Markov properties for systems described by PDE's and first order representations

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Abstract

The relation between Markovianity and representability by means of first order PDE's is investigated. We consider two versions, weak- and strong-Markovianity. The weak version has been introduced in [7] and conjectured to correspond to first order representations. We provide a counterexample to this conjecture. For finite-dimensional behaviors, strong-Markovianity is proven to be indeed equivalent to the representability by means of first order PDE's.

Key words: behaviors, PDE's, Markov property, first order representation

1 Introduction

Representing a dynamical system by means of first order differential or difference equations, not only guarantees easier recursive computations, but, in some cases, also allows to capture the system memory. Indeed, as shown in [2], the representability of a linear system with $\mathbb{R}$ or $\mathbb{Z}$ as time-axis by means of first order linear equations is equivalent to the one-dimensional Markov property. A dynamical system with $\mathbb{R}$ or $\mathbb{Z}$ as time-axis is said to be *Markovian* whenever the concatenation of two system trajectories w_1, w_2 that coincide at one point (i.e, $w_1(t) = w_2(t)$, for some t) yields a function w (coinciding with w_1 on $(-\infty, t]$ and with w_2 on $[t, +\infty)$) which is still an admissible system trajectory [2]. This is a deterministic version of the stochastic Markovianity: independence of past and future given the present.

The relation between first order representations and the memory property is quite

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different for multidimensional systems: the existing results [3,**?**] deal mainly with discrete two-dimensional (2D) (meaning that the set of independent variables is $\mathbb{Z}^2$) systems, and show that a direct generalization of the Markov property for 1D systems (which in the sequel will be referred to as the weak-Markov property) does not correspond to the representability by means of first order partial difference equations. However a stronger generalization has been introduced (the strong Markov property) which does correspond to the existence of first order representations with, in fact, special structure [5].

In this article, we consider systems described by linear constant coefficient PDE's, hence with a continuous set of independent variables equal to $\mathbb{R}^{\mathtt{n}}$. Recently, a conjecture has been presented in [7], according to which these systems are thought to behave differently from the discrete ones, and the weak-Markov property is thought to be equivalent to the representability by means of a system of first order linear PDE's. One of our purposes is to analyze this conjecture. After showing that it does not hold true, we prove that, for the particular case of *finite-dimensional* behaviors, it is a stronger version of the Markov property that indeed corresponds to representability by means of a system of first order PDE's. The question whether this result also holds for general, not necessarily finite-dimensional, behaviors of PDE's, remains open.

2 nD Markovian properties

We consider multidimensional (nD) behavioral systems that can be represented as the solution set of a system of linear PDE's with constant coefficients. Formally, let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[s_1, \cdots, s_{\mathtt{n}}]$ (the real polynomial matrices in $\mathtt{n}$ variables with $\mathtt{w}$ columns). Associate with R the following system of PDE's

$$\boxed{R(\tfrac{\partial}{\partial x_1}, \ldots, \tfrac{\partial}{\partial x_{\mathtt{n}}})w = 0.} \tag{1}$$

We define the behavior to be the set of solutions of this system of PDE's. There are many, more or less equivalent ways: $\mathcal{C}^\infty$, distributions, etc. For the purposes of this paper it is convenient to consider the continuous solutions. Hence

$$\mathfrak{B} = \{w \in \mathcal{C}^0(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid \text{(1) holds in the distributional sense}\}.$$

As $\mathfrak{B}$ is the kernel of a partial differential operator, we refer to it as a *kernel behavior*, and denote it as $\ker\Big(R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}})\Big)$. The PDE (1) is called a *kernel representation* of $\mathfrak{B} = \ker\Big(R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}})\Big)$.

As mentioned in the introduction, the question which we investigate is the connection between the fact that a behavior $\mathfrak{B}$ is Markovian (in a sense to be made precise soon) and the possibility of representing it as the kernel of a system of *first order*

PDE's

$$\boxed{R_0 w + R_1 \tfrac{\partial}{\partial x_1} w + \ldots + R_{\mathtt{n}} \tfrac{\partial}{\partial x_{\mathtt{n}}} w = 0} \tag{2}$$

We consider two versions of Markovianity. The first is the one used in [7]. We call it *weak- Markovianity*. Define Π to be the set of 3-way partitions (S_-, S_0, S_+) of $\mathbb{R}^{\mathtt{n}}$ such that S_- and S_+ are open and S_0 is closed; given a partition $\pi = (S_-, S_0, S_+) \in \Pi$ and a pair of trajectories (w_-, w_+) that coincide on S_0, define the *concatenation* of (w_-, w_+) along π as the trajectory $w_- \wedge |_\pi w_+$ that coincides with w_- on $S_0 \cup S_-$ and with w_+ on $S_0 \cup S_+$.

Definition 1 *A multidimensional behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}^{\mathtt{n}}}$ is said to be* weak-Markovian *if for any partition $\pi \in \Pi$ and any pair of trajectories $w_-, w_+ \in \mathfrak{B}$ such that $w_{-|_{S_0}} = w_{+|_{S_0}}$, the trajectory $w_- \wedge |_\pi w_+$ is also an element of $\mathfrak{B}$.*

The second version of Markovianity is called strong-Markovianity. It requires concatenability along partitions of linear subspaces (it is introduced with linear systems in the background). Given a subspace $S \subseteq \mathbb{R}^{\mathtt{n}}$, let Π_S be the set of 3-way partitions (S_-, S_0, S_+) of S such that S_- and S_+ are open (in S) and S_0 is closed (in S).

Definition 2 *A multidimensional behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}^{\mathtt{n}}}$ is said to be* strong-Markovian *if for any subspace S, any partition $\pi_S \in \Pi_S$, and any pair of trajectories $w_-, w_+ \in \mathfrak{B}|_S$ such that $w_{-|_{S_0}} = w_{+|_{S_0}}$, the trajectory $w_- \wedge |_\pi w_+$ is an element of $\mathfrak{B}|_S$.*

Obviously, strong-Markovianity implies weak-Markovianity. Note that strong-Markovianity coincides with weak-Markovianity for one-dimensional behaviors, and both can therefore be regarded as a generalization of the 1D Markov property.

Let $\mathfrak{B}$ be a behavior defined by a first order PDE (1). It is easy to see that this implies weak-Markovianity. The question arises whether a behavior as (1) that is weak-Markovian admits an equivalent first order representation (2) (equivalent in the sense that they have the same behavior). We provide a counterexample showing that, contrary as was put forward in [7], this converse does not hold true. The analogous questions arise for strong-Markovianity. Do first order PDE's generate behaviors that are strong-Markovian? Do strongly Markovian behaviors of PDE's (1) admit equivalent first order representations (2)? We will prove that for *finite dimensional* behaviors, strong-Markovianity and first order representability are indeed equivalent.

3 Weak Markovianity and first order representations

The next example shows that, similar to what happens in the discrete case, the direct generalization of the one-dimensional Markov property does not necessarily lead to the desired type of first order representations, implying that the conjecture in [7]

is false.

Consider the behavior $\mathfrak{B} \subseteq \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$\mathfrak{B} = span\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^x \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e^y \begin{bmatrix} 0 \\ 1 \end{bmatrix}, e^{x+y} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \} \tag{3}$$

Obviously,

$$\mathfrak{B} = \ker \Big(R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \Big),$$

with

$$R(s_1, s_2) = \begin{bmatrix} (s_1 - 1)(s_2 - 1) & -(s_1 - 1)(s_2 - 1) \\ 0 & s_1(s_2 - 1) \\ s_2(s_1 - 1) & 0 \\ s_1 s_2 & s_1 s_2 \end{bmatrix}$$

We will show that this behavior is weak-Markovian, but does not allow a first order representation of the form (2).

In order to check that $\mathfrak{B}$ is weak-Markovian, we show that if two trajectories w_1 and w_2 in $\mathfrak{B}$ coincide on two different points (x_1, y_1) and (x_2, y_2) of $\mathbb{R}^2$, then they are the same trajectory. This obviously implies that any two trajectories coinciding on a set S_0 of a partition $\pi = (S_-, S_0, S_+) \in \Pi$ are concatenable in $\mathfrak{B}$. Assume that

$$w_1(x, y) = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_1 e^x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_1 e^y \begin{bmatrix} 0 \\ 1 \end{bmatrix} + d_1 e^{x+y} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$w_2(x, y) = a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_2 e^x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^y \begin{bmatrix} 0 \\ 1 \end{bmatrix} + d_2 e^{x+y} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

are two trajectories in $\mathfrak{B}$ such that $w_1(x_1, y_1) = w_2(x_1, y_1)$ and $w_1(x_2, y_2) = w_2(x_2, y_2)$, with $(x_1, y_1) \neq (x_2, y_2)$. This means that:

$$\begin{aligned}
(a_1 - a_2) + (b_1 - b_2)e^{x_1} + (d_1 - d_2)e^{x_1+y_1} &= 0 \\
(a_1 - a_2) + (c_1 - c_2)e^{y_1} - (d_1 - d_2)e^{x_1+y_1} &= 0 \\
(a_1 - a_2) + (b_1 - b_2)e^{x_2} + (d_1 - d_2)e^{x_2+y_2} &= 0 \\
(a_1 - a_2) + (c_1 - c_2)e^{y_2} - (d_1 - d_2)e^{x_2+y_2} &= 0.
\end{aligned}$$

or, equivalently,

$$\underbrace{\begin{bmatrix} 1 & e^{x_1} & 0 & e^{x_1+y_1} \\ 1 & 0 & e^{y_1} & -e^{x_1+y_1} \\ 1 & e^{x_2} & 0 & e^{x_2+y_2} \\ 1 & 0 & e^{y_2} & -e^{x_2+y_2} \end{bmatrix}}_{=:A} \begin{bmatrix} a_1 - a_2 \\ b_1 - b_2 \\ c_1 - c_2 \\ d_1 - d_2 \end{bmatrix} = 0. \tag{4}$$

Since $\det(A) = e^{x_1+y_1+x_2+y_2}[e^{-(x_1-x_2)}(e^{x_1-x_2}-1)^2+e^{-(y_1-y_2)}(e^{y_1-y_2}-1)^2]$, which is clearly nonzero for $(x_1, y_1) \neq (x_2, y_2)$, we conclude that the only solution of (4) is the zero solution. In other words, we must have $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$, which means that $w_1 = w_2$ as claimed.

We next show that $\mathfrak{B}$ does not allow a first order representation. For that purpose we assume, to the contrary, that there exist real matrices R_0, R_1 and R_2, with two columns and the same number of rows, such that $\mathfrak{B} = \ker(R_0 + R_1\frac{\partial}{\partial x} + R_2\frac{\partial}{\partial y})$. Since the elements of the generating set in (3) are then obviously in $\ker(R_0 + R_1\frac{\partial}{\partial x} + R_2\frac{\partial}{\partial y})$, we have that

$$R_0\begin{bmatrix}1\\1\end{bmatrix} = 0,\ \ (R_0+R_1)\begin{bmatrix}1\\0\end{bmatrix} = 0,\ \ (R_0+R_2)\begin{bmatrix}0\\1\end{bmatrix} = 0,\ \ (R_0+R_1+R_2)\begin{bmatrix}1\\-1\end{bmatrix} = 0.$$

Therefore, there exist column vectors X, Y such that

$$R_0 + R_1 s_1 + R_2 s_2 = [X(1-s_1) + Y s_2 \qquad X(s_2-1) + Y s_1] = [X \qquad Y]Q(s_1, s_2),$$

with

$$Q(s_1, s_2) = \begin{bmatrix} 1-s_1 & s_2-1 \\ s_2 & s_1 \end{bmatrix}.$$

Consequently, $\ker\left(Q(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\right) \subseteq \ker(R_0 + R_1\frac{\partial}{\partial x} + R_2\frac{\partial}{\partial y})$. But this contradicts the fact that $\mathfrak{B}$ is finite-dimensional, since $\ker\left(Q(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\right)$ contains infinitely many linearly independent trajectories of the form $w(x, y) = e^{\alpha x+\beta y}w_0$, with (α, β) roots of $\det\left(Q(s_1, s_2)\right) = -s_1^2 - s_2^2 + s_1 + s_2$ and $0 \neq w_0 \in \mathbb{R}^2$ the associated solution of $Q(\alpha, \beta)w_0 = 0$. In this way we conclude that the given behavior cannot be represented by means of a set of first order PDE's.

This example suggests that in order to guarantee first order representability one should consider a stronger version of the Markov property. We will now examine if strong-Markovianity achieves this.

4 PDE's with a finite dimensional behavior

In this section, we examine finite-dimensional behaviors. Of course, if the solution set of (1) is finite dimensional, all its solutions are in $\mathcal{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$, and it allows very

special representations, as stated in the following result. In here we use the notion of a latent variable representation, a standard notion from the behavioral theory.

Proposition 1 *Let $\mathfrak{B} \subseteq \mathcal{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ be a finite-dimensional* n*D behavior that is the kernel of a PDE. Then it can be represented by a latent variable model of the form*

$$\begin{cases} \frac{\partial}{\partial x_1} z = A_1 z \\ \qquad \vdots \\ \frac{\partial}{\partial x_{\mathtt{n}}} z = A_{\mathtt{n}} z \\ w \quad = Cz \end{cases} \tag{5}$$

where $A_1, \ldots, A_{\mathtt{n}}$ are square pairwise commuting matrices of size $\mathtt{N} = \dim(z)$, $z \in \mathcal{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{N}})$ is the latent variable, and $w \in \mathcal{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ is the system variable. Note that $z(x_1, \ldots, x_{\mathtt{n}}) = Ce^{A_1 x_1 + \cdots + A_{\mathtt{n}} x_{\mathtt{n}}} z(0, \ldots, 0)$. Moreover, $(C; A_1, \ldots, A_{\mathtt{n}})$ can be taken to be observable, in the sense that if $Ce^{A_1 x_1 + \ldots + A_{\mathtt{n}} x_{\mathtt{n}}} z(0, \ldots, 0) = 0$ for all $x_i \in \mathbb{R}$, $i = 1, \ldots, \mathtt{n}$, then $z(0, \ldots, 0) = 0$.

Proof.
We use the results of [8]. Assume that $\mathfrak{B} \subseteq \mathcal{U}$ is a finite-dimensional nD kernel behavior. Then it admits a kernel representation with $R(s_1, \ldots, s_{\mathtt{n}})$ weakly zero prime, and hence there exist nD polynomial matrices $U_i(s_1, \ldots, s_{\mathtt{n}})$ such that

$$U_i(s_1, \ldots, s_{\mathtt{n}}) R(s_1, \ldots, s_{\mathtt{n}}) = D_i(s_i),$$

where $D_i(s_i) = d_i(s_i) I_{\mathtt{w} \times \mathtt{w}}$ for $i = 1, \ldots, n$. This implies that $\mathfrak{B} \subseteq \tilde{\mathfrak{B}}$, with $\tilde{\mathfrak{B}}$ described by

$$d_1(\frac{\partial}{\partial x_1}) w = 0, \ldots, d_{\mathtt{n}}(\frac{\partial}{\partial x_{\mathtt{n}}}) w = 0.$$

Define a vector function $\tilde{z}$ whose components are the partial derivatives $\frac{\partial^{\ell_1 + \ldots + \ell_{\mathtt{n}}}}{\partial x_1^{\ell_1} \ldots \partial x_{\mathtt{n}}^{\ell_{\mathtt{n}}}} w$ for $\ell_i = 0, \ldots, \, deg(d_i) - 1$. It is not difficult to check that this yields a latent variable representation for $\tilde{\mathfrak{B}}$ of the form

$$\begin{cases} \frac{\partial}{\partial x_1} \tilde{z} = F_1 \tilde{z} \\ \qquad \vdots \\ \frac{\partial}{\partial x_{\mathtt{n}}} \tilde{z} = F_{\mathtt{n}} \tilde{z} \\ w \quad = H \tilde{z}, \end{cases} \tag{6}$$

with real commuting matrices $F_1, \ldots, F_{\mathtt{n}}$. Therefore, $w \in \mathfrak{B}$ if and only if it satisfies (6) together with the equation $R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}}) w = 0$. Let $R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}}) = \sum_{j_1, \ldots, j_{\mathtt{n}} = 0}^{J_1, \ldots, J_{\mathtt{n}}} \frac{\partial^{j_1 + \ldots + j_{\mathtt{n}}}}{\partial x_1^{j_1} \ldots \partial x_{\mathtt{n}}^{j_{\mathtt{n}}}} R_{(j_1, \ldots, j_{\mathtt{n}})}$. Taking (6) into account, the equation $R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}}) w =$

0 becomes

$$\underbrace{(\sum_{j_1,\dots,j_{\mathtt{n}}=0}^{J_1,\dots,J_{\mathtt{n}}} R_{(j_1,\dots,j_{\mathtt{n}})} H F_1^{j_1}\dots F_{\mathtt{n}}^{j_{\mathtt{n}}})}_{=:K}\,\tilde{z}=0.$$

In this way the following latent variable representation for $\mathfrak{B}$ is obtained:

$$\begin{cases} \frac{\partial}{\partial x_1}\tilde{z} = F_1\tilde{z} \\ \qquad\vdots \\ \frac{\partial}{\partial x_{\mathtt{n}}}\tilde{z} = F_{\mathtt{n}}\tilde{z} \\ K\tilde{z} = 0 \\ w \quad = H\tilde{z}. \end{cases}$$

It follows from Proposition 2 in [6] that there exists a nonsingular real matrix T such that

$$T^{-1}F_iT = \begin{bmatrix} F_i^{11} & 0 \\ F_i^{21} & F_i^{22} \end{bmatrix},\ \ i=1,\dots,n,\ \ KT=[K_1\ \ 0],$$

with $(K_1; F_1^{11},\dots,F_{\mathtt{n}}^{11})$ observable. Thus, partitioning $T^{-1}\tilde{z}=col(\tilde{z}_1,\tilde{z}_2)$ accordingly, the equations for $\tilde{z}$ become:

$$\begin{cases} \frac{\partial}{\partial x_i}\tilde{z}_1 = F_i^{11}\tilde{z}_1 \\ \frac{\partial}{\partial x_i}\tilde{z}_2 = F_i^{21}\tilde{z}_1 + F_i^{22}\tilde{z}_2 \quad i=1,\dots,\mathtt{n} \\ K_1\tilde{z}_1 = 0 \end{cases}$$

which, by observability, is equivalent to

$$\begin{cases} \tilde{z}_1 \quad\ = 0 \\ \frac{\partial}{\partial x_i}\tilde{z}_2 = F_i^{22}\tilde{z}_2 \quad i=1,\dots,\mathtt{n}. \end{cases}$$

On the other hand, the equation $w=H\tilde{z}$ can be written as $w=H_2\tilde{z}_2$, where H_2 is such that $HT=[H_1 \quad H_2]$. Renaming $z=\tilde{z}_2$, $A_i=F_i^{22}$ and $C=H_2$, we obtain the following exact description for the dynamics of w:

$$\begin{cases} \frac{\partial}{\partial x_1}z = A_1z \\ \vdots \\ \frac{\partial}{\partial x_{\mathtt{n}}}z = A_{\mathtt{n}}z \\ w \quad = Cz, \end{cases} \tag{7}$$

where $A_1, \ldots, A_{\mathtt{n}}$ are still pairwise commuting matrices.

The fact that $(C; A_1, \ldots, A_{\mathtt{n}})$ in (7) can be taken to be observable follows again from Proposition 2 in [6]. This yields Proposition 1. ■

5 Strong-Markovianity and first order representations

It turns out that if, in addition to being finite-dimensional, $\mathfrak{B}$ has the strong-Markov property, then the matrix C in (5) can be shown to be injective.

Lemma 1 *Let $\mathfrak{B} \subseteq \mathcal{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ be a finite-dimensional $\mathtt{n}D$ behavior that is the kernel of a PDE. If $\mathfrak{B}$ is strong-Markovian then it can be represented by a latent variable model of the form (5) where the matrix C has full column rank.*

Proof.
By Proposition 1, $\mathfrak{B}$ has a latent variable representation of the form (5), with $(C; A_1, \ldots, A_{\mathtt{n}})$ observable. Note that in this case $\mathfrak{B} = \{w : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}^{\mathtt{w}} \mid w(x_1, \ldots, x_{\mathtt{n}}) = Ce^{A_1x_1+\ldots+A_{\mathtt{n}}x_{\mathtt{n}}}\bar{z},\ \bar{z} \in \mathbb{R}^{\mathtt{N}}\}$.

We start by showing that if $\mathfrak{B}$ is strong-Markovian then, for $\mathtt{k} = 1, \ldots, \mathtt{n}-1$, the behaviors $\mathfrak{B}_{\mathtt{k}} := \{w : \mathbb{R}^{\mathtt{n}-\mathtt{k}+1} \to \mathbb{R}^{\mathtt{w}} \mid w(x_{\mathtt{k}}, \ldots, x_{\mathtt{n}}) = Ce^{A_{\mathtt{k}}x_{\mathtt{k}}+\ldots+A_{\mathtt{n}}x_{\mathtt{n}}}\bar{z},\ \bar{z} \in \mathbb{R}^{\mathtt{N}}\}$ are also strong-Markovian with $(C; A_{\mathtt{k}}, \ldots, A_{\mathtt{n}})$ observable. Strong-Markovianity of $\mathfrak{B}_{\mathtt{k}}$ follows immediately from the definition. We now prove observability, by considering the case $\mathtt{k} = 2$, and proceeding by induction. Suppose that $z^*, z^{**} \in \mathbb{R}^{\mathtt{N}}$ are such that

$$Ce^{A_2x_2+\ldots+A_{\mathtt{n}}x_{\mathtt{n}}}z^* = Ce^{A_2x_2+\ldots+A_{\mathtt{n}}x_{\mathtt{n}}}z^{**}, \text{ for all } x_{\mathtt{i}} \in \mathbb{R},\ \mathtt{i} = 2, \ldots, \mathtt{n}.$$

Then the trajectories $w_*(x_1, x_2, \ldots, x_{\mathtt{n}}) = Ce^{A_1x_1+A_2x_2+\ldots+A_{\mathtt{n}}x_{\mathtt{n}}}z^*$ and $w_{**}(x_1, x_2, \ldots, x_{\mathtt{n}}) = Ce^{A_1x_1+A_2x_2+\ldots+A_{\mathtt{n}}x_{\mathtt{n}}}z^{**}$ of $\mathfrak{B}$ coincide on $S_0 = \{(x_1, \ldots, x_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{n}} \mid x_1 = 0\}$. If $\mathfrak{B}$ is strong-Markovian, this implies that $\hat{w} = w_* \wedge_{(S_-, S_0, S_+)} w_{**}$ (where $S_- = \{(x_1, \ldots, x_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{n}} \mid x_1 < 0\}$ and $S_+ = \{(x_1, \ldots, x_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{n}} \mid x_1 > 0\}$) is a trajectory of $\mathfrak{B}$, i.e., there exists $\hat{z} \in \mathbb{R}^{\mathtt{N}}$ such that $\hat{w}(x_1, \ldots, x_{\mathtt{n}}) = Ce^{A_1x_1+A_2x_2+\ldots+A_{\mathtt{n}}x_{\mathtt{n}}}\hat{z}$. Since $\hat{w}$ coincides with w_* in S_- and with w_{**} in S_+, the observability of $(C; A_1, \ldots, A_{\mathtt{n}})$ implies that

$$z^* = \hat{z} = z^{**},$$

and hence that $(C; A_2, \ldots, A_{\mathtt{n}})$ is indeed observable.

We conclude in particular that the behavior of

$$\frac{\partial}{\partial x_{\mathtt{n}}} z^{\mathtt{n}} = A_{\mathtt{n}} z^{\mathtt{n}} \qquad w^0(x_{\mathtt{n}}) = Cz^{\mathtt{n}}(x_{\mathtt{n}}),$$

is strong-Markovian and observable. However by the results of the 1D case [2] this implies that C has full column rank. ■

The previous lemma allows to state the main result of this paper.

Theorem 1 *Let $\mathfrak{B} \subseteq \mathcal{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ be a finite-dimensional* n*D behavior that is the kernel of a PDE. Then it is strong-Markovian if and only if it can be represented by means of partial differential equations of the form*

$$\begin{bmatrix} (\frac{\partial}{\partial x_1} I_{\mathtt{N}} - A_1)E \\ (\frac{\partial}{\partial x_2} I_{\mathtt{N}} - A_2)E \\ \vdots \\ (\frac{\partial}{\partial x_{\mathtt{n}}} I_{\mathtt{N}} - A_{\mathtt{n}})E \\ F \end{bmatrix} w = 0, \tag{8}$$

where $A_1, A_2, \ldots, A_{\mathtt{n}}$ are square pairwise commuting matrices and the matrix $V = \begin{bmatrix} E \\ F \end{bmatrix}$ is invertible.

Proof.
Assume now that $\mathfrak{B}$ can be represented by a model of the type (5) with C having full column rank. Let E be a left-inverse of C and F a suitable matrix such that $V = \begin{bmatrix} E \\ F \end{bmatrix}$ is invertible. Notice that equations (5) yield (8).

Conversely, let $\mathfrak{B}$ have a representation as (8). In a suitable basis in $\mathbb{R}^{\mathtt{w}}$, these equations look like

$$\begin{cases} \begin{bmatrix} (\frac{\partial}{\partial x_1} I_{\mathtt{N}} - A_1) \\ (\frac{\partial}{\partial x_2} I_{\mathtt{N}} - A_2) \\ \vdots \\ (\frac{\partial}{\partial x_{\mathtt{n}}} I_{\mathtt{N}} - A_{\mathtt{n}}) \end{bmatrix} w_1 = 0 \\ w_2 = 0, \\ w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \end{cases} \tag{9}$$

The w_1-behavior $\mathfrak{B}_1$ consists of all the expressions of the form

$$w_1(x_1, \ldots, x_{\mathtt{n}}) = e^{A_1 x_1 \cdots A_{\mathtt{n}} x_{\mathtt{n}}} z, z \in \mathbb{R}^{\mathtt{w}_1}.$$

If suffices to prove that $\mathfrak{B}_1$ is strong-Markovian. But this is easy: any two trajectories which coincide on a subspace, have the same value at $x_1 = \cdots = x_{\mathtt{n}} = 0$, and hence coincide, since $z = w_1(0, \ldots, 0)$. ■

Remark As many of the results that characterize system properties, due to the non-uniqueness of behavior representations, this theorem does not allow an immediate test to check whether a given behavior is or is not strong-Markovian. Such a test can however be obtained based on the following considerations. Note that the proof of Lemma 1 shows that if $\mathfrak{B}$ is strong-Markovian, then every observable $(C; A_1, \ldots, A_{\mathtt{n}})$ representation of $\mathfrak{B}$ is such that C has full column rank. Moreover, it is easy to see that the converse also holds true. This allows to check whether a given finite-dimensional behavior $\mathfrak{B}$ is or not strong-Markovian by constructing an observable $(C; A_1, \ldots, A_{\mathtt{n}})$ representation (which can be done as in the proof of Proposition 1) and checking whether C has full column rank. In the affirmative case, a first order representation for $\mathfrak{B}$ can be obtained as explained in the proof of Theorem 1.

6 Conclusion

In this paper the conjecture of [7] on the correspondence between the nD weak-Markov property and first order representability for PDE was proven to be false. In order to obtain equivalence with first order representability, a strong-Markov property has been introduced, which can still be viewed as a generalization of 1D Markovianity to higher dimensions. For finite-dimensional behaviors this property was shown to be indeed equivalent to the representability by means of first order PDE's, suggesting that this constitutes a suitable extension of Markovianity to the nD case.

Acknowledgments

The research o the first author is partially supported by the *Unidade de Investigação Matemática e Aplicações* (UIMA), University of Aveiro, Portugal, through the *Programa Operacional "Ciência e Tecnologia e Inovação"* (POCTI) of the *Fundação para a Ciência e Tecnologia* (FCT), co-financed by the European Union fund FEDER.

The research of the second author is supported by the Belgian Federal Government under the DWTC program Interuniversity Attraction Poles, Phase V, 2002–2006, Dynamical Systems and Control: Computation, Identification and Modelling, by the KUL Concerted Research Action (GOA) MEFISTO–666, and by several grants en projects from IWT-Flanders and the Flemish Fund for Scientific Research.

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