A BEHAVIORAL APPROACH TO ESTIMATION AND OBSERVER DESIGN WITH APPLICATIONS TO STATE-SPACE MODELS

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Abstract. The observer design problem is here investigated in the context of linear left shift invariant discrete behaviors, whose trajectories have support on the nonnegative axis $\mathbb{Z}_+$. Necessary and sufficient conditions for the existence of a (consistent or not) dead-beat observer of some relevant variables from some measured ones, in the presence of some unmeasured (and irrelevant) variables, are introduced, and a complete parametrization of all dead-beat observers is given. A characterization of those behaviors which admit non-consistent dead-beat observers is also provided. Equivalent conditions for the existence of causal dead-beat observers are then derived. Finally, several classical problems addressed for state-space models, like state estimation, the design of unknown input observers or the design of fault detectors and identifiers (possibly in the presence of disturbances), are casted in this general framework, and the aforementioned equivalent conditions and parametrizations are tailored to all these special instances.

Key words. Behavior, nilpotent autonomous system, observability, reconstructibility, observer, unknown input observer (UIO), fault detector and isolator (FDI).

1. Introduction

The original theory of state observers was concerned with the problem of reconstructing (or estimating) the state from the corresponding inputs and outputs. This problem has been later generalized in various ways, and in relatively recent years there has been a great deal of research aiming at designing state observers in the presence of unknown inputs (disturbances) [9, 16, 20].

Another research issue, which originated in the eighties and flourished in the nineties [3, 10], but still represents a very lively research topic [4, 5] is the fault detection and isolation (FDI) problem. The problem of detecting and identifying the faults affecting the system functioning, possibly in the presence of disturbances, is naturally stated and addressed as an estimation problem.

The last decade has witnessed a renewed interest for these two issues. In a few recent papers, estimation problems and observer synthesis, in a deterministic context, have been investigated for wider classes of dynamic systems, described either in a behavioral setting or by means of polynomial/rational models, thus enlightening interesting connections between the problem solutions obtained via different approaches [6, 8, 18, 19].

The goal of this paper is twofold: as a first step, we aim to fully explore the behavioral approach to a generic (deterministic) estimation problem for linear time invariant discrete

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systems. As a consequence, we will be then able to provide an extremely powerful setting were all classical estimation problems for linear time invariant (discrete-time) state space models can be casted, thus providing a uniform technique for testing the problems solvability and, when such problems are solvable, a homogeneous parametrization of all (transfer matrices which provide) the desired solutions. More in detail, in the first part of the paper we extend the analysis started in [18, 19] and further explore the observer design problem for linear time-invariant (discrete-time) dynamic systems that are described in behavioral terms by means of a set of difference equations. Specifically, we will consider a dynamic system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^{\mathbb{w}}, \mathfrak{B})$, whose system variable $\mathbf{w}$ is naturally split into three subvectors, i.e., $\mathbf{w}^T = [\, \mathbf{w}_r^T \quad \mathbf{w}_m^T \quad \mathbf{w}_i^T \,]^T$, where $\mathbf{w}_m$ represents the set of measured variables, $\mathbf{w}_r$ the (unmeasured) variables which are "relevant" for our estimation problem (actually, the target of our estimation problem), and $\mathbf{w}_i$ the set of variables which are unmeasured but "irrelevant", in the sense that we are not interested in evaluating them. Moreover, the system trajectories satisfy a difference equation of the following type

$$[\, R_r(\sigma) \quad -R_m(\sigma) \quad -R_i(\sigma) \,] \begin{bmatrix} \mathbf{w}_r(t) \\ \mathbf{w}_m(t) \\ \mathbf{w}_i(t) \end{bmatrix} = 0, \quad t \in \mathbb{Z}_+,$$

for suitable polynomial matrices R_r, R_m, and R_i, in the left shift operator σ. In this context, the natural goal is that of designing an observer of $\mathbf{w}_r$ based on the knowledge of $\mathbf{w}_m$, such that (s.t.) its estimation error goes to zero in a finite number of steps, independently of $\mathbf{w}_i$. Moreover, the concept of (consistent) dead-beat observer (DBO) is introduced, several new equivalent conditions for the existence of a consistent/non-consistent DBO are given, and a characterization of those systems which admit also non-consistent DBOs is provided. A complete parametrization of DBOs is also given.
In the second part of the paper, these general results will be applied to state-space models for formalizing, and hence solving, a wide variety of classical estimation problems (state estimation, state estimation in the presence of disturbances, fault detection and isolation, ...), which therefore will turn out to be rather trivial instances of the general problem addressed in the behavioral setting.

A preliminary version of these results can be found in [2].

Before entering the core of the paper, it is convenient to introduce some notation. We consider here polynomial matrices with entries in $\mathbb{R}[z]$ (and, occasionally, polynomial matrices with entries in $\mathbb{R}[z^{-1}]$ or Laurent (L-polynomial, for short) polynomial matrices, having entries in $\mathbb{R}[z, z^{-1}]$). A polynomial matrix $M(z) \in \mathbb{R}[z]^{p \times q}$ is *right monomic* [6, 7] if it is of full column rank and the GCD of its maximal order minors is a monomial. $M(z)$ is right monomic if and only if it admits a Laurent polynomial left inverse or, equivalently, the diophantine equation

$$X(z)M(z) = z^N I_q,$$

in the unknown polynomial matrix $X(z)$, is solvable for some nonnegative integer N. Finally, $M(z)$ is right monomic if and only if its invariant factors (in the associated Smith form) [11] are all monomials. *Right prime* matrices are special cases of right monomic matrices. Indeed, right primeness definition and characterizations can be obtained by simply replacing in the previous definition and equivalent conditions the word "monomial" with "unit" and the integer N by zero.

Left monomic and *left prime* matrices are similarly defined and characterized.

The concepts of *left annihilator* and, in particular, of *minimal left annihilator* (MLA, for short) of a given polynomial matrix $M(z)$ have been introduced in [13] (see also [1]) and can be summarized as follows: if $M(z)$ is a $p \times q$ polynomial matrix of rank r, a polynomial matrix $H(z)$ is a left annihilator of $M(z)$ if $H(z)M(z) = 0$. A left annihilator $H_m(z)$ of $M(z)$ is an MLA if it is of full row rank and for any other left annihilator $H(z)$ of $M(z)$ we have $H(z) = P(z)H_m(z)$ for some polynomial matrix $P(z)$. It can be easily proved that an MLA always exists (if $M(z)$ is of full row rank, it coincides with the empty matrix), it is a $(p-r) \times p$ left prime matrix and is uniquely determined modulo a unimodular left factor.

Right annihilators and *minimal right annihilators* (MRAs) can be similarly defined and enjoy analogous properties.

In the following, for the sake of simplicity, the size of any vector will be denoted by means of the same typewritten letter that is used for denoting the vector itself. In other words, $\mathtt{w}_m := \dim(\mathbf{w}_m), \mathtt{w}_r := \dim(\mathbf{w}_r), \mathtt{u} := \dim(\mathbf{u}), \mathtt{x} := \dim(\mathbf{x})$, etc.

In the paper, all trajectories will be assumed defined on the set $\mathbb{Z}_+$ of nonnegative integers. The left (backward) shift operator on $(\mathbb{R}^{\mathtt{v}})^{\mathbb{Z}_+}$, the set of trajectories defined on $\mathbb{Z}_+$ and taking values in $\mathbb{R}^{\mathtt{v}}$, is defined as

$$\sigma : (\mathbb{R}^{\mathtt{v}})^{\mathbb{Z}_+} \to (\mathbb{R}^{\mathtt{v}})^{\mathbb{Z}_+} : (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \cdots) \mapsto (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots).$$

If $M(z) = \sum_{i=0}^{L} M_i z^i \in \mathbb{R}[z]^{p \times q}$ is a polynomial matrix, we associate with it the polynomial matrix operator $M(\sigma) = \sum_{i=0}^{L} M_i \sigma^i$. It can be proved that $M(\sigma)$ describes an injective map from $(\mathbb{R}^q)^{\mathbb{Z}_+}$ to $(\mathbb{R}^p)^{\mathbb{Z}_+}$ if and only if $M(z)$ is a right prime matrix, and a surjective map if and only if $M(z)$ is of full row rank.

2. Basic results about infinite support behaviors in $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$

In this section we briefly summarize some basic definitions and results about linear left shift invariant behaviors, whose trajectories have support in $\mathbb{Z}_+$. Further details on the subject can be found in [15, 17, 18].

A behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ is said to be *linear* if it is a vector subspace (over $\mathbb{R}$) of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$, and *left shift invariant* if $\sigma\mathfrak{B} \subseteq \mathfrak{B}$. A linear left shift invariant behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ is *complete* if for every sequence $\tilde{\mathbf{w}} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$, the condition $\tilde{\mathbf{w}}|_{\mathcal{S}} \in \mathfrak{B}|_{\mathcal{S}}$ for every finite set $\mathcal{S} \subset \mathbb{Z}_+$ implies $\tilde{\mathbf{w}} \in \mathfrak{B}$, where $\tilde{\mathbf{w}}|_{\mathcal{S}}$ denotes the restriction to $\mathcal{S}$ of the trajectory $\tilde{\mathbf{w}}$ and $\mathfrak{B}|_{\mathcal{S}}$ the set of all restrictions to $\mathcal{S}$ of behavior trajectories.

Linear left shift invariant complete behaviors are kernels of polynomial matrices in the left shift operator σ, which amounts to saying that the trajectories $\mathbf{w} = \{\mathbf{w}(t)\}_{t \in \mathbb{Z}_+}$ of $\mathfrak{B}$ can be identified with the set of solutions in $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ of a system of difference equations

$$R_0\mathbf{w}(t) + R_1\mathbf{w}(t+1) + \cdots + R_L\mathbf{w}(t+L) = 0, \qquad t \in \mathbb{Z}_+, \tag{2.1}$$

with $R_i \in \mathbb{R}^{p \times \mathtt{w}}$, and hence described by the equation

$$R(\sigma)\mathbf{w} = 0, \tag{2.2}$$

where $R(z) := \sum_{i=0}^{L} R_i z^i$ belongs to $\mathbb{R}[z]^{p\times \mathtt{w}}$. In the sequel, a linear left shift invariant complete behavior will be simply referred to as a *behavior*. Also, for a behavior $\mathfrak{B}$ described as in (2.2), we will adopt the short-hand notation $\mathfrak{B} = \ker R(\sigma)$.

It can be shown that $\ker R_1(\sigma) \subseteq \ker R_2(\sigma)$ if and only if $R_2(z) = P(z)R_1(z)$ for some polynomial matrix $P(z)$.

A behavior $\mathfrak{B} = \ker R(\sigma) \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ is said to be *autonomous* if there exists $\nu \in \mathbb{Z}_+$ such that if $\mathbf{w}_1, \mathbf{w}_2 \in \mathfrak{B}$ and $\mathbf{w}_1(t) = \mathbf{w}_2(t)$ for $t \in [0, \nu]$, then $\mathbf{w}_1 = \mathbf{w}_2$. A behavior $\mathfrak{B} = \ker R(\sigma)$, with $R(z) \in \mathbb{R}[z]^{p\times \mathtt{w}}$, is autonomous if and only if $R(z)$ is of full column rank [18].

Every autonomous behavior in $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ can be expressed as $\ker R(\sigma)$ for some nonsingular square polynomial matrix $R(z)$. Autonomous behaviors for which there exists some $N \in \mathbb{N}$ such that all their trajectories have (compact) supports included in $[0, N-1]$ are called *nilpotent autonomous* and they are kernels of polynomial matrix operators $R(\sigma)$ corresponding to right monomic matrices [18]. In particular, if $R(z)$ is nonsingular square, then $\ker R(\sigma)$ is nilpotent if and only if $\det R(z) = c \cdot z^N$, for some $c \in \mathbb{R}\setminus\{0\}$ and some $N \in \mathbb{Z}_+$. Of course, $\ker R(\sigma)$ is the zero behavior if and only if $\det R(z) = c \neq 0$, namely $R(z)$ is unimodular.

It is worthwhile to remark a significant difference between behaviors defined on $\mathbb{Z}_+$ and the more traditional ones, defined on $\mathbb{Z}$. When dealing with autonomous behaviors defined on $\mathbb{Z}$, nilpotent behaviors cannot arise. In fact, the only finite support trajectory of an autonomous behavior defined on $\mathbb{Z}$ is the zero one, and the kernel (on $\mathbb{Z}$) of a monomic matrix coincides with the zero behavior.

3. Observability and reconstructibility

Consider a dynamic system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, whose behavior $\mathfrak{B}$ is described as in (2.2), for some polynomial matrix $R(z)$. Independently of the physical meaning of the system variables which are grouped together in the vector $\mathbf{w}$, when dealing with any type of estimation problem a first natural distinction is introduced between measured variables, denoted by $\mathbf{w}_m$, and unmeasured variables. These latter, in turn, may be naturally split into the subvector of all system variables which are (unmeasured and) the target of our estimation problem (the "relevant" variables for the specific estimation problem), $\mathbf{w}_r$, and the subvector of all variables which are both unmeasured (for instance because they represent disturbances or modeling errors) and "irrelevant" for our estimation problem. We refer to such a subvector as $\mathbf{w}_i$. As a consequence, the vector $\mathbf{w}$ is naturally split into three subvectors as follows:

$$\mathbf{w}(t) = \begin{bmatrix} \mathbf{w}_r(t) \\ \mathbf{w}_m(t) \\ \mathbf{w}_i(t) \end{bmatrix}.$$

The polynomial matrix $R(z)$ can be accordingly block-partitioned, thus leading to the following description of the behavior trajectories:

$$[\, R_r(\sigma) \quad -R_m(\sigma) \quad -R_i(\sigma)\,] \begin{bmatrix} \mathbf{w}_r(t) \\ \mathbf{w}_m(t) \\ \mathbf{w}_i(t) \end{bmatrix} = 0, \qquad t \in \mathbb{Z}_+, \tag{3.1}$$

or, equivalently

$$R_r(\sigma)\mathbf{w}_r(t) = R_m(\sigma)\mathbf{w}_m(t) + R_i(\sigma)\mathbf{w}_i(t), \qquad t \in \mathbb{Z}_+. \tag{3.2}$$

With respect to this partition of the system variables, the notions of observability and reconstructibility are easily introduced as follows.

DEFINITION 3.1. *[18, 19] Given a dynamic system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ whose behavior is described as in (3.2), we say that $\mathbf{w}_r$ is* reconstructible *from $\mathbf{w}_m$, if $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i), (\bar{\mathbf{w}}_r, \mathbf{w}_m, \bar{\mathbf{w}}_i) \in \mathfrak{B}$ implies that there exists $N \in \mathbb{Z}_+$ such that $\mathbf{w}_r(t) - \bar{\mathbf{w}}_r(t) = 0$, $\forall\ t \geq N$. In particular, when $N = 0$, $\mathbf{w}_r$ is said to be* observable *from $\mathbf{w}_m$. Σ is said to be* reconstructible *(*observable*) if every trajectory $\mathbf{w}_r$ is reconstructible (observable) from the corresponding $\mathbf{w}_m$.*

Characterizations of reconstructibility and observability have been obtained in [18]. It is worthwhile to remark that when a system is reconstructible a common nonnegative integer N can be found such that all relevant trajectories can be exactly evaluated (from the corresponding measured trajectories) after N steps. So, the index N does not depend on the specific pair $(\mathbf{w}_r, \mathbf{w}_m)$ but represents a system property.

Consider the dynamic system Σ described by (3.2), with $\mathbf{w}_m$ the measured variable, $\mathbf{w}_r$ the to-be-estimated variable and $\mathbf{w}_i$ the irrelevant one. A dead-beat observer (DBO) of $\mathbf{w}_r$ from $\mathbf{w}_m$ is a system that, corresponding to every trajectory $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i)$ in $\mathfrak{B}$, produces an estimate $\hat{\mathbf{w}}_r$ of the trajectory $\mathbf{w}_r$ (based on the measured variable $\mathbf{w}_m$ alone), that coincides with the sequence $\mathbf{w}_r$ except, possibly, in a finite number of initial time instants. In particular, a dead-beat observer for Σ which produces an estimate $\hat{\mathbf{w}}_r$ of $\mathbf{w}_r$ which coincides with $\mathbf{w}_r$ at each time instant $t \in \mathbb{Z}_+$ (and hence is not affected by any "estimation error") is an "exact" observer. These notions are formalized in the following definition.

DEFINITION 3.2. *[18] Consider the dynamic system Σ, whose behavior $\mathfrak{B}$ is described as in (3.2). The system represented by the difference equation*

$$Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m, \tag{3.3}$$

with $P(z)$ and $Q(z)$ polynomial matrices of suitable dimensions, is said to be

- *a* dead-beat observer *(DBO) of $\mathbf{w}_r$* from *$\mathbf{w}_m$* for *Σ if*
  - *(a) for every $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i) \in \mathfrak{B}$ there exists $\hat{\mathbf{w}}_r$ such that $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ satisfies (3.3), and*
  - *(b) there exists $N \in \mathbb{Z}_+$ such that for every $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i)$ in $\mathfrak{B}$ and $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ satisfying (3.3), we have $\mathbf{w}_r(t) - \hat{\mathbf{w}}_r(t) = 0$ for every $t \geq N$;*
- *a* consistent dead-beat observer *(cDBO) of $\mathbf{w}_r$* from *$\mathbf{w}_m$* for *Σ if it is a dead-beat observer and for every $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i)$ in $\mathfrak{B}$ the trajectory $(\mathbf{w}_r, \mathbf{w}_m)$ always satisfies (3.3);*
- *an* exact observer *(EO) of $\mathbf{w}_r$* from *$\mathbf{w}_m$* for *Σ if (a) holds, and (b) holds for $N = 0$.*

Remarks i) In the sequel, as the roles of $\mathbf{w}_m$, $\mathbf{w}_r$ and $\mathbf{w}_i$ will always be the same, we will refer to the observers of $\mathbf{w}_r$ from $\mathbf{w}_m$ for Σ simply as the observers for Σ.

ii) For an observer described by (3.3), the difference variable $\mathbf{e} := \mathbf{w}_r - \hat{\mathbf{w}}_r$ represents the *estimation error*. So, the previous definitions can be paraphrasized by saying that an observer is dead-beat (exact) if the estimation error trajectories belong to a nilpotent autonomous behavior (to the zero behavior).

iii) The concept of consistent DBO may sound somewhat strange and redundant. However, simple examples prove that this is not the case. In fact, if we consider the simple system

$$\begin{bmatrix} \sigma \\ 1 \end{bmatrix} w_r(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_m(t), \tag{3.4}$$

it is easily seen that $w_r(t) = 0, \forall t \geq 1$. Consequently, $\hat{w}_r(t) = 0, t \geq 0$, represents a DBO for the system. However, it is not consistent, since all trajectories $(w_r(t), w_m(t))$ which are identically zero for $t \geq 1$, but at $t = 0$ take a different value, i.e., $(w_r(0), w_m(0)) = (a, a), a \neq 0$, belong to the system behavior but do not satisfy the observer equations. As we will see, however, if a DBO exists then also a cDBO may be found. Of course, a similar distinction does not make sense when dealing with exact observers, which are by definition consistent.

Necessary and sufficient conditions for the existence of dead-beat or exact observers are given in Theorem 3.4. Before addressing the theorem details, however, it is worthwhile to introduce a short lemma.

LEMMA 3.3. *Given a dynamic system Σ, whose behavior $\mathfrak{B}$ is described as in (3.2),*

i) if Σ is reconstructible (observable), the polynomial matrix $R_r(z)$ is right monomic (right prime);

ii) if Σ admits a DBO (EO) (3.3), the polynomial matrix $Q(z)$ appearing in (3.3) is right monomic (right prime).

Proof. If $R_r(z)$ were not right monomic (right prime), there would be an infinite support (nonzero) trajectory $\mathbf{w}_r \in \ker R_r(\sigma)$. Since the trajectory $(0, 0, 0)$ trivially belongs to $\mathfrak{B}$ and so does $(\mathbf{w}_r, 0, 0)$, this would contradict the reconstructibility (observability) of Σ. The proof of ii) is almost identical to the proof of i). □

The following theorem provides an extensive characterization of those systems which admit DBOs, thus significantly extending the results obtained in [18] and [19].

THEOREM 3.4. *Consider a dynamic system, whose behavior $\mathfrak{B}$ is described as in (3.2), and let $H_i(z)$ denote a minimal left annihilator of $R_i(z)$. The following facts are equivalent:*

ia) there exists a consistent DBO for Σ;

ib) there exists a DBO for Σ;

ii) $\Gamma(z) := H_i(z)R_r(z)$ is right monomic;

iii) the polynomial matrix $R_r(z)$ is right monomic and there exists a polynomial matrix $L(z)$ satisfying

$$L(z)R_r(z) = z^N I, \tag{3.5}$$

$$L(z)R_i(z) = 0, \tag{3.6}$$

for some $N \in \mathbb{Z}_+$;

iv) $\mathfrak{B}$ is reconstructible.

Proof. ia) $\Rightarrow$ ib) Obvious.

ib) $\Rightarrow$ ii) Suppose that there exists a DBO described as in (3.3), and let $N \in \mathbb{Z}_+$ be a nonnegative integer such that condition (b) in the DBO definition is satisfied, namely

$$\sigma^N \mathbf{w}_r(t) - \sigma^N \hat{\mathbf{w}}_r(t) = \mathbf{w}_r(t+N) - \hat{\mathbf{w}}_r(t+N) = 0, \qquad t \in \mathbb{Z}_+.$$

Upon premultiplying equation (3.3) by $\sigma^N I_{\mathtt{w}_r}$ and replacing $\hat{\mathbf{w}}_r$ with $\mathbf{w}_r$, it easily follows that

$$[\,\sigma^N Q(\sigma) \quad -\sigma^N P(\sigma)\,] \begin{bmatrix} \mathbf{w}_r(t) \\ \mathbf{w}_m(t) \end{bmatrix} = 0,$$

which can be rewritten as

$$[\,\sigma^N Q(\sigma) \quad -\sigma^N P(\sigma) \quad 0\,] \begin{bmatrix} \mathbf{w}_r(t) \\ \mathbf{w}_m(t) \\ \mathbf{w}_i(t) \end{bmatrix} = 0.$$

As this condition holds (at any time instant $t \in \mathbb{Z}_+$) for any $(\mathbf{w}_r(t), \mathbf{w}_m(t), \mathbf{w}_i(t)) \in \mathfrak{B}$, then

$$\ker\,[\,R_r(\sigma) \quad -R_m(\sigma) \quad -R_i(\sigma)\,] \subseteq \ker\,[\,\sigma^N Q(\sigma) \quad -\sigma^N P(\sigma) \quad 0\,].$$

Therefore a polynomial matrix $T(z)$ exists such that

$$T(z)\,[\,R_r(z) \quad -R_m(z) \quad -R_i(z)\,] = [\,z^N Q(z) \quad -z^N P(z) \quad 0\,],$$

i.e.

$$\begin{cases} T(z)R_r(z) &= z^N Q(z), \\ T(z)R_m(z) &= z^N P(z), \\ T(z)R_i(z) &= 0. \end{cases}$$

The last of the above conditions ensures that $T(z)$ is a left annihilator of $R_i(z)$ and hence $T(z) = S(z)H_i(z)$ for some polynomial matrix $S(z)$. Consequently,

$$S(z)\,[H_i(z)R_r(z)] = z^N Q(z).$$

Since $Q(z)$ is right monomic, by Lemma 3.3, a polynomial matrix $V(z)$ exists such that $V(z)Q(z) = z^k I_{\mathtt{w}_r}$ for some $k \in \mathbb{Z}_+$, and hence

$$V(z)S(z)\,[H_i(z)R_r(z)] = z^{k+N} I_{\mathtt{w}_r}$$

showing that $\Gamma(z) = H_i(z)R_r(z)$ is, in turn, right monomic.

ii) $\Rightarrow$ iii) Suppose, now, that $\Gamma(z)$ is right monomic and hence there exists a polynomial matrix $S(z)$ such that $S(z)\Gamma(z) = z^N I_{\mathtt{w}_r}$, for some $N \in \mathbb{Z}_+$, namely (3.5) holds with $L(z) := S(z)H_i(z)$. By definition of $H_i(z)$, $L(z)$ satisfies also (3.6).

iii) $\Rightarrow$ ia) Suppose that there exists a polynomial matrix $L(z)$ satisfying (3.5) and (3.6), and let $(\mathbf{w}_r(t), \mathbf{w}_m(t), \mathbf{w}_i(t))$ be any trajectory in $\mathfrak{B}$, and hence satisfying (3.1). Premultiplication by $L(\sigma)$ leads to

$$[\,\sigma^N I_{\mathtt{w}_r} \quad -L(\sigma)R_m(\sigma) \quad 0\,] \begin{bmatrix} \mathbf{w}_r(t) \\ \mathbf{w}_m(t) \\ \mathbf{w}_i(t) \end{bmatrix} = 0. \tag{3.7}$$

We aim to show that the polynomial matrices $Q(z) = z^N I_{\mathtt{w}_r}$ and $P(z) = L(z)R_m(z)$ define a consistent DBO (3.3). Actually, condition (a) is always satisfied by simply choosing $\hat{\mathbf{w}}_r = \mathbf{w}_r$, since

$$[\,Q(\sigma) \quad -P(\sigma)\,] \begin{bmatrix} \mathbf{w}_r(t) \\ \mathbf{w}_m(t) \end{bmatrix} = 0.$$

This ensures, in particular, the observer consistency. On the other hand, for any other choice of $\hat{\mathbf{w}}_r$ such that

$$[\,Q(\sigma) \quad -P(\sigma)\,] \begin{bmatrix} \hat{\mathbf{w}}_r(t) \\ \mathbf{w}_m(t) \end{bmatrix} = 0,$$

one immediately obtains

$$Q(\sigma)\,[\mathbf{w}_r(t) - \hat{\mathbf{w}}_r(t)] = \sigma^N e(t) = 0,$$

thus proving (b).

iii) $\Rightarrow$ iv) Suppose that there exists a polynomial matrix $L(z)$ satisfying (3.5) and (3.6), and let $(\mathbf{w}_r(t), \mathbf{w}_m(t), \mathbf{w}_i(t))$ and $(\bar{\mathbf{w}}_r(t), \mathbf{w}_m(t), \bar{\mathbf{w}}_i(t))$ be two trajectories in $\mathfrak{B}$, and hence satisfying (3.1). Premultiplication by $L(\sigma)$ leads to

$$[\,\sigma^N I \quad -L(\sigma)R_m(\sigma) \quad 0\,] \begin{bmatrix} \mathbf{w}_r(t) \\ \mathbf{w}_m(t) \\ \mathbf{w}_i(t) \end{bmatrix} = 0,$$

$$[\,\sigma^N I \quad -L(\sigma)R_m(\sigma) \quad 0\,] \begin{bmatrix} \bar{\mathbf{w}}_r(t) \\ \mathbf{w}_m(t) \\ \bar{\mathbf{w}}_i(t) \end{bmatrix} = 0.$$

So, by subtracting the second equation to the first one, we get

$$\sigma^N\,[\mathbf{w}_r(t) - \bar{\mathbf{w}}_r(t)] = 0,$$

thus ensuring reconstructibility.

iv) $\Rightarrow$ ii) Factorize $R_i(z)$ as $R_i(z) = R_{i1}(z)R_{i2}(z)$, with R_{i1} of size say $p \times k$ and right prime and R_{i2} of full row rank k, and column border $R_{i1}(z)$ up to a unimodular matrix by means of some (right prime) polynomial matrix $C(z)$. Set

$$U(z) := [\,R_{i1}(z) \quad C(z)\,]^{-1}.$$

It is easy to see that $[\,0 \quad I_{p-k}\,]U(z)$ is an MLA of $R_{i1}(z)$, and hence of $R_i(z)$, and we can assume w.l.o.g. that it coincides with $H_i(z)$, so that

$$U(z) = \begin{bmatrix} S(z) \\ H_i(z) \end{bmatrix}.$$

So, the behavior $\mathfrak{B}$ is equivalently described as

$$\begin{bmatrix} S(\sigma)R_r(\sigma) \\ H_i(\sigma)R_r(\sigma) \end{bmatrix} \mathbf{w}_r = \begin{bmatrix} S(\sigma)R_m(\sigma) \\ H_i(\sigma)R_m(\sigma) \end{bmatrix} \mathbf{w}_m + \begin{bmatrix} R_{i2}(\sigma) \\ 0 \end{bmatrix} \mathbf{w}_i. \tag{3.8}$$

As a consequence, a trajectory $(\bar{\mathbf{w}}_r, 0, \bar{\mathbf{w}}_i)$ belongs to $\mathfrak{B}$ if and only if

$$\Gamma(\sigma)\bar{\mathbf{w}}_r = H_i(\sigma)R_r(\sigma)\bar{\mathbf{w}}_r = 0 \text{ and } S(\sigma)R_r(\sigma)\bar{\mathbf{w}}_r = R_{i2}(\sigma)\bar{\mathbf{w}}_i.$$

By the full row rank property of $R_{i2}(z)$, $R_{i2}(\sigma)$ defines a surjective map, and hence for any $\bar{\mathbf{w}}_r \in \ker \Gamma(\sigma)$ there exists $\bar{\mathbf{w}}_i$ which satisfies the second equation. So, if $\Gamma(z)$ were not right monomic, there would be an infinite support $\bar{\mathbf{w}}_r \in \ker \Gamma(\sigma)$, and a trajectory $\bar{\mathbf{w}}_i$ such that $(\bar{\mathbf{w}}_r, 0, \bar{\mathbf{w}}_i)$ belongs to $\mathfrak{B}$. So, both $(0,0,0)$ and $(\bar{\mathbf{w}}_r, 0, \bar{\mathbf{w}}_i)$ would belong to $\mathfrak{B}$, thus contradicting reconstructibility. Therefore $\Gamma(z)$ has to be right monomic. □

Remark It is worth enlightening two limit cases of the previous theorem.

- When no irrilevant variables are involved in the behavior description (i.e. R_i is the empty matrix), then H_i reduces to the identity matrix and hence the existence of a DBO (EO) is equivalent to the right monomicity (primeness) of $R_r(z)$.
- When R_i is of full row rank, then H_i is the empty matrix. When so, Theorem 3.4 can be read in a negative sense, since none of the conditions ia), ib), ii), iii) and iv) can be satisfied.

Theorem 3.4 easily leads to the following corollary.

COROLLARY 3.5. *Consider a dynamic system, whose behavior $\mathfrak{B}$ is described as in (3.2), and let $H_i(z)$ denote a minimal left annihilator of $R_i(z)$. The following facts are equivalent:*

i) there exists an EO for Σ;

ii) $\Gamma(z)$ is right prime;

iii) the polynomial matrix $R_r(z)$ is right prime and there exists a polynomial matrix $L(z)$ satisfying $L(z)R_r(z) = I_{\mathtt{w}_r}$ and $L(z)R_i(z) = 0$;

iv) $\mathfrak{B}$ is observable.

4. A parametrization of all dead-beat (exact) observers

Consider, again, a behavior $\mathfrak{B}$, described as in (3.2), and suppose that it admits a DBO (EO) and hence any of the equivalent conditions of Theorem 3.4 is satisfied. It entails no loss of generality assuming that $[\, R_r(z) \quad -R_m(z) \quad -R_i(z)\,]$ is of full row rank p. We aim to provide a complete parametrization of all (consistent or not) DBOs (EOs) for $\mathfrak{B}$. To this end, we first recall the concept of equivalent observers and a useful technical lemma [19].

Given a DBO (EO), its behavior $\hat{\mathfrak{B}}$ is the set of all solutions $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ of the difference equation (3.3). Among all the trajectories of $\hat{\mathfrak{B}}$, however, we are interested only in those that are produced corresponding to the trajectories of $\mathfrak{B}$, namely in the set $\{(\hat{\mathbf{w}}_r, \mathbf{w}_m) \in \hat{\mathfrak{B}} : \mathbf{w}_m \in \mathcal{P}_m\mathfrak{B}\}$, where $\mathcal{P}_m\mathfrak{B} := \{\mathbf{w}_m : \exists\, \mathbf{w}_r, \mathbf{w}_i \text{ s.t. } (\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i) \in \mathfrak{B}\}$. So, by assuming this point of view, it seems reasonable to regard as *equivalent* two observers (3.3), for the same system Σ, not if their behaviors $\hat{\mathfrak{B}}_1$ and $\hat{\mathfrak{B}}_2$ coincide, but if their behaviors satisfy the following condition

$$\{(\hat{\mathbf{w}}_r, \mathbf{w}_m) \in \hat{\mathfrak{B}}_1 : \mathbf{w}_m \in \mathcal{P}_m\mathfrak{B}\} = \{(\hat{\mathbf{w}}_r, \mathbf{w}_m) \in \hat{\mathfrak{B}}_2 : \mathbf{w}_m \in \mathcal{P}_m\mathfrak{B}\}.$$

It is worthwhile underlying that, of course, two equivalent observers are either both consistent or both non-consistent. We can now introduce the following result about equivalent observers.

LEMMA 4.1. *[19] If $Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m$ is a DBO (in particular, an EO) for Σ, there exists an equivalent DBO (EO) $\bar{Q}(\sigma)\hat{\mathbf{w}}_r = \bar{P}(\sigma)\mathbf{w}_m$ with $\bar{Q}$ of full row rank and hence, by Lemma 3.3, nonsingular square and monomic (unimodular).*

The reason for this lemma is that, from now on, we will steadily focus on the parametrization of all those observers whose matrix $Q(z)$ is nonsingular square. Aiming at this goal, it is convenient to reduce the original behavior description to a more suitable one.

By resorting to the same reasoning adopted within the proof of Theorem 3.4, we can adopt for the behavior $\mathfrak{B}$ the equivalent description

$$\begin{bmatrix} S(\sigma)R_r(\sigma) \\ \Gamma(\sigma) \end{bmatrix} \mathbf{w}_r = \begin{bmatrix} S(\sigma)R_m(\sigma) \\ \Phi(\sigma) \end{bmatrix} \mathbf{w}_m + \begin{bmatrix} R_{i2}(\sigma) \\ 0 \end{bmatrix} \mathbf{w}_i, \tag{4.1}$$

where $R_{i2}(z)$ is of full row rank, $\Gamma(z) = H_i(z)R_r(z)$ and $\Phi(z) := H_i(z)R_m(z)$. Let, now, $U(z)$ be a unimodular matrix such that

$$U(z)\Gamma(z) = \begin{bmatrix} \Delta(z) \\ 0 \end{bmatrix},$$

with $\Delta(z)$ nonsingular square, and conformably partition

$$U(z)\Phi(z) = \begin{bmatrix} L_1(z) \\ L_0(z) \end{bmatrix}.$$

Clearly, $\Delta(z)$ is a square monomic (unimodular) matrix. The behavior $\mathfrak{B}$ can then be equivalently described as follows:

$$\begin{bmatrix} S(\sigma)R_r(\sigma) \\ \Delta(\sigma) \\ 0 \end{bmatrix} \mathbf{w}_r = \begin{bmatrix} S(\sigma)R_m(\sigma) \\ L_1(\sigma) \\ L_0(\sigma) \end{bmatrix} \mathbf{w}_m + \begin{bmatrix} R_{i2}(\sigma) \\ 0 \\ 0 \end{bmatrix} \mathbf{w}_i. \tag{4.2}$$

Notice that $R_{i2}(\sigma)$ defines a surjective map, and hence

$$\begin{aligned} \mathcal{P}_{r,m}\mathfrak{B} &:= \{(\mathbf{w}_r, \mathbf{w}_m) : \exists\ \mathbf{w}_i \text{ s.t. } (\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i) \in \mathfrak{B}\} = \ker \begin{bmatrix} \Gamma(\sigma) & -\Phi(\sigma) \end{bmatrix} \\ &= \ker \begin{bmatrix} \Delta(\sigma) & -L_1(\sigma) \\ 0 & -L_0(\sigma) \end{bmatrix}. \end{aligned}$$

By suitably adjusting the proof of Theorem 5.4 of [18], we can easily obtain the following parametrization of all consistent DBOs (EOs)[1].

THEOREM 4.2. *[18] Consider a system Σ whose behavior $\mathfrak{B}$ is described as in (4.2), with $R_{i2}(z)$ of full row rank and $\Delta(z)$ square monomic (unimodular). If P and Q are polynomial matrices, with Q nonsingular square, then $Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m$ is a consistent (exact) dead-beat observer for Σ if and only if*

$$\begin{bmatrix} Q(z) & -P(z) \end{bmatrix} = \begin{bmatrix} Y(z) & X(z) \end{bmatrix} \begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -L_0(z) \end{bmatrix}, \tag{4.3}$$

with $Y(z)$ a monomic (unimodular) polynomial matrix and $X(z)$ a polynomial matrix.

We aim now to provide an extension of the previous parametrization to the whole class of DBOs, thus including also non-consistent DBOs.

THEOREM 4.3. *Consider a system Σ whose behavior $\mathfrak{B}$ is described as in (4.2), with $R_{i2}(z)$ of full row rank and $\Delta(z)$ square monomic. If P and Q are polynomial matrices, with Q nonsingular square, then $Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m$ is a DBO for Σ if and only if*

$$\begin{bmatrix} Q(z) & -P(z) \end{bmatrix} = \begin{bmatrix} Y(z, z^{-1}) & X(z, z^{-1}) \end{bmatrix} \begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -L_0(z) \end{bmatrix}, \tag{4.4}$$

with $Y(z, z^{-1})$ and $X(z, z^{-1})$ L-polynomial matrices such that $Y(z, z^{-1})\Delta(z)$ is (square polynomial and) monomic.

Proof. Assume, first, that the polynomial pair $(Q(z), P(z))$ satisfies (4.4) and $Q(z)$ is square monomic, and let $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i)$ be any trajectory in $\mathfrak{B}$. Clearly, $Q(z)$ defines a surjective map and hence, corresponding to the assigned $\mathbf{w}_m$, there exists $\hat{\mathbf{w}}_r$ such that

$$Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m.$$

We aim, now, to show that there exists $N \in \mathbb{Z}_+$ such that, for any such $\hat{\mathbf{w}}_r$, $\mathbf{w}_r(t) - \hat{\mathbf{w}}_r(t) = 0, \forall\ t \geq N$. To this end, let $k \in \mathbb{Z}_+$ be a nonnegative integer such that

$$\bar{Y}(z) := z^k \cdot Y(z, z^{-1}) \qquad \bar{X}(z) := z^k \cdot X(z, z^{-1})$$

[1] It is worthwhile to remark that in [18, 19] the possibility of resorting to non-consistent DBOs had not been contemplated. So, all results and parametrizations appearing there implicitly assume consistency.

are both polynomial matrices. Clearly, any such $\hat{\mathbf{w}}_r$ satisfies the difference equation

$$\sigma^k Q(\sigma)\hat{\mathbf{w}}_r = \sigma^k P(\sigma)\mathbf{w}_m,$$

which defines, by Theorem 4.2, a consistent DBO. Consequently, $\hat{\mathbf{w}}_r$ coincides with $\mathbf{w}_r$ after a finite number of steps.

Conversely, suppose that the polynomial pair $(Q(z), P(z))$ defines a DBO and, according to Definition 3.2, let N be a nonnegative integer such that $\mathbf{w}_r(t) - \hat{\mathbf{w}}_r(t) = 0, \forall\, t \geq N$, or, equivalently, $\sigma^N(\mathbf{w}_r(t) - \hat{\mathbf{w}}_r(t)) = 0, \forall\, t \geq 0$. Clearly, all trajectories $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ satisfying

$$Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m,$$

also satisfy

$$\sigma^N Q(\sigma)\hat{\mathbf{w}}_r = \sigma^N P(\sigma)\mathbf{w}_m, \tag{4.5}$$

and hence

$$\sigma^N Q(\sigma)\mathbf{w}_r = \sigma^N P(\sigma)\mathbf{w}_m.$$

So, (4.5) represents a consistent DBO and this ensures, by Theorem 4.2, that polynomial matrices $\bar{Y}(z)$ and $\bar{X}(z)$ can be found such that

$$[\, z^N \cdot Q(z) \quad -z^N \cdot P(z)\,] = [\, \bar{Y}(z) \quad \bar{X}(z)\,] \begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -L_0(z) \end{bmatrix}.$$

Consequently, (4.4) holds for

$$Y(z, z^{-1}) = z^{-N} \cdot \bar{Y}(z) \qquad X(z, z^{-1}) = z^{-N} \cdot \bar{X}(z).$$

□

Remarks i) It is worthwhile noticing that since $[\, R_r(z) \quad -R_m(z) \quad -R_i(z)\,]$ was supposed to be of full row rank p, then also

$$Z(z) := \begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -L_0(z) \end{bmatrix}$$

is of full row rank. This allows to establish a bijective correspondence between polynomial pairs $(Q(z), P(z))$ and the corresponding pairs $(Y, X) \in \mathbb{R}[z, z^{-1}]^{\mathtt{w}_r \times \mathtt{w}_r} \times \mathbb{R}[z, z^{-1}]^{\mathtt{w}_r \times \ell}$ in (4.4), ℓ denoting the number of rows of $L_0(z)$.

ii) Notice that any square L-polynomial matrix $Y(z, z^{-1})$ such that $Y(z, z^{-1})\Delta(z)$ is a polynomial monomic matrix must satisfy $\det Y(z, z^{-1}) = z^r$, for some integer r.

iii) An equivalent parametrization of all DBOs can be easily obtained by referring to the behavior description (4.1). Indeed, if P and Q are polynomial matrices, with Q nonsingular square, then $Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m$ is a DBO for Σ if and only if

$$[\, Q(z) \quad -P(z)\,] = Y(z, z^{-1})\,[\, \Gamma(z) \quad -\Phi(z)\,], \tag{4.6}$$

with $Y(z, z^{-1})$ an L-polynomial matrix such that $Y(z, z^{-1})\Gamma(z)$ is (square polynomial and) monomic, while $Y(z, z^{-1})\Phi(z)$ is polynomial. On the other hand, if we are interested

in consistent DBOs, then the above parametrization is still true, provided that $Y(z, z^{-1})$ is strictly polynomial.

iv) One may wonder why we are interested in non-consistent DBOs when, under the same conditions, we can always resort to consistent ones. The only reason that may lead to choose this solution is complexity. Indeed, by choosing L-polynomial matrices X and Y, instead of polynomial ones, we may reduce the degree of the polynomial matrices $Q(z)$ and $P(z)$, and this leads to an autoregressive model of lower complexity. This fact is enlightened, for instance, by the simple example (3.4) we provided in section 3. Further examples, supporting this claim, will be provided in section 6.

v) In the behavioral setting the target of the estimation problem is simply that of obtaining a kernel description of some (consistent or not) DBO for the given system Σ. When applying these results to the state-space setting (in section 6), however, we will face the problem of obtaining, when possible, a state-space realization of such a DBO. To this end, a preliminary goal is that of investigating the structure of the DBO transfer matrices in order to verify when there exist proper rational ones among them. This will be the object of section 5. In particular, one may wonder whether by relaxing the consistency constraint one may enlarge the class of DBO transfer matrices, thus increasing the chances to obtain DBOs endowed with a proper rational transfer matrix.

As an important corollary of Theorem 4.3, we easily verify that the class of the transfer matrices of all DBOs coincides with the class of the transfer matrices of all consistent DBOs, since it is immediate to obtain, as we did within the proof, a consistent DBO starting from a generic one. Specifically, the DBO transfer matrices are parametrized, according to (4.3) or to (4.4) as

$$\hat{W}(z) = Q(z)^{-1}P(z) = \Delta^{-1}(z)L_1(z) + \Delta^{-1}(z)Y^{-1}(z, z^{-1})X(z, z^{-1})L_0(z)$$

as $Y(z, z^{-1})$ and $X(z, z^{-1})$ vary over the set of all Laurent polynomial matrices of suitable sizes (under the constraint that $Q(z) = Y(z, z^{-1})\Delta(z)$ is polynomial and square monomic, and the corresponding $P(z)$ is polynomial, too). Upon setting

$$\hat{W}_0(z) := \Delta^{-1}(z)L_1(z),$$

which can be seen as a "particular" transfer matrix, and noting that $T(z, z^{-1}) := \Delta^{-1}(z)\ Y^{-1}(z, z^{-1})X(z, z^{-1})$ is an arbitrary Laurent polynomial matrix[2], the previous parametrization can be rewritten as

$$\hat{W}(z) = \hat{W}_0(z) + T(z, z^{-1})L_0(z), \qquad T(z, z^{-1}) \in \mathbb{R}[z, z^{-1}]^{\mathtt{w}_r \times \ell}. \tag{4.7}$$

So, consistency is just a property that affects the free observer evolution, as it depends on the choice of the initial samples of $\hat{\mathbf{w}}_r$, but not the forced evolution, which is uniquely determined by the observer transfer matrix and by the specific $\mathbf{w}_m$.

As a further corollary of Theorem 4.3 above, one may deduce conditions for a system Σ, whose behavior $\mathfrak{B}$ is described as in (4.2), to admit non-consistent DBOs. As one

[2]Of course, if X and Y are L-polynomial matrices, while Δ is monomic, T is an L-polynomial matrix. On the other hand, it is not hard to show that if Δ is monomic and T is an arbitrary L-polynomial matrix, then we can always find L-polynomial matrices X and Y such that $\Delta(z)T(z, z^{-1}) = Y^{-1}(z, z^{-1})X(z, z^{-1})$ and the corresponding Q and P are polynomial matrices with Q square monomic.

easily deduces by comparing Theorem 4.2 and Theorem 4.3, this is the case if and only if there is a polynomial pair $(Q(z), P(z))$, with $Q(z)$ monomic, which is obtained as in (4.4) corresponding to a pair (Y, X) which belongs to $\mathbb{R}[z, z^{-1}]^{\mathtt{w}_r \times \mathtt{w}_r} \times \mathbb{R}[z, z^{-1}]^{\mathtt{w}_r \times \ell}$, but not to $\mathbb{R}[z]^{\mathtt{w}_r \times \mathtt{w}_r} \times \mathbb{R}[z]^{\mathtt{w}_r \times \ell}$.

THEOREM 4.4. *Consider a system Σ whose behavior $\mathfrak{B}$ is described as in (4.2), with $\Delta(z)$ monomic, and let $d(z)$ be the greatest common divisor (g.c.d.) of the maximal (namely, $(\mathtt{w}_r + \ell)$th) order minors of $Z(z) = \begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -L_0(z) \end{bmatrix}$. The following facts are equivalent:*

i) the class of DBOs coincides with the class of consistent DBOs;

ii) $d(0) \neq 0$.

Proof. By the full row rank assumption on the matrix $Z(z)$ (see the previous Remark i)) and the assumption that $\Delta(z)$ is square monomic, this amounts to showing that there exists a non-polynomial pair (Y, X) in $\mathbb{R}[z, z^{-1}]^{\mathtt{w}_r \times \mathtt{w}_r} \times \mathbb{R}[z, z^{-1}]^{\mathtt{w}_r \times \ell}$ such that the corresponding pair $(Q(z), P(z))$ is polynomial, with $Q(z)$ monomic, if and only if $z | d(z)$.

Suppose, first, that there exists a strictly L-polynomial pair (Y, X) in $\mathbb{R}[z, z^{-1}]^{\mathtt{w}_r \times \mathtt{w}_r} \times \mathbb{R}[z, z^{-1}]^{\mathtt{w}_r \times \ell}$ such that the corresponding pair $(Q(z), P(z))$ is polynomial, with $Q(z)$ monomic, and let S_i be a $q \times (\mathtt{w}_r + \ell)$ selection matrix that singles out the ith maximal order minor of $Z(z)$. Then

$$\begin{aligned} [\, Q(z) \quad -P(z) \,]\, S_i \,\mathrm{adj}(Z(z)S_i) &= [\, Y(z, z^{-1}) \quad X(z, z^{-1}) \,]\, Z(z) S_i \,\mathrm{adj}(Z(z)S_i) \\ &= [\, Y(z, z^{-1}) \quad X(z, z^{-1}) \,]\, \det(Z(z)S_i). \end{aligned}$$

Since the lefthand side is polynomial, this means that the least common denominator, say $z^k, k > 0$, of the entries of $[\, Y(z, z^{-1}) \quad X(z, z^{-1}) \,]$ satisfies $z^k \mid \det(Z(z)S_i)$ for every index i and hence $z^k | d(z)$.

Conversely, suppose that $d(z) = z^k \cdot \tilde{d}(z)$, for some positive integer k and some polinomial $\tilde{d}(z)$ satisfying $\tilde{d}(0) \neq 0$. Upon factorizing $Z(z)$ as

$$Z(z) = T(z)\bar{Z}(z),$$

where $T(z)$ is nonsingular square with $\det T(z) = z^k$, while $\bar{Z}(z)$ is of full row rank, one can always choose a unimodular matrix $U(z)$ such that

$$U(z)\bar{Z}(s) \begin{bmatrix} I_{\mathtt{w}_r} \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{\Delta}(z) \\ 0 \end{bmatrix},$$

with $\bar{\Delta}(z)$ nonsingular square and monomic (as it is not hard to see that it is a right divisor of $\Delta(z)$). Consequently, we can assume w.l.o.g.

$$\bar{Z}(z) = \begin{bmatrix} \bar{\Delta}(z) & -\bar{L}_1(z) \\ 0 & -\bar{L}_0(z) \end{bmatrix}$$

and this implies that

$$T(z) = \begin{bmatrix} T_{11}(z) & T_{12}(z) \\ 0 & T_{22}(z) \end{bmatrix},$$

with $T_{11}(z), T_{22}(z)$ nonsingular square and $\det T_{11}(z) \cdot \det T_{22}(z) = z^k$. If $\det T_{11}(z)$ is a multiple of z, then by assuming

$$[\, Y(z,z^{-1}) \quad X(z,z^{-1}) \,] = [\, T_{11}(z)^{-1} \quad -T_{11}(z)^{-1}T_{12}(z)T_{22}(z)^{-1} \,],$$

which is Laurent polynomial but not polynomial, we get

$$\begin{aligned}
&[\, Q(z) \quad -P(z) \,] = [\, Y(z,z^{-1}) \quad X(z,z^{-1}) \,] \begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -L_0(z) \end{bmatrix} \\
&= [\, T_{11}(z)^{-1} \quad -T_{11}(z)^{-1}T_{12}(z)T_{22}(z)^{-1} \,] \begin{bmatrix} T_{11}(z) & T_{12}(z) \\ 0 & T_{22}(z) \end{bmatrix} \begin{bmatrix} \bar{\Delta}(z) & -\bar{L}_1(z) \\ 0 & -\bar{L}_0(z) \end{bmatrix} \\
&= [\, I_{\mathtt{w}_r} \quad 0 \,] \begin{bmatrix} \bar{\Delta}(z) & -\bar{L}_1(z) \\ 0 & -\bar{L}_0(z) \end{bmatrix} = [\, \bar{\Delta}(z) \quad -\bar{L}_1(z) \,],
\end{aligned}$$

which corresponds to a non-consistent DBO. If $\det T_{11}(z)$ is not a multiple of z, then $\det T_{22}(z) = z^k$, and we can also resort to the following factorization:

$$\begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -L_0(z) \end{bmatrix} = \begin{bmatrix} I_{\mathtt{w}_r} & 0 \\ 0 & T_{22}(z) \end{bmatrix} \begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -\bar{L}_0(z) \end{bmatrix}.$$

So, by assuming

$$[\, Y(z,z^{-1}) \quad X(z,z^{-1}) \,] = [\, I_{\mathtt{w}_r} \quad T_{22}(z)^{-1} \,],$$

which is Laurent polynomial but not polynomial, we get

$$\begin{aligned}
[\, Q(z) \quad -P(z) \,] &= [\, Y(z,z^{-1}) \quad X(z,z^{-1}) \,] \begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -L_0(z) \end{bmatrix} \\
&= [\, I_{\mathtt{w}_r} \quad T_{22}(z)^{-1} \,] \begin{bmatrix} I_{\mathtt{w}_r} & 0 \\ 0 & T_{22}(z) \end{bmatrix} \begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -\bar{L}_0(z) \end{bmatrix} \\
&= [\, I_{\mathtt{w}_r} \quad I_\ell \,] \begin{bmatrix} \Delta(z) & -L_1(z) \\ 0 & -\bar{L}_0(z) \end{bmatrix} = [\, \Delta(z) \quad -L_1(z) - \bar{L}_0(z) \,],
\end{aligned}$$

which corresponds to a non-consistent DBO. □

Remark It is worthwhile to underline that the test one has to perform on the system matrices, in order to verify whether non-consistent DBOs exist, simply consists in verifying that the rank of

$$\begin{bmatrix} \Delta(0) & -L_1(0) \\ 0 & -\bar{L}_0(0) \end{bmatrix}$$

or, equivalently, of

$$[\, \Gamma(0) \quad -\Phi(0) \,],$$

is less than $\mathtt{w}_r + \ell$. This is the case, for instance, of the example (3.4) provided in section 3, where

$$[\, \Gamma(z) \quad -\Phi(z) \,] = \begin{bmatrix} z & 0 \\ 1 & -1 \end{bmatrix}$$

and hence

$$[\, \Gamma(0) \quad -\Phi(0) \,] = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

Actually, as already seen, the system admits a non-consistent DBO.

5. Causal dead-beat observers

As mentioned in the previous section, if the task we have in mind is simply that of obtaining a "behavioral approach" to the solution of various types of estimation problems, and a parametric description of all available solutions, the results of the previous sections already provide satisfactory answers. If we aim at applying the previous general results to the specific problems one may address in the state-space setting, however, it is extremely important to investigate the existence of a DBO which admits a state-space realization. This requires the observer (rational) transfer matrix $\hat{W}(z) := Q^{-1}(z)P(z)$ to be proper. On the other hand, since a DBO is characterized by a monomic $Q(z)$, this implies that $\hat{W}(z)$ is proper if and only if it is a polynomial matrix in the negative powers of z, i.e. $\hat{W}(z) = \bar{W}(z^{-1})$, for some suitable polynomial matrix $\bar{W}(z)$. This amounts to saying that $\hat{W}(z)$ defines a finite memory filter.

The parameterization (4.6) can be fruitfully exploited to investigate this issue. Since we already remarked that the class of DBOs does not provide additional transfer matrices with respect to those obtained corresponding to cDBOs, we will assume that the matrix Y appearing in (4.6) is polynomial and hence denote it by $Y(z)$. If we refer to the behavior description (4.1), and hence to the following description of $\mathcal{P}_{r,m}\mathfrak{B}$

$$\Gamma(\sigma)\mathbf{w}_r = \Phi(\sigma)\mathbf{w}_m, \tag{5.1}$$

it entails no loss of generality assuming that

$$[\,\Gamma(z) \quad -\Phi(z)\,] \in \mathbb{R}[z]^{(\mathtt{w}_r+\ell)\times(\mathtt{w}_r+\mathtt{w}_m)} \tag{5.2}$$

is a row reduced matrix [11] with row degrees $\mu_1, \mu_2, \ldots, \mu_{\mathtt{w}_r+\ell}$. Of course, $\Gamma(z)$ is supposed to be right monomic. So, the general expression of the observer transfer matrix is

$$\hat{W}(z) = [Y(z)\Gamma(z)]^{-1}\,[Y(z)\Phi(z)]\,,$$

with $Y(z)$ a polynomial matrix such that $Y(z)\Gamma(z)$ is square and monomic.

The characterization of those behaviors which admit a consistent DBO endowed with a proper transfer matrix, obtained in [19], can be easily adjusted to the case when irrelevant variables are involved in the behavior description, thus leading to the following result[3].

THEOREM 5.1. *[19] Consider a dynamic system Σ with behavior $\mathfrak{B}$ described as in (4.1) and assume that $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$. Suppose that*

$$[\,\Gamma(z) \quad -\Phi(z)\,] \in \mathbb{R}[z]^{(\mathtt{w}_r+\ell)\times(\mathtt{w}_r+\mathtt{w}_m)}$$

is row reduced with row degrees $\mu_1, \mu_2, \ldots, \mu_{\mathtt{w}_r+\ell}$, so that

$$[\,\Gamma(z) \quad -\Phi(z)\,] = \begin{bmatrix} z^{\mu_1} & & & \\ & z^{\mu_2} & & \\ & & \ddots & \\ & & & z^{\mu_{\mathtt{w}_r+\ell}} \end{bmatrix} [\,\Gamma_0 + z^{-1}\cdot\bar{\Gamma}(z^{-1}) \quad -\Phi_0 - z^{-1}\cdot\bar{\Phi}(z^{-1})\,]\,, \tag{5.3}$$

[3] An explicit proof of this theorem has been presented in [2].

where Γ_0 *and* Φ_0 *are constant matrices,* $[\,\Gamma_0 \quad -\Phi_0\,]$ *is of full row rank, and* $\bar{\Gamma}(z^{-1})$ *and* $\bar{\Phi}(z^{-1})$ *are polynomial matrices in* z^{-1} *of suitable sizes. A necessary and sufficient condition for the existence of a consistent DBO endowed with a proper transfer matrix* $\hat{W}(z)$ *is that* Γ_0 *is of full column rank.*

Remark. It is worthwhile remarking (see [19, 2] for the details) that the assumption that the polynomial matrix (5.2) is row reduced plays a role only in the necessity part of the proof of the previous theorem. Actually, if (5.2) is row reduced and a proper DBO exists, then Γ_0 is necessarily of full column rank. However, if we start with a representation corresponding to a polynomial matrix(5.2) which is not row reduced, but Γ_0 is of full column rank, then a causal DBO exists. Notice that since the proof is a constructive one, it is easy to explicitly obtain such a DBO. Clearly, if Γ_0 is not of full column rank in a row reduced description, it cannot exhibit this property in any other representation, so occasionally (see the following section) one does not need to resort to a row reduced representation in order to claim that a causal DBO exists.

6. Applications to state-space models

In this section we will show how the observer theory, here developed within the behavioral approach, allows to treat in a homogeneous way several classical estimation problems for state-space systems. To this end we will consider the most general expression of a state-space model considered in the literature (in a deterministic setting), including not only the usual state, input and output variables, but also disturbances and additive faults. Once we will cast the state-space model in the behavioral framework, by differently choosing the measured, the relevant and the irrelevant variables, we will be able to formalize the following traditional problems:

1. the state estimation when neither disturbances nor faults affect the system;

2. the state estimation when only disturbances affect the system. This leads to the well-known concept of *unknown input observer* (UIO);

3. the fault detection and isolation when no disturbance affects the system (but faults, of course, do) (FDI);

4. the fault detection and isolation in the presence of disturbances (dFDI).

A general state-space model is described by the following equations:

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B_u\mathbf{u}(t) + B_d\mathbf{d}(t) + B_f\mathbf{f}(t), \tag{6.1}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D_u\mathbf{u}(t) + D_d\mathbf{d}(t) + D_f\mathbf{f}(t), \qquad t \in \mathbb{Z}_+ \tag{6.2}$$

where $\mathbf{x}$ denotes the state, $\mathbf{u}$ the controlled input, $\mathbf{y}$ the measured output, $\mathbf{d}$ the disturbance (i.e., the uncontrollable input) and $\mathbf{f}$ the fault. The state-space model (6.1)-(6.2)

can be rewritten in behavioral form as

$$\begin{bmatrix} \sigma I_{\mathbf{x}} - A & 0 & -B_u & -B_d & -B_f \\ C & -I_{\mathbf{y}} & D_u & D_d & D_f \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \\ \mathbf{u}(t) \\ \mathbf{d}(t) \\ \mathbf{f}(t) \end{bmatrix} = 0, \qquad t \in \mathbb{Z}_+. \tag{6.3}$$

It is worthwhile to remark that the polynomial matrix involved in the system description is always of full row rank.

Before applying the results of the previous section to the aforementioned problems, it is worthwhile to briefly point out a few facts:

- In section 3 we have derived a simple means for deciding whether a particular estimation problem is solvable or not, i.e. whether DBOs and/or EOs are available. It amounts to testing whether a suitable polynomial matrix $\Gamma(z)$ is right monomic (right prime).
- Whenever the answer is positive, namely $\Gamma(z)$ is right monomic (right prime), we may resort to the parametrization of the DBOs given in (4.3) or in (4.6) and to the parametrization of the corresponding transfer matrices given in (4.7).
- Having in mind a state-space realization of $\hat{W}(z)$, deciding whether a causal DBO (i.e., endowed with a transfer matrix which is polynomial in z^{-1}) exists or not, becomes a crucial point. The solution is given by the full column rank test on the matrix Γ_0, developed in the previous section. When a causal DBO exists, we can realize it by means of a finite memory system of the form

$$\mathbf{v}(t+1) = F\mathbf{v}(t) + G\mathbf{w}_m(t), \quad \hat{\mathbf{w}}_r(t) = H\mathbf{v}(t) + J\mathbf{w}_m(t).$$

- Whenever causal DBOs are not available, we may still resort to state-space realizations. However these systems will provide only a "delayed" estimation, in the sense that if $\hat{W}(z) = z^i \cdot \tilde{W}(z^{-1})$, with $\tilde{W}(z^{-1})$ a polynomial matrix in the variable z^{-1} and i a positive integer, we can realize $\tilde{W}(z^{-1})$ by means of a state-space model. The only difference with the "causal case" resides in the fact that the DBO output will be $\hat{\mathbf{w}}_r(t-i)$, instead of $\hat{\mathbf{w}}_r(t)$. In other words, the estimation is performed with a fixed delay of i steps.

6.1. Standard state estimation

If neither faults $\mathbf{f}$ nor disturbances $\mathbf{d}$ affect the system, we are reduced to the case of plain state estimation from the controlled input and the measured output. When so, the relevant variable is $\mathbf{w}_r = \mathbf{x}$, the available measurements are $\mathbf{w}_m = \begin{bmatrix} \mathbf{y}^T & \mathbf{u}^T \end{bmatrix}^T$, and there are no irrelevant variables $\mathbf{w}_i$. The behavioral equation takes the form

$$\left[\begin{array}{c|cc} \sigma I_{\mathbf{x}} - A & 0 & -B_u \\ C & -I_{\mathbf{y}} & D_u \end{array}\right] \begin{bmatrix} \mathbf{x} \\ - \\ \mathbf{y} \\ \mathbf{u} \end{bmatrix} = 0. \tag{6.4}$$

As previously remarked, in this case $R_i(z)$ is the empty matrix and hence an MLA of $R_i(z)$ is $H_i(z) = I_{\mathtt{x}+\mathtt{y}}$, while

$$R_r(z) = \begin{bmatrix} zI_{\mathtt{x}} - A \\ C \end{bmatrix} =: \mathcal{O}(z).$$

So, reconstructibility (observability), and hence the existence of a dead-beat (an exact) state observer, corresponds to the right monomicity (right primeness) of the PBH observability matrix $\mathcal{O}(z)$, a well-known result [11, 12, 14]. When so, both causal and non-causal DBO (EO) can be constructed. Indeed, the polynomial matrix

$$[\,\Gamma(z) \quad -\Phi(z)\,] = \left[\begin{array}{c|cc} zI_{\mathtt{x}} - A & 0 & -B_u \\ C & -I_{\mathtt{y}} & D_u \end{array}\right]$$

is row reduced and the constant matrix

$$\Gamma_0 = \begin{bmatrix} I_{\mathtt{x}} \\ C \end{bmatrix}$$

is of full column rank. Consequently, DBOs endowed with a proper transfer matrix always exist.

Note that $[\,\Gamma(0) \quad -\Phi(0)\,]$ is of full row rank if and only if $[\,A \quad B\,]$ is. So, nonconsistent DBOs exist if and only if the state $\mathbf{x}$ can be partitioned (possibly after a change of basis) as $\mathbf{x} = [\,\mathbf{x}_1^T \quad \mathbf{x}_2^T\,]^T$, where the evolution of the first subvector $\mathbf{x}_1$ is independent of $\mathbf{u}$ and vanishes in a finite number of steps. Indeed, in this case, the choice $\hat{\mathbf{x}}_1(t) = 0$, together with a DBO for $\mathbf{x}_2(t)$ alone, allows to implement a non-consistent DBO of lower complexity w.r.t. the complexity of any consistent DBO. In particular, when A is a nilpotent matrix and $B = 0$, $\hat{\mathbf{x}}(t) = 0$ represents a (static) non-consistent DBO of minimal complexity (see Remark iv) in section 4).

Example 1. Consider a state-space model (6.4) with $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $C = [\,0 \quad 1\,]$, and assume that no controlled inputs act on the system. From $H_i(z) = I_3$, it follows that

$$\Gamma(z) = R_r(z) = \begin{bmatrix} zI_2 - A \\ C \end{bmatrix} = \begin{bmatrix} z & -1 \\ -1 & z-1 \\ 0 & 1 \end{bmatrix}, \; \Phi(z) = R_m(z) = \begin{bmatrix} 0 \\ I_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

By applying the unimodular matrix

$$U(z) = \begin{bmatrix} 0 & -1 & z-1 \\ 0 & 0 & 1 \\ 1 & z & 1+z-z^2 \end{bmatrix}$$

one may obtain the behavior description (4.2) with

$$\Delta(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, L_1(z) = \begin{bmatrix} z-1 \\ 1 \end{bmatrix}, L_0(z) = 1 + z - z^2.$$

Notice that the constraint $L_0(\sigma)\mathbf{y}(t) = 0$, namely $\mathbf{y}(t+2) - \mathbf{y}(t+1) - \mathbf{y}(t) = 0, t \geq 0$, is just the auto-regressive equation satisfied by the free output evolution. The DBO transfer matrix parametrization leads to

$$\hat{W}(z) = \begin{bmatrix} z-1 \\ 1 \end{bmatrix} + \begin{bmatrix} p(z, z^{-1}) \\ q(z, z^{-1}) \end{bmatrix} [1 + z - z^2]$$

where $p(z, z^{-1})$ and $q(z, z^{-1})$ are arbitrary Laurent polynomials. The causality condition is satisfied (as it may be seen by direct inspection) if and only if $p(z, z^{-1}) = z^{-1}[1 + z^{-1}\bar{p}(z^{-1})]$ and $q(z, z^{-1}) = z^{-2}\bar{q}(z^{-1})$, with $\bar{p}, \bar{q}$ arbitrary polynomials in the variable z^{-1} alone.

As interesting special cases, it is worth mentioning:

1. when $p = q = 0$ then $\hat{W}(z) = \begin{bmatrix} z-1 \\ 1 \end{bmatrix}$. A corresponding DBO is $\hat{\mathbf{x}}(t) = \begin{bmatrix} \sigma - 1 \\ 1 \end{bmatrix} \mathbf{y}(t)$, namely $\hat{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{y}(t+1) - \mathbf{y}(t) \\ \mathbf{y}(t) \end{bmatrix}$, which represents a non-causal EO;

2. when $p = z^{-1}$ and $q = 0$, then $\hat{W}(z) = \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}$. A corresponding DBO is $\sigma\hat{\mathbf{x}}(t) = \begin{bmatrix} 1 \\ \sigma \end{bmatrix} \mathbf{y}(t)$, namely $\hat{\mathbf{x}}(t+1) = \begin{bmatrix} \mathbf{y}(t) \\ \mathbf{y}(t+1) \end{bmatrix}$, which represents a causal DBO (coinciding with the classical reduced order dead-beat observer).

3. when $p = z^{-1}$ and $q = z^{-2}$, then $\hat{W}(z) = \begin{bmatrix} z^{-1} \\ z^{-1} + z^{-2} \end{bmatrix}$. A corresponding DBO is $\sigma^2\hat{\mathbf{x}}(t) = \begin{bmatrix} \sigma \\ \sigma + 1 \end{bmatrix} \mathbf{y}(t)$, namely $\hat{\mathbf{x}}(t+2) = \begin{bmatrix} \mathbf{y}(t+1) \\ \mathbf{y}(t+1) + \mathbf{y}(t) \end{bmatrix}$, which is causal and coincides with the classical Luenberger DBO of gain matrix $L^T = [\,-1 \quad -1\,]$.

6.2. Unknown input observers (UIOs)

When faults $\mathbf{f}$ are not contemplated, but disturbances $\mathbf{d}$ affect the system dynamics, we are reduced to the problem of designing an UIO: the relevant variable is $\mathbf{w}_r = \mathbf{x}$, while the available measurements are $\mathbf{w}_m = \begin{bmatrix} \mathbf{y}^T & \mathbf{u}^T \end{bmatrix}^T$. The irrelevant variables are of course represented by the disturbances $\mathbf{w}_i = \mathbf{d}$. The behavioral equations can be block-partitioned in the following form

$$\left[\begin{array}{c|cc|c} \sigma I_{\mathtt{x}} - A & 0 & -B_u & -B_d \\ C & -I_{\mathtt{y}} & D_u & D_d \end{array}\right] \begin{bmatrix} \mathbf{x} \\ - \\ \mathbf{y} \\ \mathbf{u} \\ - \\ \mathbf{d} \end{bmatrix} = 0. \tag{6.5}$$

Upon introducing an MLA of $R_i(z) = \begin{bmatrix} B_d \\ -D_d \end{bmatrix}$, which can always be assumed a constant matrix so that $H_i(z) = [\,H_{iB} \quad H_{iD}\,]$, a dead-beat (an exact) UIO exists if and only if the polynomial matrix

$$[\,H_{iB} \quad H_{iD}\,]\, R_r(z) = H_{iB}(zI_{\mathtt{x}} - A) + H_{iD}C =: \Gamma_x(z)$$

is right monomic (right prime). This requires, in particular, both that the state-space model is reconstructible (observable) in the classical sense and that $\mathtt{x}+\mathtt{y}-\mathrm{rank}\begin{bmatrix} B_d \\ -D_d \end{bmatrix} \geq$

$\mathtt{x}$, namely $\mathtt{y} \geq \text{rank} \begin{bmatrix} B_d \\ -D_d \end{bmatrix}$. In this case

$$[\,\Gamma(z) \quad -\Phi(z)\,] = [\,H_{iB} \quad H_{iD}\,] \left[\begin{array}{c|cc} zI_{\mathtt{x}} - A & 0 & -B_u \\ C & -I_{\mathtt{y}} & D_u \end{array}\right]$$

is not necessarily row reduced. Moreover, causal (dead-beat or exact) UIOs may not exist, as shown in the following example.

Example 2. Consider a state-space model (6.5) with

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ B_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D_d = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

which represents an observable system devoid of controlled inputs but affected by disturbances. In this case

$$R_r(z) = \begin{bmatrix} zI_2 - A \\ C \end{bmatrix} = \begin{bmatrix} z & 0 \\ -1 & z \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ R_m(z) = \begin{bmatrix} 0 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$R_i(z) = \begin{bmatrix} B_d \\ -D_d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ H_i(z) = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\Gamma(z) = \begin{bmatrix} -1 & z \\ 0 & 1 \end{bmatrix}$ is unimodular, $\Delta(z) = \Gamma(z)$, $L_1(z) = \Phi(z) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, while $L_0(z) = \emptyset$. The last relation means that $\mathbf{u}(t)$ and $\mathbf{y}(t)$ are completely arbitrary evolutions (* cosa vuoi dire esattamente ??? *) . The DBO transfer matrix $\hat{W}(z) = \Delta^{-1}(z)L_1(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$ is uniquely determined and is not a proper rational matrix, so a corresponding DBO $\hat{\mathbf{x}}(t) = \begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix} \mathbf{y}(t)$ is a non-causal EO. There is no possibility of performing real time estimation, since only $\hat{\mathbf{x}}(t-1)$ can be computed at time t. In fact, $[\Gamma(z) \quad -\Phi(z)]$ is row reduced and $\Gamma_0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ is not of full column rank.

Another interesting problem, even though less explored in the literature, is that of obtaining estimates both for the state and for the disturbance: in this case the relevant variable is $\mathbf{w}_r = [\mathbf{x}^T\ \mathbf{d}^T]^T$, the measured variable is $\mathbf{w}_m = \left[\mathbf{y}^T\ \mathbf{u}^T\right]^T$ and no irrelevant variables are involved in the system description. This situation coincides, as a matter of fact, with the first FDI problem analyzed in section 6.3, below, provided that the disturbance $\mathbf{d}(t)$ is regarded as a fault.

6.3. Fault detection and isolation (FDI)

Suppose, first, that disturbances $\mathbf{d}$ may be neglected. When so, we may face to two interesting problems: the first problem is the design of an *observer-based FDI*, which corresponds to assuming as relevant variables both $\mathbf{x}$ and $\mathbf{f}$, i.e. $\mathbf{w}_r = \left[\mathbf{x}^T\ \mathbf{f}^T\right]^T$, while using as measurements $\mathbf{w}_m = \left[\mathbf{y}^T\ \mathbf{u}^T\right]^T$. If so, no irrelevant variables appear in the

system description, i.e. $\mathbf{w}_i = \emptyset$, and the MLA of $R_i(z) = \emptyset$ is $H_i(z) = I_{\mathtt{x+y}}$. The behavioral description can be block-partitioned as follows

$$\left[\begin{array}{cc|cc} \sigma I_{\mathtt{x}} - A & -B_f & 0 & -B_u \\ C & D_f & -I_{\mathtt{y}} & D_u \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \mathbf{f} \\ - \\ \mathbf{y} \\ \mathbf{u} \end{array}\right] = 0,$$

and a dead-beat (exact) FDI exists if and only if the system matrix [14]

$$R_r(z) = \left[\begin{array}{cc} zI_{\mathtt{x}} - A & -B_f \\ C & D_f \end{array}\right] =: \Gamma_{x,f}(z)$$

is right monomic (right prime).

The second problem one may want to address is the design of an FDI which allows to estimate just the faults, disregarding the state evolution (*standard FDI*). In this case $\mathbf{f}$ becomes the only relevant variable $\mathbf{w}_r$, while $\mathbf{x}$ becomes the irrelevant variable $\mathbf{w}_i$, and we can write

$$\left[\begin{array}{c|cc|c} -B_f & 0 & -B_u & \sigma I_{\mathtt{x}} - A \\ D_f & -I_{\mathtt{y}} & D_u & C \end{array}\right] \left[\begin{array}{c} \mathbf{f} \\ - \\ \mathbf{u} \\ \mathbf{y} \\ - \\ \mathbf{x} \end{array}\right] = 0.$$

Now $R_i(z)$ is just the PBH observability matrix and once we select any left coprime matrix fraction description $D_L(z)^{-1}N_L(z)$ of the state to output transfer matrix $C(zI_{\mathtt{x}} - A)^{-1}$, we get $H_i(z) = [\,-N_L(z) \quad D_L(z)\,]$ as an MLA of $R_i(z)$. Consequently, a dead-beat (exact) FDI exists if and only if the following matrix

$$H_i(z) \left[\begin{array}{c} -B_f \\ D_f \end{array}\right] = N_L(z)B_f + D_L(z)D_f =: \Gamma_f(z)$$

is right monomic (right prime). Notice that this requires that $\left[\begin{array}{c} B_f \\ -D_f \end{array}\right]$ is of full column rank and that $\mathtt{y} \geq \mathtt{f}$.

6.4. Fault detection and isolation in presence of disturbances (dFDI)

Similarly to the previous subsection, two different FDI problems in the presence of disturbances may be considered: one may be interested in estimating both $\mathbf{x}$ and $\mathbf{f}$ (*observer-based dFDI* problem), i.e. $\mathbf{w}_r = \left[\mathbf{x}^T \ \mathbf{f}^T\right]^T$, making use of the measurements $\mathbf{w}_m = \left[\mathbf{y}^T \ \mathbf{u}^T\right]^T$, and disregarding $\mathbf{w}_i = \mathbf{d}$. When so, the behavioral equation takes the

form

$$\left[\begin{array}{cc|cc|c} \sigma I_{\mathbf{x}} - A & -B_f & 0 & -B_u & -B_d \\ C & D_f & -I_{\mathbf{y}} & D_u & D_d \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \mathbf{f} \\ - \\ \mathbf{y} \\ \mathbf{u} \\ - \\ \mathbf{d} \end{array}\right] = 0$$

Upon denoting by $H_i(z) = [\, H_{iB} \quad H_{iD} \,]$ (a constant matrix) an MLA of $R_i(z) := \left[\begin{array}{c} B_d \\ -D_d \end{array}\right]$, the existence of an observer-based FDI which produces exact estimates of both the state and the fault after a finite number of steps (after 0 steps) corresponds to the right monomicity (primeness) of

$$[\, H_{iB} \quad H_{iD} \,] \left[\begin{array}{cc} zI_{\mathbf{x}} - A & -B_f \\ C & D_f \end{array}\right] =: \Gamma_{x,f}(z).$$

The other case corresponds to the problem of estimating the faults, from the input and output measurements, by neglecting the state dynamics and the disturbances (*standard dFDI* problem). In this case $\mathbf{w}_r = \mathbf{f}$, $\mathbf{w}_m = \left[\mathbf{y}^T \ \mathbf{u}^T\right]^T$ and $\mathbf{w}_i = \left[\mathbf{x}^T \ \mathbf{d}^T\right]^T$. Consequently, we can write

$$\left[\begin{array}{c|cc|cc} -B_f & 0 & -B_u & \sigma I_{\mathbf{x}} - A & -B_d \\ D_f & -I_{\mathbf{y}} & D_u & C & D_d \end{array}\right] \left[\begin{array}{c} \mathbf{f} \\ - \\ \mathbf{y} \\ \mathbf{u} \\ - \\ \mathbf{x} \\ \mathbf{d} \end{array}\right] = 0.$$

The polynomial matrix $H_i(z)$ represents, in this case, an MLA of the system matrix

$$R_i(z) = \left[\begin{array}{cc} zI_{\mathbf{x}} - A & -B_d \\ C & D_d \end{array}\right],$$

and the existence of a (non-observer based) dead-beat (exact) FDI in the presence of disturbances is equivalent to the right monomicity (primeness) of

$$H_i(z)R_r(z) = H_i(z) \left[\begin{array}{c} -B_f \\ D_f \end{array}\right] =: \Gamma_f(z).$$

In order to better enlighten various aspects of the FDI and dFDI problems (both in their observer-based and in their standard versions), which can be obtained in this behavioral framework, let us consider the following concluding example.

Example 3. Consider a state-space model (6.3) with $A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$, $C = [\, 0 \quad 1 \,]$, $B_d = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$, $D_d = [\, 0 \,]$, $B_f = \left[\begin{array}{c} 0 \\ a \end{array}\right]$, $D_f = [\, 1 - a \,]$, $a \in \mathbb{R}$, and assume that no controlled input acts on the system.

Let us first consider the case when disturbances may be neglected (this amounts to assuming that B_d and D_d are replaced by empty matrices). For determining whether an observer-based FDI exists, we evaluate

$$\Gamma_{x,f}(z) = \begin{bmatrix} z & -1 & 0 \\ 0 & z & -a \\ 0 & 1 & 1-a \end{bmatrix}, \ \Phi(z) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and since $\det \Gamma_{x,f}(z) = z[z(1-a)+a]$, $\Gamma_{x,f}(z)$ is monomic (and hence the problem is solvable) if and only if $a = 0$ or $a = 1$. Notice, however, that for $a = 0, 1$, $\Gamma_{x,f}$ is square monomic but not unimodular, and hence EOs are not available.

Also, $\Delta(z) = \Gamma_{x,f}(z), L_1(z) = \Phi(z), L_0(z) = \emptyset$. So, the DBO transfer matrix is uniquely determined as $\hat{W}(z) = \Gamma_{x,f}^{-1}(z)\Phi(z)$.

If $a = 0$, then $\hat{W}(z) = [\,0 \quad 0 \quad 1\,]^T$, which corresponds to $\hat{\mathbf{x}}(t) = 0$, $\hat{\mathbf{f}}(t) = \mathbf{y}(t)$. This is a causal DBO, and in fact $[\,\Gamma_{x,f}(z) \quad -\Phi(z)\,]$ is row-reduced, with $\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ of full column rank.

On the other hand, for $a = 1$, $\hat{W}(z) = [\,z^{-1} \quad 1 \quad z\,]^T$, $\hat{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{y}(t-1) \\ \mathbf{y}(t) \end{bmatrix}$, $\hat{\mathbf{f}}(t) = \mathbf{y}(t+1)$, which represents a non-causal DBO, in agreement with the fact that $\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is now not of full column rank.

If we are interested in estimating the fault $\mathbf{f}$ alone (namely we search for an standard FDI), we can choose as a left coprime matrix fraction description $D_L(z)^{-1}N_L(z)$ of $C(zI_{\mathbf{x}} - A)^{-1}$ the one associated with

$$D_L(z) = [\,z\,], \qquad N_L(z) = [\,0 \quad 1\,].$$

Consequently, $[\,\Gamma_f(z) \quad -\Phi(z)\,] = [\,z(1-a)+a \quad -z\,]$. As before, a necessary condition for the problem solvability is that the real parameter a takes only the values 0 or 1.

If $a = 0$, then $\hat{W}(z) = 1$, i.e. $\hat{\mathbf{f}}(t) = \mathbf{y}(t)$, which represents a causal DBO (but not an EO). In fact, $\Gamma_0 = [1]$ is trivially of full column rank. On the other hand, if $a = 1$, then $\hat{W}(z) = z$ and $\hat{\mathbf{f}}(t) = \mathbf{y}(t+1)$, which is a non-causal EO (indeed, in this case, $\Gamma_0 = [0]$).

In this specific example, therefore, estimating $(\mathbf{x}, \mathbf{f})$ or $\mathbf{f}$ alone lead to the same result for $\hat{\mathbf{f}}(t)$, but in general the case can occur that $(\mathbf{x}, \mathbf{f})$ cannot be estimated (for instance, if the pair (A, C) does not correspond to a reconstructible system) while $\mathbf{f}$ can.

Now, we consider the disturbed FDI problem. For the observer-based dFDI, we have $[\,H_{iB} \quad H_{iD}\,] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so that $\Gamma_{x,f}(z) = \begin{bmatrix} 0 & z & -a \\ 0 & 1 & 1-a \end{bmatrix}$. As this matrix is not of full column rank, the estimation problem for the pair $(\mathbf{x}, \mathbf{f})$ is not solvable.

We may now try to estimate $\mathbf{f}$ alone. This requires to determine an MLA $H_i(z)$ of the polynomial matrix $R_i(z) = \begin{bmatrix} z & -1 & 1 \\ 0 & z & 0 \\ 0 & 1 & 0 \end{bmatrix}$. A possible choice is $H_i(z) = [\,0 \quad 1 \quad -z\,]$.

Correspondingly, we get $[\,\Gamma_f(z) \quad -\Phi(z)\,] = [\,-[a+z(1-a)] \quad -z\,]$. Therefore the problem is solvable, again, only for $a = 0, 1$.

If $a = 0$, $\Gamma_0 = [-1]$ and, in fact, $\hat{W}(z) = 1$ represents a causal DBO (but not an EO). For $a = 1$, $\Gamma_0 = [0]$, a causal dDBO does not exist, however $\hat{W}(z) = z$ represents a non-causal EO.

It might be somewhat surprising that estimating $(\mathbf{x}, \mathbf{f})$ is impossible, even though (A, C) is a reconstructible pair and an estimate for $\mathbf{f}$ (when $a = 0, 1$) can be obtained. Actually, there is no contradiction, since once $\mathbf{f}(t)$ has been estimated, it plays the role of a measured variable, in the estimation problem for $\mathbf{x}$. However, if we assume $\mathbf{w}_r = \mathbf{x}$, $\mathbf{w}_m = [\,\mathbf{y}^T \quad \mathbf{u}^T \quad \mathbf{f}^T\,]^T$, $\mathbf{w}_i = \mathbf{d}$, we have to solve an UIO design problem, for which standard reconstructibility is only a necessay condition. In fact, by resorting to this variable partition, we have

$$R_i(z) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; R_m(z) = \begin{bmatrix} 0 & 0 \\ 0 & a \\ 1 & a-1 \end{bmatrix}, \; R_r(z) = \begin{bmatrix} z & -1 \\ 0 & z \\ 0 & 1 \end{bmatrix},$$

and by choosing $H_i(z) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ we obtain $\Gamma(z) = \begin{bmatrix} 0 & z \\ 0 & 1 \end{bmatrix}$, which is not of full column rank. Therefore this problem is unsolvable, too, in perfect agreement with the previous conclusions.

Remark. To conclude, it is worthwhile noticing that all the characterizations provided in this section have been expressed in terms of polynomial matrices which never involve the two constant matrices B_u and D_u which weight the controlled input contribution to the system dynamics. This result is well-known and very intuitive, as one can always compensate the effect of the controlled input when trying to estimate the other variables.

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