\Pi_{\boldsymbol{H}} \in \mathbb{C}^{R(L+1) \times R(L+1)}$ *is a permutation matrix of the form* (21).

Corollary 3.3 states that if we ignore the Khatri-Rao product in $\mathbf{H}^* \odot \mathbf{H}$, then due to the block-Hankel structure of $\mathbf{H}$ it is still possible to recover $\mathbf{H}^* \odot \mathbf{H}$ up to a block-diagonal matrix from $\mathrm{Col}\,(\mathbf{H}^* \odot \mathbf{H}) = \mathrm{Col}\,(\mathbf{Y})$.

We will now generalize proposition 3.2 to the case of arbitrary filter orders.

Theorem 3.4. *Consider the cumulant tensor* $\mathcal{T} \in \mathbb{C}^{M \times M(2L_1+1) \times M(2L_1+1) \times M(2L_1+1)}$ *in* (7) *with* $L_1 \geq L_2 \geq \cdots \geq L_R$ *and matrix representation* $\boldsymbol{Y} = (\boldsymbol{H}^* \odot \boldsymbol{H})\, \boldsymbol{D}^T$, *where* $\boldsymbol{H} \in \mathbb{C}^{(2L_1+1) \times (\sum_{r=1}^{R} L_r + R)}$ *is of the form* (8) *and* $\boldsymbol{D} \in \mathbb{C}^{2(2L_1+1) \times (\sum_{r=1}^{R} L_r + R)}$ *is of the form* (14). *Assume that there are* N *distinct filter orders such that*

$$\boldsymbol{P}^{(n)} = \begin{bmatrix} \boldsymbol{p}_0^{(1)} & \cdots & \boldsymbol{p}_0^{(R_n)} \\ \vdots & \ddots & \vdots \\ \boldsymbol{p}_{L_n}^{(1)} & \cdots & \boldsymbol{p}_{L_n}^{(R_n)} \end{bmatrix} \in \mathbb{C}^{M(L_n+1) \times R_n}$$

with $L_1 > \cdots > L_N$. *Let*

$$\boldsymbol{P} = \left[\begin{array}{c|c|c|c} \boldsymbol{P}^{(1)} & \begin{matrix} \boldsymbol{P}^{(2)} \\ \boldsymbol{0}_{M(L_1-L_2),R_2} \end{matrix} & \cdots & \begin{matrix} \boldsymbol{P}^{(N)} \\ \boldsymbol{0}_{M(L_1-L_N),R_N} \end{matrix} \end{array}\right] \in \mathbb{C}^{M(L_1+1) \times R} \tag{29}$$

where $\sum_{n=1}^{N} R_n = R$ *and assume that the matrices* $\boldsymbol{D}$, $(\boldsymbol{H}^* \odot \boldsymbol{H}) \in \mathbb{C}^{M^2(2L_1+1)^2 \times (\sum_{r=1}^{R} L_r + R)}$,

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{P}D_1(\boldsymbol{P}^*) \\ \vdots \\ \boldsymbol{P}D_M(\boldsymbol{P}^*) \\ \boldsymbol{P}_0 D_{M+1}(\boldsymbol{P}^*) \\ \vdots \\ \boldsymbol{P}_0 D_{M(L_1+1)}(\boldsymbol{P}^*) \end{bmatrix} \in \mathbb{C}^{M^2(2L_1+1)\times R} \quad \textit{and} \quad \boldsymbol{T} = \begin{bmatrix} \boldsymbol{P}_{L_1} D_1(\boldsymbol{P}^*) \\ \vdots \\ \boldsymbol{P}_{L_1} D_{ML_1}(\boldsymbol{P}^*) \\ \boldsymbol{P}D_{ML+1}(\boldsymbol{P}^*) \\ \vdots \\ \boldsymbol{P}D_{M(L_1+1)}(\boldsymbol{P}^*) \end{bmatrix} \in \mathbb{C}^{M^2(2L_1+1)\times R}$$

have full column rank, where

$$\begin{aligned} \boldsymbol{P}_0 &= \boldsymbol{P}(1:M,:) \in \mathbb{C}^{M\times R} \\ \boldsymbol{P}_L &= \boldsymbol{P}(ML_1+1:M(L_1+1),:) \in \mathbb{C}^{M\times R}. \end{aligned}$$

Assume also that the CPD $\boldsymbol{Z} = (\boldsymbol{P}^* \odot \boldsymbol{P})\boldsymbol{A}$ *is essentially unique, where* $\boldsymbol{A} \in \mathbb{C}^{R\times R}$ *is a nonsingular matrix. Let* $\widehat{\boldsymbol{H}} \in \mathbb{C}^{(2L_1+1)\times(\sum_{r=1}^{R} L_r+R)}$ *be another block-Hankel matrix with the same structure as* $\boldsymbol{H}$*, then* $Col\left(\widehat{\boldsymbol{H}}^* \odot \widehat{\boldsymbol{H}}\right) \subseteq Col\left(\boldsymbol{H}^* \odot \boldsymbol{H}\right) = Col\left(\boldsymbol{Y}\right)$ *if and only if* $\widehat{\boldsymbol{H}}^{(r)} = \boldsymbol{H}^{(r)}\alpha_r$*, where* $\alpha_r \in \mathbb{C}$, $\forall r \in [1, R]$ *and up to a permutation of the block matrices* $\widehat{\boldsymbol{H}}^{(r)}$*.*

Proof. The proof is in principle the same as for proposition 3.2. However, due to the nonuniform partition of the involved matrices the proof becomes quite tedious. Hence, we will just sketch it by an example. Let $R_1 = 2, L_1 = 2$, $R_2 = 1, L_1 = 1, R_3 = 1$ and $L_3 = 0$, then

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_0^{(1)} & \mathbf{p}_0^{(2)} & \mathbf{p}_0^{(3)} & \mathbf{p}_0^{(4)} \\ \mathbf{p}_1^{(1)} & \mathbf{p}_1^{(2)} & \mathbf{p}_1^{(3)} & \mathbf{0}_M \\ \mathbf{p}_2^{(1)} & \mathbf{p}_2^{(2)} & \mathbf{0}_M & \mathbf{0}_M \end{bmatrix}.$$

Assume that $\left(\widehat{\mathbf{H}}^* \odot \widehat{\mathbf{H}}\right)\Pi_{\mathbf{H}} = (\mathbf{H}^* \odot \mathbf{H})\,\Pi_{\mathbf{H}}\mathrm{Blockdiag}\left(\mathbf{A}_{\Pi\Delta}, \mathbf{A}_{\Pi\Delta}(1:3,1:3), \mathbf{A}_{\Pi\Delta}(1:2,1:2)\right)$,

where

$$\Pi_{\mathbf{H}} = \left[\begin{array}{ccccc|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \in \mathbb{C}^{(\sum_{r=1}^{R} L_r + R) \times (\sum_{r=1}^{R} L_r + R)},$$

$$\mathbf{A}_{\Pi\Delta} = \Pi\Delta \in \mathbb{C}^{R \times R},$$

and $\Pi \in \mathbb{C}^{R \times R}$ is a permutation matrix and $\Delta \in \mathbb{C}^{R \times R}$ is a diagonal matrix, then obviously $\mathrm{Col}\left(\widehat{\mathbf{H}}^{*} \odot \widehat{\mathbf{H}}\right) \subseteq \mathrm{Col}\left(\mathbf{H}^{*} \odot \mathbf{H}\right) = \mathrm{Col}\left(\mathbf{Y}\right)$. Conversely, assume that $\mathrm{Col}\left(\widehat{\mathbf{H}}^{*} \odot \widehat{\mathbf{H}}\right) \subseteq \mathrm{Col}\left(\mathbf{H}^{*} \odot \mathbf{H}\right) = \mathrm{Col}\left(\mathbf{Y}\right)$, then there exists a nonsingular matrix $\mathbf{M} \in \mathbb{C}^{9 \times 9}$ such that

$$\widehat{\mathbf{Z}} = \left(\widehat{\mathbf{H}}^{*} \odot \widehat{\mathbf{H}}\right) \Pi_{\mathbf{H}} = \left(\mathbf{H}^{*} \odot \mathbf{H}\right) \Pi_{\mathbf{H}} \mathbf{M} = \mathbf{Z}\mathbf{M}. \tag{30}$$

We have

$$\mathbf{Z} = \left[\mathbf{Z}^{(0)}, \mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}\right] = \left(\mathbf{H}^{*} \odot \mathbf{H}\right) \Pi_{\mathbf{H}}$$

with

$$\mathbf{Z}^{(0)} = \left[\begin{array}{c} \mathbf{0}_{2M,4} \\ \mathbf{P}^{*} \end{array}\right] \odot \left[\begin{array}{c} \mathbf{0}_{2M,4} \\ \mathbf{P} \end{array}\right], \mathbf{Z}^{(1)} = \left[\begin{array}{c} \mathbf{0}_{M,3} \\ \mathbf{P}^{*}(:, 1:3) \\ \mathbf{0}_{M,3} \end{array}\right] \odot \left[\begin{array}{c} \mathbf{0}_{M,3} \\ \mathbf{P}(:, 1:3) \\ \mathbf{0}_{M,3} \end{array}\right], \mathbf{Z}^{(2)} = \left[\begin{array}{c} \mathbf{P}^{*}(:, 1:2) \\ \mathbf{0}_{2M,2} \end{array}\right] \odot \left[\begin{array}{c} \mathbf{P}(:, 1:2) \\ \mathbf{0}_{2M,2} \end{array}\right].$$

In the first part of the proof we will capitalize on the block-banded structure of $\mathbf{Z}$ and $\widehat{\mathbf{Z}}$. Partition $\mathbf{M}$ as follows

$$\mathbf{M} = \left[\begin{array}{ccc} \mathbf{M}^{(1,1)} & \mathbf{M}^{(1,2)} & \mathbf{M}^{(1,3)} \\ \mathbf{M}^{(2,1)} & \mathbf{M}^{(2,2)} & \mathbf{M}^{(2,3)} \\ \mathbf{M}^{(3,1)} & \mathbf{M}^{(3,2)} & \mathbf{M}^{(3,3)} \end{array}\right]$$

where

$$\begin{array}{lll} \mathbf{M}^{(1,1)} \in \mathbb{C}^{4 \times 4} & \mathbf{M}^{(1,2)} \in \mathbb{C}^{4 \times 3} & \mathbf{M}^{(1,3)} \in \mathbb{C}^{4 \times 2} \\ \mathbf{M}^{(2,1)} \in \mathbb{C}^{3 \times 4} & \mathbf{M}^{(2,2)} \in \mathbb{C}^{3 \times 3} & \mathbf{M}^{(2,3)} \in \mathbb{C}^{3 \times 2} \\ \mathbf{M}^{(3,1)} \in \mathbb{C}^{2 \times 4} & \mathbf{M}^{(3,2)} \in \mathbb{C}^{2 \times 3} & \mathbf{M}^{(3,3)} \in \mathbb{C}^{2 \times 2} \end{array}$$

As in the proof of proposition 3.2, then due to the assumption that the matrices $\mathbf{S}$ and $\mathbf{T}$ have full column rank and the banded structure of $\mathbf{Z}$ and $\widehat{\mathbf{Z}}$ we conclude that

$$\mathbf{M} \quad = \quad \text{blockdiag}\left(\mathbf{M}^{(1,1)}, \mathbf{M}^{(2,2)}, \mathbf{M}^{(3,3)}\right).$$

Note that

$$\widehat{\mathbf{Z}}(:, 1:4) = \mathbf{Z}^{(0)}\mathbf{M}^{(1,1)} = \begin{bmatrix} \mathbf{0}_{LM,R} \\ \mathbf{P}^* \end{bmatrix} \odot \begin{bmatrix} \mathbf{0}_{LM,R} \\ \mathbf{P} \end{bmatrix} \mathbf{M}^{(1,1)}.$$

Since we assume that the CPD $(\mathbf{P}^* \odot \mathbf{P})\,\mathbf{M}^{(1,1)}$ is essentially unique this implies that the ambiguity matrix $\mathbf{M}^{(1,1)}$ reduces to $\mathbf{M}^{(1,1)} = \Pi\Delta$, where $\Pi \in \mathbb{C}^{R\times R}$ is a permutation matrix and $\Delta \in \mathbb{C}^{R\times R}$ is a diagonal matrix.

Finally, due to the block-Hankel structure of $\mathbf{Z}$ and $\widehat{\mathbf{Z}}$ the matrix $\mathbf{M}$ further reduces to

$$\mathbf{M} \quad = \quad \text{blockdiag}\left(\mathbf{A}_{\Pi\Delta}, \mathbf{A}_{\Pi\Delta}(1:3, 1:3), \mathbf{A}_{\Pi\Delta}(1:2, 1:2)\right).$$

□

3.2 Generic Uniqueness Result for MIMO FIR Identication

In order to apply proposition 3.2 in practice we need to figure out when the conditions made in it are generically satisfied. We limit the exposition to the case when $L_r = L$, $\forall r \in [1, R]$ and $2 \leq M, L \leq 6$, but the result can of course be generalized. The generic uniqueness result is stated in theorem 3.9 and it is based on the following lemmas.

Lemma 3.5. *Given an analytic function $f : \mathbb{C}^n \to \mathbb{C}$. If there exists an element $\boldsymbol{x} \in \mathbb{C}^n$ such that $f(\boldsymbol{x}) \neq 0$, then the set $\{\boldsymbol{x} \mid f(\boldsymbol{x}) = 0\}$ is of Lebesgue measure zero.*

Proof. For a proof, see for instance [6]. □

Lemma 3.6. *Let the matrix $\boldsymbol{D} \in \mathbb{C}^{2M^2(2L+1)\times R(L+1)}$ of the form* (14)*. If $2 \leq M, L \leq 6$ and R is less or equal than the value indicated in table 1, then $\boldsymbol{D}$ has generically full column rank.*

Proof. Due to lemma 3.5 we just need to find one generic example where the statement made in the lemma holds. This was done numerically. □

Next, we show that if $M^2(2L+1) \geq R$, then the matrices $\mathbf{S}$, $\mathbf{T}$ and $\mathbf{H}^* \odot \mathbf{H}$ have generically full column rank.

$M\backslash L$	2	3	4	5	6
2	12	13	13	14	14
3	27	29	30	31	32
4	48	52	54	56	57
5	75	81	85	87	89
6	108	117	122	126	128

Table 1: The entry of the given pair (M, L) corresponds to the maximal value of R for which the matrix $\mathbf{D}$ has full column rank.

Lemma 3.7. *Let $\boldsymbol{P} \in \mathbb{C}^{M(L+1)\times R}$ with $M^2(2L+1) \geq R$, then the matrices $\boldsymbol{H}^* \odot \boldsymbol{H} \in \mathbb{C}^{M^2(2L+1)^2 \times R(L+1)}$, where $\boldsymbol{H} \in \mathbb{C}^{M(2L+1)\times R(L+1)}$ is of the form* (8)*, and*

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{P}D_1(\boldsymbol{P}^*) \\ \vdots \\ \boldsymbol{P}D_M(\boldsymbol{P}^*) \\ \boldsymbol{P}_0 D_{M+1}(\boldsymbol{P}^*) \\ \vdots \\ \boldsymbol{P}_0 D_{M(L+1)}(\boldsymbol{P}^*) \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}_0^* \odot \boldsymbol{P} \\ \overline{\boldsymbol{P}}^* \odot \boldsymbol{P}_0 \end{bmatrix} \in \mathbb{C}^{M^2(2L+1)\times R} \tag{31}$$

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{P}_L D_1(\boldsymbol{P}^*) \\ \vdots \\ \boldsymbol{P}_L D_{ML}(\boldsymbol{P}^*) \\ \boldsymbol{P}D_{ML+1}(\boldsymbol{P}^*) \\ \vdots \\ \boldsymbol{P}D_{M(L+1)}(\boldsymbol{P}^*) \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{P}}^* \odot \boldsymbol{P}_L \\ \boldsymbol{P}_L^* \odot \boldsymbol{P} \end{bmatrix} \in \mathbb{C}^{M^2(2L+1)\times R} \tag{32}$$

have generically full column rank, where

$$\begin{aligned} \overline{\boldsymbol{P}} &= \boldsymbol{P}(M+1 : M(L+1), :) \in \mathbb{C}^{ML\times R} \\ \underline{\boldsymbol{P}} &= \boldsymbol{P}(1 : ML, :) \in \mathbb{C}^{ML\times R} \\ \boldsymbol{P}_0 &= \boldsymbol{P}(1 : M, :) \in \mathbb{C}^{M\times R} \\ \boldsymbol{P}_L &= \boldsymbol{P}(ML+1 : M(L+1), :) \in \mathbb{C}^{M\times R}. \end{aligned}$$

Proof. Consider the Krylov matrix

$$\mathbf{P} = \left[\mathbf{f}, \mathbf{A}\mathbf{f}, \ldots, \mathbf{A}^{R-1}\mathbf{f}\right] \in \mathbb{C}^{M(L+1)\times R},$$

where $\mathbf{A} \in \mathbb{C}^{M(L+1)\times M(L+1)}$ and $\mathbf{f} \in \mathbb{C}^{M(L+1)}$, then

$$\begin{aligned}
\overline{\mathbf{P}} &= \left[\bar{\mathbf{f}}, \overline{\mathbf{A}}\mathbf{f}, \ldots, \overline{\left(\mathbf{A}^{R-1}\right)}\mathbf{f}\right] \\
\underline{\mathbf{P}} &= \left[\underline{\mathbf{f}}, \underline{\mathbf{A}}\mathbf{f}, \ldots, \underline{\left(\mathbf{A}^{R-1}\right)}\mathbf{f}\right] \\
\mathbf{P}_0 &= \left[\mathbf{f}_0, \mathbf{A}_0\mathbf{f}, \ldots, \left(\mathbf{A}^{R-1}\right)_0 \mathbf{f}\right] \\
\mathbf{P}_L &= \left[\mathbf{f}_L, \mathbf{A}_L\mathbf{f}, \ldots, \left(\mathbf{A}^{R-1}\right)_L \mathbf{f}\right]
\end{aligned}$$

with

$$\begin{aligned}
\bar{\mathbf{f}} &= \mathbf{f}(M+1 : M(L+1), :) \quad , \quad \overline{(\mathbf{A}^r)} = (\mathbf{A}^r)(M+1 : M(L+1), :) \\
\underline{\mathbf{f}} &= \mathbf{f}(1 : ML, :) \quad , \quad \underline{(\mathbf{A}^r)} = (\mathbf{A}^r)(1 : ML, :) \\
\mathbf{f}_0 &= \mathbf{f}(1 : M, :) \quad , \quad (\mathbf{A}^r)_0 = (\mathbf{A}^r)(1 : M, :) \\
\mathbf{f}_L &= \mathbf{f}(M+1, M(L+1), :) \quad , \quad (\mathbf{A}^r)_L = (\mathbf{A}^r)(M+1, M(L+1), :) .
\end{aligned}$$

Thus

$$\begin{bmatrix} \mathbf{P}_0^* \odot \mathbf{P} \\ \overline{\mathbf{P}}^* \odot \mathbf{P}_0 \end{bmatrix} = \begin{bmatrix} \left(\mathbf{f}_0^* \otimes \mathbf{f}\right), \left(\mathbf{A}_0^* \otimes \mathbf{A}\right)(\mathbf{f}^* \otimes \mathbf{f}), \ldots, \left(\overline{\left(\mathbf{A}_0^{R-1}\right)^*} \otimes \mathbf{A}^{R-1}\right)(\mathbf{f}^* \otimes \mathbf{f}) \\ \left(\bar{\mathbf{f}}^* \otimes \mathbf{f}_0\right), \left(\overline{\mathbf{A}}^* \otimes \mathbf{A}_0\right)(\mathbf{f}^* \otimes \mathbf{f}), \ldots, \left(\overline{\left(\mathbf{A}^{R-1}\right)^*} \otimes \left(\mathbf{A}^{R-1}\right)_0\right)(\mathbf{f}^* \otimes \mathbf{f}) \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} \underline{\mathbf{P}}^* \odot \mathbf{P}_L \\ \mathbf{P}_L^* \odot \mathbf{P} \end{bmatrix} = \begin{bmatrix} \left(\underline{\mathbf{f}}^* \otimes \mathbf{f}_L\right), \left(\underline{\mathbf{A}}^* \otimes \mathbf{A}_L\right)(\mathbf{f}^* \otimes \mathbf{f}), \ldots, \left(\underline{\left(\mathbf{A}^{R-1}\right)^*} \otimes \left(\mathbf{A}^{R-1}\right)_L\right)(\mathbf{f}^* \otimes \mathbf{f}) \\ (\mathbf{f}^* \otimes \mathbf{f}), \left(\mathbf{A}_L^* \otimes \mathbf{A}\right)(\mathbf{f}^* \otimes \mathbf{f}), \ldots, \left(\underline{\left(\mathbf{A}^{R-1}\right)^*}_L \otimes \mathbf{A}^{R-1}\right)(\mathbf{f}^* \otimes \mathbf{f}) \end{bmatrix} . \tag{34}$$

By inserting $\mathbf{A} = D_1(\mathbf{d}) \in \mathbb{C}^{M(L+1)\times M(L+1)}$ with $\mathbf{d} = [d_1, \ldots, d_{M(L+1)}] \in \mathbb{C}^{1\times M(L+1)}$ and $\mathbf{f} = \mathbf{1}_{M(L+1)}$ into (33) and (34) we get

$$\begin{aligned}
\begin{bmatrix} \mathbf{P}_0^* \odot \mathbf{P} \\ \overline{\mathbf{P}}^* \odot \mathbf{P}_0 \end{bmatrix} &= \begin{bmatrix} \mathbf{1}_M \otimes \mathbf{1}_{M(L+1)} & \mathbf{d}_0^* \otimes \mathbf{d} & \ldots & \left(\mathbf{d}_0^*\right)^{R-1} \otimes \mathbf{d}^{R-1} \\ \mathbf{1}_{ML} \otimes \mathbf{1}_M & \overline{\mathbf{d}}^* \otimes \mathbf{d}_0 & \ldots & \left(\overline{\mathbf{d}}^*\right)^{R-1} \otimes \mathbf{d}_0^{R-1} \end{bmatrix} \\
&= \left[\begin{array}{cccc} 1 & d_1^* d_1 & \cdots & (d_1^* d_1)^{R-1} \\ 1 & d_1^* d_2 & \cdots & (d_1^* d_2)^{R-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & d_M^* d_{M(L+1)} & \cdots & (d_M^* d_{M(L+1)})^{R-1} \\ \hline 1 & d_{M+1}^* d_1 & \cdots & (d_{M+1}^* d_1)^{R-1} \\ 1 & d_{M+1}^* d_2 & \cdots & (d_{M+1}^* d_2)^{R-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & d_{M(L+1)}^* d_M & \cdots & (d_{M(L+1)}^* d_M)^{R-1} \end{array}\right]
\end{aligned} \tag{35}$$

and

$$\begin{bmatrix} \underline{\mathbf{P}}^* \odot \mathbf{P}_L \\ \mathbf{P}_L^* \odot \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{1}_{ML} \otimes \mathbf{1}_M & \underline{\mathbf{d}}^* \otimes \mathbf{d}_L & \dots & \left(\underline{\mathbf{d}}^*\right)^{R-1} \otimes \mathbf{d}_L^{R-1} \\ \mathbf{1}_M \otimes \mathbf{1}_{M(L+1)} & \mathbf{d}_L^* \otimes \mathbf{d} & \dots & \left(\mathbf{d}_L^*\right)^{R-1} \otimes \mathbf{d}^{R-1} \end{bmatrix}$$

$$= \left[\begin{array}{cccc} 1 & d_1^* d_1 & \cdots & (d_1^* d_1)^{R-1} \\ 1 & d_1^* d_2 & \cdots & (d_1^* d_2)^{R-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & d_{ML}^* d_{M(L+1)} & \cdots & (d_{ML}^* d_{M(L+1)})^{R-1} \\ \hline 1 & d_{ML+1}^* d_1 & \cdots & (d_{ML+1}^* d_1)^{R-1} \\ 1 & d_{ML+1}^* d_2 & \cdots & (d_{ML+1}^* d_2)^{R-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & d_{M(L+1)}^* d_{M(L+1)} & \cdots & (d_{M(L+1)}^* d_{M(L+1)})^{R-1} \end{array}\right]. \quad (36)$$

Obviously (35) and (36) are transposed Vandermonde matrices. If the generators $\{d_m^* d_n\}$ of the transposed Vandermonde matrices (35) and (36) are distinct, then there generically are $\min(R, M^2(2L+1))$ linearly independent column vectors of the transposed Vandermonde matrices (35) and (36). Due to lemma 3.5 we can now conclude that the matrices (35) and (36) have generically full column rank when $M^2(2L+1) \geq R$. Hence, from (35), (36) and lemma 3.5 we can conclude that the matrices $\mathbf{S}$ and $\mathbf{T}$ have generically full column rank. As observed in the proof of proposition 3.2 this also implies that $\mathbf{H}^* \odot \mathbf{H}$ has generically full column rank when $M^2(2L+1) \geq R$. □

Lemma 3.8. *Consider the CPD $\boldsymbol{X} = (\boldsymbol{P}^* \odot \boldsymbol{P})\boldsymbol{A}$, where $\boldsymbol{P} \in \mathbb{C}^{M(L+1)\times R}$ and $\boldsymbol{A} \in \mathbb{C}^{R\times R}$ is a nonsingular matrix. Then the CPD of $\boldsymbol{X}$ is essentially unique in the generic case if*

$$2R(R-1) \leq M^2 (L+1)^2 (M(L+1)-1)^2 .$$

Proof. The result follows directly from theorem 1.2 in subsection 1.2 and proposition 2.4 in [13]. □

We are now ready to present a generic version of proposition 3.2 for the case when $2 \leq M, L \leq 6$ and it will be stated in the following theorem.

Theorem 3.9. *Consider the cumulant tensor $\mathcal{T} \in \mathbb{C}^{M\times M(2L+1)\times M(2L+1)\times M(2L+1)}$ in* (7) *with $L = L_r$, $\forall r \in [1, R]$ and matrix representation $\boldsymbol{Y} = (\boldsymbol{H}^* \odot \boldsymbol{H})\boldsymbol{D}^T$, where $\boldsymbol{H} \in \mathbb{C}^{M(2L+1)\times R(L+1)}$ is of the form* (8) *and $\boldsymbol{D} \in \mathbb{C}^{2M^2(2L+1)\times R(L+1)}$ is of the form* (14)*. If $2 \leq M, L \leq 6$ and R is less or equal than the value indicated in table 2, then CPD of $\mathcal{T}$ is essentially unique in the generic case.*

$M\backslash L$	2	3	4	5	6
2	12	13	13	14	14
3	27	29	30	31	32
4	48	52	54	56	57
5	75	81	85	87	89
6	108	117	122	126	128

Table 2: The entry of the given pair (M, L) corresponds to the maximal value of R for which the CPD of $\mathcal{T}$ is essentially unique in the generic case.

Proof. The result follows directly from proposition 3.2 and lemmas 3.6, 3.7 and 3.8. □

In order to appreciate the new generic uniqueness result for underdetermined MIMO FIR identification we will compare it with the uniqueness results obtained in [16] and [21]. Roughly speaking, in [16] it was observed that if

$$M(L+1) \geq R\,, \tag{37}$$

then the CPD of $\mathcal{T}$ in (7) is essentially unique in the generic case. When $2 \leq M, L \leq 6$ and R is less or equal than the value indicated in table 3, then condition (37) guarantees that the CPD of $\mathcal{T}$ is essentially unique in the generic case.

$M\backslash L$	2	3	4	5	6
2	6	8	10	12	14
3	9	12	15	18	21
4	12	16	20	24	28
5	15	20	25	30	35
6	18	24	30	36	42

Table 3: The entry of the given pair (M, L) corresponds to the maximal value of R for which the condition (37) guarantees that the CPD of $\mathcal{T}$ is essentially unique in the generic case.

In [21], then under some approximations, it was observed that if

$$3\min(M, R) + \min(2M, R) \geq 2R + 3\,, \tag{38}$$

then the CPD of $\mathcal{T}$ in (7) is essentially unique in the generic case. Note that this condition does not take the filter order L into account. When $2 \leq M, L \leq 6$ and R is less or equal than the value indicated in table 4, then condition (38) guarantees that the CPD of $\mathcal{T}$ is essentially unique in the generic case.

$M \backslash L$	2	3	4	5	6
2	3	3	3	3	3
3	6	6	6	6	6
4	8	8	8	8	8
5	11	11	11	11	11
6	13	13	13	13	13

Table 4: The entry of the given pair (M, L) corresponds to the maximal value of R for which the condition (38) guarantees that the CPD of $\mathcal{T}$ is essentially unique in the generic case.

By inspecting the tables 2, 3 and 4 it is clear that the presented uniqueness results are significantly more relaxed than the existing ones. We also observe that condition (38) is very conservative and the reason is that it does not take the filter order L into account. The reason why the proposed uniqueness results yield better results than condition (37) is that it better takes the Khatri-Rao product structure and the banded block-Hankel structure of the CPD of the tensor $\mathcal{T}$ in (7) into account.

4 Computation of MIMO FIR Channels based on Cumulants

Subsection 4.1 and 4.2 present channel and noise subspace methods that transform the original MIMO FIR channel estimation problem into a SMD problem involving a congruence transform. Subsection 4.3 briefly reviews how to solve a SMD problem. Finally, in subsection 4.4 we present a simple Alternating Least Squares (ALS)-type method which can be used to refine the estimates obtained by the subspace methods.

4.1 Channel Subspace Method

Let

$$\mathbf{D} = \begin{bmatrix} \mathbf{0}_{L,L+1} \\ \mathbf{J}_{L+1} \end{bmatrix} \in \mathbb{C}^{(2L+1)\times(L+1)}$$

and

$$\begin{aligned} \mathbf{C}^{(k,m)} &= \Pi^{m-1}\mathbf{D} \otimes \mathbf{e}_m^{(M)} \in \mathbb{C}^{M(2L+1)\times(L+1)}, \\ \mathbf{D}^{(n)} &= \Pi^{n-1}\mathbf{D} \qquad\qquad \in \mathbb{C}^{(2L+1)\times(L+1)}, \end{aligned}$$

where

$$\Pi = \begin{bmatrix} \mathbf{0}_{2L}^T & 1 \\ \mathbf{I}_{2L} & \mathbf{0}_{2L} \end{bmatrix} \in \mathbb{C}^{(2L+1)\times(2L+1)}$$

and $\mathbf{e}_m^{(M)} \in \mathbb{C}^M$ is a unit vector with unit entry at position m and zero elsewhere. Then

$$\begin{aligned} \mathbb{C}^{M^2(2L+1)^2\times R(L+1)} \ni \mathbf{Z} &= (\mathbf{H}^* \odot \mathbf{H})\,\Pi_{\mathbf{H}} \\ &= \sum_{k=1}^{L+1}\sum_{m=1}^{M}\sum_{n=1}^{L+1}\left(\mathbf{C}^{(k,m)} \odot \mathbf{D}^{(n)}\right) \otimes \left(\mathbf{P}_{L+2-n} D_m\left(\mathbf{P}^*_{L+2-k}\right)\right) \\ &= \sum_{k=1}^{L+1}\sum_{m=1}^{M}\sum_{n=1}^{L+1}\mathbf{E}^{(k,m,n)} \otimes \left(\mathbf{P}_{L+2-n} D_m\left(\mathbf{P}^*_{L+2-k}\right)\right), \end{aligned} \tag{39}$$

where $\Pi_{\mathbf{H}} \in \mathbb{C}^{R(L+1)\times R(L+1)}$ is the permutation matrix of the form (21) and

$$\mathbf{P}_l = \left[\mathbf{p}_l^{(1)}, \ldots, \mathbf{p}_l^{(R)}\right] \in \mathbb{C}^{M\times R}, \quad \forall l \in [1, L+1], \tag{40}$$

and

$$\mathbf{E}^{(k,m,n)} = \left(\mathbf{C}^{(k,m)} \odot \mathbf{D}^{(n)}\right) \in \mathbb{C}^{M(2L+1)^2\times(L+1)}. \tag{41}$$

Furthermore, let

$$\mathbf{E} = \left[\mathrm{Vec}\left(\mathbf{E}^{(L+1,1,L+1)}\right) \otimes \mathbf{I}_M, \mathrm{Vec}\left(\mathbf{E}^{(L+1,1,L)}\right) \otimes \mathbf{I}_M, \ldots, \mathrm{Vec}\left(\mathbf{E}^{(1,M,1)}\right) \otimes \mathbf{I}_M\right] \in \mathbb{C}^{M^2(2L+1)^2(L+1)\times(L+1)^2M^2},$$

and let us partition $\mathbf{Z}$ and $\mathbf{H}$ as follows

$$\begin{aligned} \mathbf{Z} &= [\mathbf{Z}_1, \ldots \mathbf{Z}_{L+1}], \quad \mathbf{Z}_m \in \mathbb{C}^{M^2(2L+1)^2\times R} \quad \forall m \in [1, L+1], \\ \mathbf{H} &= [\mathbf{H}_1, \ldots \mathbf{H}_{L+1}], \quad \mathbf{H}_m \in \mathbb{C}^{M^2(2L+1)\times R} \quad \forall m \in [1, L+1], \end{aligned}$$

then (39) is equal to

$$\mathbf{E}\left(\mathbf{P}^{*}\odot\mathbf{P}\right)=\begin{bmatrix}\mathbf{Z}_{1}\\ \vdots\\ \mathbf{Z}_{L+1}\end{bmatrix}, \tag{42}$$

where

$$\mathbf{P}=\left[\mathbf{P}_{1}^{T},\ldots,\mathbf{P}_{L+1}^{T}\right]^{T}\in\mathbb{C}^{ML\times R}. \tag{43}$$

Let the column vectors of $\mathbf{U}\in\mathbb{C}^{M^{2}(2L+1)^{2}\times R(L+1)}$ constitute a basis for Col $(\mathbf{H}^{*}\odot\mathbf{H})$, then there exists a nonsingular matrix $\mathbf{M}\in\mathbb{C}^{R(2L+1)\times R(2L+1)}$ such that

$$\mathbf{H}^{*}\odot\mathbf{H}=\mathbf{U}\mathbf{M}. \tag{44}$$

Let us partition $\mathbf{M}$ as follows

$$\mathbf{M}=[\mathbf{M}_{1},\ldots,\mathbf{M}_{L+1}],\quad \mathbf{M}_{l}\in\mathbb{C}^{R(L+1)\times R},$$

then due to relations (42) and (44) we get

$$\begin{bmatrix}\mathbf{Z}_{1}\\ \vdots\\ \mathbf{Z}_{L+1}\end{bmatrix}=(\mathbf{I}_{L+1}\otimes\mathbf{U})\begin{bmatrix}\mathbf{M}_{1}\\ \vdots\\ \mathbf{M}_{L+1}\end{bmatrix}\Leftrightarrow\mathbf{E}\left(\mathbf{P}^{*}\odot\mathbf{P}\right)=(\mathbf{I}_{L+1}\otimes\mathbf{U})\begin{bmatrix}\mathbf{M}_{1}\\ \vdots\\ \mathbf{M}_{L+1}\end{bmatrix}. \tag{45}$$

Equation (45) is equivalent to

$$[\mathbf{I}_{L+1}\otimes\mathbf{U},-\mathbf{E}]\begin{bmatrix}\mathbf{M}_{1}\\ \vdots\\ \mathbf{M}_{L+1}\\ \mathbf{P}^{*}\odot\mathbf{P}\end{bmatrix}=\mathbf{G}\begin{bmatrix}\mathbf{M}_{1}\\ \vdots\\ \mathbf{M}_{L+1}\\ \mathbf{P}^{*}\odot\mathbf{P}\end{bmatrix}=\mathbf{0}, \tag{46}$$

where

$$\mathbf{G}=[\mathbf{I}_{L+1}\otimes\mathbf{U},-\mathbf{E}]\in\mathbb{C}^{M^{2}(2L+1)^{2}(L+1)\times\left(R(L+1)^{2}+(L+1)^{2}M^{2}\right)}.$$

If the conditions stated in corollary 3.3 are satisfied, then we know that the solution (46) is unique up to a nonsingular matrix $\mathbf{A}\in\mathbb{C}^{R\times R}$. Let $\mathbf{Q}$ denote the Least Squares (LS) estimate of the relaxed version of $\mathbf{P}^{*}\odot\mathbf{P}$ obtained from (46), then it remains to solve

$$\mathbf{Q}=\left(\mathbf{P}^{*}\odot\mathbf{P}\right)\mathbf{A}. \tag{47}$$

The problem of solving (47) will be addressed in subsection 4.3.

4.2 Noise Subspace Method

Instead of looking for both the parameter matrix $\mathbf{P}$ and the linking matrix $\mathbf{M}$ the noise subspace method presented in this subsection only looks for $\mathbf{P}$. In the context of blind system identification it can be understood an extension of the blind Single-Input-Multiple-Output (SIMO) FIR identification method proposed in [9] to the MIMO FIR case. A necessary condition for this method is that $M^2(2L+1)^2 > R(L+1)$.

Assume that $\mathbf{H}^* \odot \mathbf{H}$ has full column rank, then $\dim\left(\mathrm{Col}\,(\mathbf{H}^* \odot \mathbf{H})^{\perp}\right) = M^2(2L+1)^2 - R(L+1) = Q$. Let the column vectors of $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_Q] \in \mathbb{C}^{M^2(L+1)^2 \times Q}$ constitute a basis for $\mathrm{Col}\,(\mathbf{H}^* \odot \mathbf{H})^{\perp}$. Due to the fact that $\mathbf{v}_i \in \mathrm{Col}\,(\mathbf{H}^* \odot \mathbf{H})^{\perp}$ we have

$$\mathbf{v}_i^H\,(\mathbf{H}^* \odot \mathbf{H}) = \mathbf{0}, \quad \forall i \in [1, Q]. \tag{48}$$

From subsection 4.1 we know that

$$(\mathbf{H}^* \odot \mathbf{H})\,\Pi_{\mathbf{H}} \;=\; \sum_{k=1}^{L+1}\sum_{m=1}^{M}\sum_{n=1}^{L+1} \mathbf{E}^{(k,m,n)} \otimes \left(\mathbf{P}_{L+2-n} D_m\left(\mathbf{P}^*_{L+2-k}\right)\right),$$

where $\mathbf{E}^{(k,m,n)} \in \mathbb{C}^{M(L+1)^2 \times (2L+1)}$, $\Pi_{\mathbf{H}} \in \mathbb{C}^{R(L+1) \times R(L+1)}$ and $\mathbf{P}_l \in \mathbb{C}^{M \times R}$ are of the forms (41), (21) and (40), respectively. Thus, from (48) we get

$$\mathbf{v}_i^H\,(\mathbf{H}^* \odot \mathbf{H})\,\Pi_{\mathbf{H}} = \mathbf{0} \Leftrightarrow \sum_{k=1}^{L+1}\sum_{m=1}^{M}\sum_{n=1}^{L+1} \left[\mathbf{E}^{(k,m,n)T} \otimes \left(\mathbf{P}_{L+2-n} D_m\left(\mathbf{P}^*_{L+2-k}\right)\right)^H\right] \mathbf{v}_i = \mathbf{0}, \quad \forall i \in [1, Q]. \tag{49}$$

Notice that $\mathrm{Unvec}\left(\left[\mathbf{E}^{(k,m,n)T} \otimes \left(\mathbf{P}_{L+2-n} D_m\left(\mathbf{P}^*_{L+2-k}\right)\right)^H\right] \mathbf{v}_i\right) = \left(\mathbf{P}_{L+2-n} D_m\left(\mathbf{P}^*_{L+2-k}\right)\right)^H \mathrm{Unvec}\,(\mathbf{v}_i)\,\mathbf{E}^{(k,m,n)}$ and let $\mathbf{V}_i = \mathrm{Unvec}\,(\mathbf{v}_i) \in \mathbb{C}^{M \times M(2L+1)^2}$, then from (49) we get

$$\mathrm{Unvec}\left(\mathbf{v}_i^H\,(\mathbf{H}^* \odot \mathbf{H})\,\Pi_{\mathbf{H}}\right)^H = \mathbf{0} \Leftrightarrow \sum_{k=1}^{L+1}\sum_{m=1}^{M}\sum_{n=1}^{L+1} \mathbf{E}^{(k,m,n)T} \mathbf{V}_i^H \left(\mathbf{P}_{L+2-n} D_m\left(\mathbf{P}^*_{L+2-k}\right)\right) = \mathbf{0}, \quad \forall i \in [1, Q]. \tag{50}$$

Relation (50) can be written more compactly as

$$\left[\mathbf{E}^{(L+1,1,L+1)T}\mathbf{V}_i^H, \mathbf{E}^{(L+1,1,L)T}\mathbf{V}_i^H, \ldots, \mathbf{E}^{(1,M,1)T}\mathbf{V}_i^H\right](\mathbf{P}^* \odot \mathbf{P}) = \mathbf{0}\,, \quad \forall i \in [1, Q]. \tag{51}$$

Let

$$\mathbf{W}_i = \left[\mathbf{E}^{(L+1,1,L+1)T}\mathbf{V}_i^H, \mathbf{E}^{(L+1,1,L)T}\mathbf{V}_i^H, \ldots, \mathbf{E}^{(1,M,1)T}\mathbf{V}_i^H\right] \in \mathbb{C}^{M(L+1) \times M(L+1)^2}$$

and

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 \\ \vdots \\ \mathbf{W}_Q \end{bmatrix} \in \mathbb{C}^{QM(L+1)\times M(L+1)^2},$$

then relation (51) is equal to

$$\mathbf{W}\left(\mathbf{P}^* \odot \mathbf{P}\right) = \mathbf{0}. \tag{52}$$

Assume that the conditions stated in proposition 3.2 are satisfied, then we know that any non-trivial solution to (52) is essentially unique. However, the problem of finding a non-trivial solution to (52) is hard, but according to corollary 3.3 we can relax the Khatri-Rao product structure of $\mathbf{P}^* \odot \mathbf{P}$ and instead solve

$$\mathbf{W}\mathbf{U} = \mathbf{0}. \tag{53}$$

Due to corollary 3.3 we know that the solution to (53) is unique up to a nonsingular matrix $\mathbf{A} \in \mathbb{C}^{R\times R}$. Let $\mathbf{U}$ denote the LS solution to (53), then $\mathbf{P}$ can be found from

$$\mathbf{U} = \left(\mathbf{P}^* \odot \mathbf{P}\right)\mathbf{A}^T. \tag{54}$$

The problem of solving (54) will be addressed in the following subsection.

4.3 Partially Hermitian symmetric CPD problem

We noticed that when one of the subspace methods outlined in subsection 4.1 or 4.2 is applied, then the MIMO FIR identification problem was converted into the following partially Hermitian symmetric CPD problem

$$\mathbf{U} = \left(\mathbf{P}^* \odot \mathbf{P}\right)\mathbf{A}^T, \tag{55}$$

where $\mathbf{A} \in \mathbb{C}^{R\times R}$ is a nonsingular matrix and $\mathbf{P} \in \mathbb{C}^{M(L+1)\times R}$ is the unknown parameter matrix which we attempt to retrieve. Proposition 2.4 in [13] tells us that the matrix $\mathbf{P}^* \odot \mathbf{P}$ has generically full column rank when $M^2(L+1)^2 \geq R$. Combining this result with theorem 1.2 in subsection 1.2, then we know that as long as $M^2(L+1)^2(M(L+1)-1)^2 \geq 2R(R-1)$, then the solution to (55) is essentially unique in the generic case. Moreover, in [4] it was explained that if $M^2(L+1)^2(M(L+1)-1)^2 \geq 2R(R-1)$, then there generically exist

R symmetric matrices $\{\mathbf{X}^{(r)}\}$, R diagonal matrices $\{\mathbf{D}^{(r)}\}$ and a nonsingular matrix $\mathbf{F} \in \mathbb{C}^{R\times R}$ such that

$$\mathbf{U}\mathbf{F} = \mathbf{P}^* \odot \mathbf{P} \tag{56}$$

and

$$\begin{aligned} \mathbf{X}^{(1)} &= \mathbf{F}\mathbf{D}^{(1)}\mathbf{F}^T \\ &\vdots \\ \mathbf{X}^{(R)} &= \mathbf{F}\mathbf{D}^{(R)}\mathbf{F}^T. \end{aligned} \tag{57}$$

If $R = 2$, then (57) just corresponds to a generalized eigenvalue which can be solved by a generalized SVD. If $R > 2$, then no closed form solutions to (57) currently exist. In order to solve the SMD problem (57) when $R > 2$, we will apply a simultaneous generalized Schur decomposition method with a partial symmetry, as developed in [12]. However, any other numerical method could be applied. For instance, in [18] descent-type methods such as Gauss-Newton have been suggested.

Once $\mathbf{F}$ has been obtained from (57), then the parameter matrix $\mathbf{P}$ follows from (56) via set of R decoupled best rank-1 matrix approximation problems.

4.4 Simple ALS-type Method

Let the cumulant tensor $\mathcal{T} \in \mathbb{C}^{M\times M(2L+1)\times M(2L+1)\times M(2L+1)}$ be of the form (7) constructed from the channel vectors $\{\mathbf{h}^{(r)}\}$ with $L_r = L$, $\forall r \in [1, R]$. In order to numerically estimate the channel vectors $\{\mathbf{h}^{(r)}\}$ from $\mathcal{T}$ we will in this subsection present a very simple ALS-type algorithm. Let us ignore the symmetry of $\mathcal{T}$ and work with the model

$$\mathcal{T} = \sum_{r=1}^{R(L+1)} \mathbf{a}_r^{(1)} \circ \mathbf{a}_r^{(2)} \circ \mathbf{a}_r^{(3)} \circ \mathbf{a}_r^{(4)} ,$$

where ideally we should have

$$\mathbf{A}^{(1)} = \left[\gamma_1 \mathbf{P}^{(1)}, \ldots, \gamma_R \mathbf{P}^{(R)}\right] \quad , \quad \mathbf{A}^{(2)} = \mathbf{H} \quad , \quad \mathbf{A}^{(3)} = \mathbf{H}^* \quad , \quad \mathbf{A}^{(4)} = \mathbf{H} \quad ,$$

where $\mathbf{P}^{(r)} \in \mathbb{C}^{M\times(L+1)}$ is of the form (6) and $\mathbf{H} \in \mathbb{C}^{M(2L+1)\times R(L+1)}$ is of the form (8). Consider the following four matrix representations of the CPD of the

tensor $\mathcal{T} \in \mathbb{C}^{I_1 \times I_2 \times I_3 \times I_4}$ in (7) , where $I_1 = M$ and $I_2 = I_3 = I_4 = M(2L+1)$. Let $\mathbf{T}^{(\cdot\cdot i_3 i_4)} \in \mathbb{C}^{I_1 \times I_2}$ denote the matrix such that $\mathbf{T}^{(\cdot\cdot i_3 i_4)}_{i_1 i_2} = \mathcal{T}_{i_1 i_2 i_3 i_4}$, then

$$\mathbf{T}^{(\cdot\cdot i_3 i_4)} \quad = \quad \mathbf{A}^{(1)} D_{i_3}\left(\mathbf{A}^{(3)}\right) D_{i_4}\left(\mathbf{A}^{(4)}\right) \mathbf{A}^{(2)T}$$

and

$$\mathbb{C}^{I_1 I_3 I_4 \times I_2} \ni \mathbf{T}_{(2)} \triangleq \begin{bmatrix} \mathbf{T}^{(\cdot\cdot 11)} \\ \mathbf{T}^{(\cdot\cdot 12)} \\ \vdots \\ \mathbf{T}^{(\cdot\cdot I_3 I_4)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(1)} D_1\left(\mathbf{A}^{(3)}\right) D_1\left(\mathbf{A}^{(4)}\right) \\ \mathbf{A}^{(1)} D_1\left(\mathbf{A}^{(3)}\right) D_2\left(\mathbf{A}^{(4)}\right) \\ \vdots \\ \mathbf{A}^{(1)} D_{I_3}\left(\mathbf{A}^{(3)}\right) D_{I_4}\left(\mathbf{A}^{(4)}\right) \end{bmatrix} \mathbf{A}^{(2)T} = \left(\mathbf{A}^{(3)} \odot \mathbf{A}^{(4)} \odot \mathbf{A}^{(1)}\right) \mathbf{A}^{(2)T}$$

and

$$\mathbb{C}^{I_2 I_3 I_4 \times I_1} \ni \mathbf{T}_{(1)} \triangleq \begin{bmatrix} \mathbf{T}^{(\cdot\cdot 11)T} \\ \mathbf{T}^{(\cdot\cdot 12)T} \\ \vdots \\ \mathbf{T}^{(\cdot\cdot I_3 I_4)T} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(2)} D_1\left(\mathbf{A}^{(3)}\right) D_1\left(\mathbf{A}^{(4)}\right) \\ \mathbf{A}^{(2)} D_1\left(\mathbf{A}^{(3)}\right) D_2\left(\mathbf{A}^{(4)}\right) \\ \vdots \\ \mathbf{A}^{(2)} D_{I_3}\left(\mathbf{A}^{(3)}\right) D_{I_4}\left(\mathbf{A}^{(4)}\right) \end{bmatrix} \mathbf{A}^{(1)T} = \left(\mathbf{A}^{(3)} \odot \mathbf{A}^{(4)} \odot \mathbf{A}^{(2)}\right) \mathbf{A}^{(1)T}.$$

Similarly, let the matrices $\mathbf{T}^{(i_1 i_2 \cdot\cdot)} \in \mathbb{C}^{I_3 \times I_4}$ be constructed such that $\mathbf{T}^{(i_1 i_2 \cdot\cdot)}_{i_3 i_4} = \mathcal{T}_{i_1 i_2 i_3 i_4}$, then

$$\mathbf{T}^{(i_1 i_2 \cdot\cdot)} = \mathbf{A}^{(3)} D_{i_1}\left(\mathbf{A}^{(2)}\right) D_{i_2}\left(\mathbf{A}^{(2)}\right) \mathbf{A}^{(4)T}$$

and

$$\mathbb{C}^{I_1 I_2 I_3 \times I_4} \ni \mathbf{T}_{(4)} \triangleq \begin{bmatrix} \mathbf{T}^{(11\cdot\cdot)} \\ \mathbf{T}^{(12\cdot\cdot)} \\ \vdots \\ \mathbf{T}^{(I_1 I_2 \cdot\cdot)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(3)} D_1\left(\mathbf{A}^{(1)}\right) D_1\left(\mathbf{A}^{(2)}\right) \\ \mathbf{A}^{(3)} D_1\left(\mathbf{A}^{(1)}\right) D_2\left(\mathbf{A}^{(2)}\right) \\ \vdots \\ \mathbf{A}^{(3)} D_{I_1}\left(\mathbf{A}^{(1)}\right) D_{I_2}\left(\mathbf{A}^{(2)}\right) \end{bmatrix} \mathbf{A}^{(4)T} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)} \odot \mathbf{A}^{(3)}\right) \mathbf{A}^{(4)T}$$

and

$$\mathbb{C}^{I_1 I_2 I_4 \times I_3} \ni \mathbf{T}_{(3)} \triangleq \begin{bmatrix} \mathbf{T}^{(11\cdot\cdot)T} \\ \mathbf{T}^{(12\cdot\cdot)T} \\ \vdots \\ \mathbf{T}^{(I_1 I_2 \cdot\cdot)T} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{(4)} D_1\left(\mathbf{A}^{(1)}\right) D_1\left(\mathbf{A}^{(2)}\right) \\ \mathbf{A}^{(4)} D_1\left(\mathbf{A}^{(1)}\right) D_2\left(\mathbf{A}^{(2)}\right) \\ \vdots \\ \mathbf{A}^{(4)} D_{I_1}\left(\mathbf{A}^{(1)}\right) D_{I_2}\left(\mathbf{A}^{(2)}\right) \end{bmatrix} \mathbf{A}^{(3)T} = \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)} \odot \mathbf{A}^{(4)}\right) \mathbf{A}^{(3)T}.$$

The conditional LS update of $\mathbf{A}^{(1)}$ is equal to

$$\mathbf{A}^{(1)T} = \left(\mathbf{A}^{(3)} \odot \mathbf{A}^{(4)} \odot \mathbf{A}^{(2)}\right)^{\dagger} \mathbf{T}_{(1)} .$$

In order to find the conditional LS updates of $\mathbf{A}^{(n)}$, $n \in \{1,2,3\}$ we need reorganize the involved matrices a bit further. Recall that

$$\Pi_{\mathbf{H}}^{T} \mathbf{A}^{(n)T} = \sum_{l=0}^{L} \mathbf{E}_l^T \otimes \mathbf{P}_l^{(n)T} \in \mathbb{C}^{M(2L_1+1)\times R(L+1)}, \quad n \in \{1,2,3\},$$

where $\Pi_{\mathbf{H}} \in \mathbb{C}^{R(L+1)\times R(L+1)}$ is a permutation matrix of the form (21), $\mathbf{E}_l \in \mathbb{C}^{(2L+1)\times(L+1)}$ is a basis matrix of the form (10) and $\mathbf{P}_l^{(n)} \in \mathbf{C}^{M\times R}$. Furthermore, partition $\mathbf{T}_{(n)}$, $n \in \{2,3,4\}$ as follows

$$\mathbf{T}_{(n)} = \left[\mathbf{T}_{(n)}^{(1)}, \ldots, \mathbf{T}_{(n)}^{(2L+1)}\right], \quad \mathbf{T}_{(n)}^{(l)} \in \mathbb{C}^{M^3(2L+1)^2\times M}, \quad l \in [1, 2L+1],$$

and construct

$$\mathbf{T}_{(n)}^{[M]} = \begin{bmatrix} \mathbf{T}_{(n)}^{(1)} \\ \vdots \\ \mathbf{T}_{(n)}^{(2L+1)} \end{bmatrix} \in \mathbb{C}^{M^3(2L+1)^3\times M},$$

then we get

$$\begin{aligned}
\mathbf{T}_{(2)}^{[M]} &= \left(\mathbf{I}_{2L+1} \otimes \left(\mathbf{A}^{(3)} \odot \mathbf{A}^{(4)} \odot \mathbf{A}^{(1)}\right)\Pi_{\mathbf{H}}\right)\mathbf{E}\mathbf{P}^{(2)} \\
\mathbf{T}_{(3)}^{[M]} &= \left(\mathbf{I}_{2L+1} \otimes \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)} \odot \mathbf{A}^{(4)}\right)\Pi_{\mathbf{H}}\right)\mathbf{E}\mathbf{P}^{(3)} \\
\mathbf{T}_{(4)}^{[M]} &= \left(\mathbf{I}_{2L+1} \otimes \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)} \odot \mathbf{A}^{(3)}\right)\Pi_{\mathbf{H}}\right)\mathbf{E}\mathbf{P}^{(4)},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{E} &= \left[\mathrm{Vec}\left(\mathbf{E}_0^T\right) \otimes \mathbf{I}_R, \ldots, \mathrm{Vec}\left(\mathbf{E}_L^T\right) \otimes \mathbf{I}_R\right] \in \mathbb{C}^{(2L+1)R(L+1)\times R(L+1)} \\
\mathbf{P}^{(n)} &= \begin{bmatrix} \mathbf{P}_0^{(n)T} \\ \vdots \\ \mathbf{P}_L^{(n)T} \end{bmatrix} \in \mathbb{C}^{R(L+1)\times M}.
\end{aligned}$$

Hence, the conditional LS updates of $\mathbf{A}^{(n)}$, $n \in \{1,2,3\}$ are equal to

$$\begin{aligned}
\mathbf{P}^{(2)} &= \left(\left(\mathbf{I}_{2L+1} \otimes \left(\mathbf{A}^{(3)} \odot \mathbf{A}^{(4)} \odot \mathbf{A}^{(1)}\right)\Pi_{\mathbf{H}}\right)\mathbf{E}\right)^{\dagger} \mathbf{T}_{(2)}^{[M]} \\
\mathbf{P}^{(3)} &= \left(\left(\mathbf{I}_{2L+1} \otimes \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)} \odot \mathbf{A}^{(4)}\right)\Pi_{\mathbf{H}}\right)\mathbf{E}\right)^{\dagger} \mathbf{T}_{(3)}^{[M]} \\
\mathbf{P}^{(4)} &= \left(\left(\mathbf{I}_{2L+1} \otimes \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)} \odot \mathbf{A}^{(3)}\right)\Pi_{\mathbf{H}}\right)\mathbf{E}\right)^{\dagger} \mathbf{T}_{(4)}^{[M]}.
\end{aligned}$$

We summarize the simple ALS-type method for cumulant based identification of MIMO FIR channels in Algorithm 1[1].

As an alternative to iterative ALS-type algorithms it is also possible to develop descent-type methods such as those presented in [18], but this is beyond the scope of this paper.

Algorithm 1 Outline of ALS-type procedure for cumulant based identification of MIMO FIR channels.

Initialize: $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$, $\mathbf{A}^{(3)}$ and $\mathbf{A}^{(4)}$

Repeat until convergence

$$\mathbf{A}^{(1)} = \left(\left(\mathbf{A}^{(3)} \odot \mathbf{A}^{(4)} \odot \mathbf{A}^{(2)}\right)^{\dagger} \mathbf{T}_{(1)}\right)^{T}$$

$$\mathbf{P}^{(2)} = \left(\left(\mathbf{I}_{2L+1} \otimes \left(\mathbf{A}^{(3)} \odot \mathbf{A}^{(4)} \odot \mathbf{A}^{(1)}\right) \Pi_{\mathbf{H}}\right) \mathbf{E}\right)^{\dagger} \mathbf{T}_{(2)}^{[M]}$$

$$\mathbf{A}^{(2)} = \Pi_{\mathbf{H}} \sum_{l=0}^{L} \mathbf{E}_l \otimes \mathbf{P}_l^{(2)}$$

$$\mathbf{P}^{(3)} = \left(\left(\mathbf{I}_{2L+1} \otimes \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)} \odot \mathbf{A}^{(4)}\right) \Pi_{\mathbf{H}}\right) \mathbf{E}\right)^{\dagger} \mathbf{T}_{(3)}^{[M]}$$

$$\mathbf{A}^{(3)} = \Pi_{\mathbf{H}} \sum_{l=0}^{L} \mathbf{E}_l \otimes \mathbf{P}_l^{(3)}$$

$$\mathbf{P}^{(4)} = \left(\left(\mathbf{I}_{2L+1} \otimes \left(\mathbf{A}^{(1)} \odot \mathbf{A}^{(2)} \odot \mathbf{A}^{(3)}\right) \Pi_{\mathbf{H}}\right) \mathbf{E}\right)^{\dagger} \mathbf{T}_{(4)}^{[M]}$$

$$\mathbf{A}^{(4)} = \Pi_{\mathbf{H}} \sum_{l=0}^{L} \mathbf{E}_l \otimes \mathbf{P}_l^{(4)}$$

5 Simulation Results

6 Conclusion

Cumulant based system identification of MIMO FIR channels have a long history. However, only a few results exist on blind identification of underdetermined MIMO FIR channels. We first presented identifiability results for underdetermined MIMO FIR channels which exploited the CPD, block-banded and block-Toeplitz structure of the observed spatial cumulants. Second, we presented channel and noise subspace methods for the estimation of the MIMO FIR channel parameter matrix.

[1] Algorithm 1 can for instance be initialized by the proposed subspace methods.

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