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Integral Equations and Operator Theory

Symmetries of 2D discrete linear systems

Paula Rocha[†], Paolo Vettori[†], and Jan C. Willems[‡]

Abstract. Static symmetries of linear shift invariant 1D systems had been thoroughly investigated, and the resulting theory is now a well-established topic. Nevertheless, only partial results were available for the multidimensional case, since the extension of the theory for 1D systems proved not to be straightforward, as usual. Actually, a non trivial regularity assumption and also restrictions on the set of allowed symmetries had to be imposed. In this paper we show how it is possible to overcome these difficulties using a particular canonical form for 2D discrete linear systems.

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1. Introduction

After Noether's Theorem, the importance of the analysis of symmetries in the study of dynamical systems became more than evident. Indeed, the study of symmetries is a very important tool to comprehend and clarify many intrinsic properties of physical systems. In fact, the knowledge of the symmetries of a dynamical system often leads to a simplification of its mathematical description.

A basic but illuminating example is a dynamical system given by a collection of equal particles. As is well-known, the behavior of this kind of system can be described by the dynamics of their center of mass. The same simplified representation can be achieved without any consideration of its physical nature, just by

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analyzing the symmetry law which is here given by the invariance with respect to the exchange of particles.

The first and essential aim of our research is precisely to characterize through canonical forms the special structures of system representations that are induced by the underlying symmetries.

As regards linear systems, in the framework of the behavioral approach, the first results were presented by Fagnani and Willems [2, 3, 4] for regular behaviors. However, while in the 1D case every behavior is regular, this condition is not necessarily fulfilled by generic multidimensional systems. Moreover, only a restricted class of symmetries was allowed in the study of real-valued trajectories.

In Section 2 we introduce the class of dynamical discrete systems we deal with in this paper: the *behaviors*, sets of trajectories which can be described by a finite number of linear (ordinary or partial) difference equations. To introduce the notion of symmetric behavior, we recall the necessary concepts regarding symmetries and their representations in Section 3.

In Section 4, we provide a canonical way to write the equations which define a behavior. This canonical form is then used in Section 5 to show how the results about symmetric 1D behaviors can be extended to the 2D case without the need of any further assumption.

We conclude by showing how each 2D behavior that exhibits some symmetry, can be described by a set of equations that reflect the intrinsic structure of its symmetry.

2. Behavioral systems

We briefly recall the notion of dynamical system in the behavioral approach [12, 13] and some results that we will need later on. According to this approach a dynamical system is defined as a triple

$$\Sigma = (T, W, \mathcal{B}),$$

where T denotes the domain, W the signal space, and $\mathcal{B}$, which is a subset of $W^T = \{w : T \to W\}$, represents the set of trajectories which are allowed to occur by the definition of the system. This is called the behavior of the system. We will consider only discrete, linear, complete and shift-invariant systems. This amounts to say that: the domain is $T = \mathbb{Z}^n$, $n = 1, 2$, i.e., for 1D and 2D systems, respectively; W and $\mathcal{B}$ are vector spaces over $\mathbb{K}$ (the real or the complex field); the dimension of W is finite and $\mathcal{B}$ is closed in the topology of pointwise convergence; for any trajectory $w \in \mathcal{B}$ and $\tau \in T$ we have $\sigma^\tau w \in \mathcal{B}$ where σ^τ is the shift operator such that $(\sigma^\tau w)(t) = w(t + \tau)$.

Remark 2.1. The multi-index notation is here used to handle 1D and 2D systems in a unified way. So, if $\tau = (\tau_1, \tau_2) \in \mathbb{Z}^2$, then $\sigma^\tau = \sigma_1^{\tau_1}\sigma_2^{\tau_2}$ where σ_i are the (commuting) partial shift operators on the i-th component. Analogous is the notation for monomials: $s^\tau = s_1^{\tau_1}s_2^{\tau_2}$. In the following sections we shall be more explicit to

distinguish between 1D and 2D systems. In particular, note that for 2D systems,

$$\sigma^{\tau} w(t) = \sigma_1^{\tau_1}\sigma_2^{\tau_2} w(t_1, t_2) = w(t_1 + \tau_1, t_2 + \tau_2).$$

This way of defining the dynamics leads to a representation free theory, i.e., to a theory which does not require a specific model for the system equations, as for example the input/state/output model of classical systems theory. Indeed, it is possible to characterize the trajectories of a dynamical system in many ways. Those studied most are the kernel and image representations, which define the behavior as the kernel and, respectively, as the image of a suitable operator.

To be more precise, we introduce operators on trajectories $w \in W^T$ of the form $\sum_{i\in\mathcal{I}} R_i w(t+i)$, where $\mathcal{I} \subseteq T$ is a finite subset of the domain and R_i are constant matrices with suitable dimensions. We may write

$$\sum_{i\in\mathcal{I}} R_i w(t+i) = \sum_{i\in\mathcal{I}} R_i \sigma^i w(t) = R(\sigma, \sigma^{-1}) w(t), \tag{2.1}$$

where $R(s, s^{-1})$ is a univariate or bivariate Laurent polynomial matrix.

Using this notation, a behavior $\mathcal{B} \subseteq (\mathbb{K}^q)^T$ is defined by a kernel representation if there exists $R(s, s^{-1}) \in \mathbb{K}[s, s^{-1}]^{p\times q}$, for some $p \in \mathbb{N}$, such that

$$\mathcal{B} = \ker R(\sigma, \sigma^{-1}) = \{w \in (\mathbb{K}^q)^T : R(\sigma, \sigma^{-1}) w = 0\},$$

i.e., if $\mathcal{B}$ is the set of solutions of a matrix difference equation represented by the difference operator $R(\sigma, \sigma^{-1})$. Note that, by shift-invariance of the system, we may always suppose that the summation in (2.1) is made over indices with non-negative components and therefore we assume without loss of generality that kernel representations are polynomial matrices $R(s) \in \mathbb{K}[s]^{p\times q}$.

This representation is very general as the following theorem [8] states.

Theorem 2.2. *Every nD behavior admits a kernel representation.*

Moreover, it is possible to establish a deep relation between a behavior and any matrix providing its kernel representation that leads, for instance, to the following fundamental result about inclusion of behaviors [8, Thm. 2.61].

Theorem 2.3. *The condition* $\ker R_1(\sigma) \subseteq \ker R_2(\sigma)$ *holds if and only if there exists a Laurent polynomial matrix* $X(s, s^{-1})$ *such that* $X(s, s^{-1}) R_1(s) = R_2(s)$.

In this theorem, $X(s, s^{-1})$ has to be a Laurent polynomial matrix. Consider, for instance, the case $R_1(s) = s$ and $R_2(s) = 1$.

If $X(s, s^{-1})$ in Theorem 2.3 is unimodular, i.e., it has a polynomial inverse, say $Y(s, s^{-1})$, then $Y(s, s^{-1}) R_2(s) = R_1(s)$ and so $\ker R_1(\sigma) = \ker R_2(\sigma)$. We say that in this case R_1 and R_2 are equivalent representations. Additional hypotheses are needed to prove the converse statement.

Corollary 2.4. *If* $\ker R_1(\sigma) = \ker R_2(\sigma)$ *and both* $R_1(s)$ *and* $R_2(s)$ *have full row rank, then* $X(s, s^{-1}) R_1(s) = R_2(s)$ *with* $X(s, s^{-1})$ *Laurent unimodular matrix.*

Note that while every 1D behavior that can be defined by a kernel representation admits a full row rank kernel representation too, called *minimal*, this does not hold for 2D behaviors. Behaviors that have a full row rank kernel representation are called *regular*.

3. Symmetries and their representations

To define in a proper way the class of symmetries we will be dealing with, we first have to introduce some notions of representation theory. For more details we refer the reader to [10].

Given a finite dimensional vector space $\mathcal{W}$ over the field $\mathbb{K}$, we will denote by $\mathrm{GL}(\mathcal{W})$ the group of $\mathbb{K}$-isomorphisms of $\mathcal{W}$.

Definition 3.1. A representation of the group G on the vector space $\mathcal{W}$ is a group homomorphism

$$\rho : G \to \mathrm{GL}(\mathcal{W}),\ g \mapsto \rho_g.$$

The degree of a representation is defined by $\deg \rho = \dim \mathcal{W}$.

Remark 3.2. We suppose in this paper that G is equipped with a topology that makes it into a Hausdorff compact topological group. Even if not mentioned, every representation will be assumed to be continuous. Thus, if G is finite, the discrete topology is employed, which trivially ensures continuity.

Note that, according to the definition, ρ_g is an isomorphism of $\mathcal{W}$ onto itself for every g. With a little abuse of notation we may implicitly assume that some basis of $\mathcal{W}$ has already been fixed and therefore we will always identify ρ_g with its matrix representation.

Definition 3.3. Given a representation ρ on $\mathcal{W}$, a subspace $\mathcal{U} \subseteq \mathcal{W}$ is ρ-symmetric if $\rho\mathcal{U} \subseteq \mathcal{U}$, i.e., $\rho_g\mathcal{U} \subseteq \mathcal{U}$ for any $g \in G$.

Note that when $\mathcal{U}$ is a ρ-symmetric subspace of $\mathcal{W}$, the restrictions of ρ_g to $\mathcal{U}$ are isomorphisms of $\mathcal{U}$ and thus $\rho|_{\mathcal{U}}$ is itself a representation which is called subrepresentation of ρ. It can be proved that in the case of finite-degree representations there exists another ρ-symmetric subspace $\mathcal{V}$ such that $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$. We write also $\rho = \rho|_{\mathcal{U}} \oplus \rho|_{\mathcal{V}}$. A representation which does not admit proper symmetric subspaces, that is to say subrepresentations, is called irreducible. The decomposition of $\mathcal{W}$ into minimal symmetric subspaces gives then rise to a decomposition of ρ into irreducible subrepresentations. This decomposition becomes unique only if we identify different irreducible representations which are isomorphic — η^1 is isomorphic to η^2, or also $\eta^1 \cong \eta^2$, if there exists an isomorphism $\pi : \mathcal{W} \to \mathcal{W}$ such that $\pi\eta^1_g = \eta^2_g\pi$ for every g.

Eventually, the standard way to write such a decomposition of ρ into subrepresentations is

$$\rho = m_1\eta^1 \oplus m_2\eta^2 \oplus \cdots \oplus m_r\eta^r, \tag{3.1}$$

where the notation $m_i\eta^i$ stands for the direct sum of m_i copies of subrepresentations isomorphic to η^i.

Example. Consider the symmetric group $S_3 = \{e, (12), (13), (23), (123), (132)\}$, i.e., the group of all permutations of three elements. The most natural representation of S_3 is given by the 3×3 matrix group generated by

$$A = \rho_{(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \rho_{(123)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Note that $\rho_e = I = A^2 = B^3$, $\rho_{(13)} = AB$, $\rho_{(23)} = BA$, and $\rho_{(132)} = B^2$.

As it can be easily verified, the subspace $\mathcal{U} \subseteq \mathbb{R}^3$ generated by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ is ρ-symmetric and $\rho|_{\mathcal{U}} = 1$ is the corresponding (trivial) subrepresentation. On the other hand, for any direct summand $\mathcal{V}$ of $\mathcal{U}$, $\rho = \rho|_{\mathcal{U}} \oplus \rho|_{\mathcal{V}}$ is a decomposition of ρ into irreducible subrepresentations.

To write the representation $\eta = \rho|_{\mathcal{V}}$ in matrix form, it is necessary to fix a basis of $\mathcal{V}$. If, e.g. , $\mathcal{V} = \left\langle \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\rangle$, it follows that η is generated by

$$\eta_{(12)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \eta_{(123)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}. \tag{3.2}$$

The theory that was just exposed can be used to define and analyze symmetries of dynamical systems. A quite general approach would be to choose $\mathcal{W} = W^T$, the set of trajectories. Nevertheless, in this paper we deal with a simpler class of symmetries, called *static symmetries*, since they act on the coordinates of the trajectories, i.e., $\mathcal{W} = W$, the signal space of the behavior. Therefore, the representations are homomorphism of the type

$$\rho : G \to \mathrm{GL}(W).$$

In this case, if $W = \mathbb{K}^q$, a representation of G is a family of invertible matrices $\rho_g \in \mathbb{K}^{q \times q}$ that act on trajectories as one would expect. Actually, for any $w \in W^T$, $\rho_g w \in W^T$ is such that

$$\rho_g w : T \to W,\ t \mapsto \rho_g(w(t))\ \forall g \in G.$$

Definition 3.4. Given a representation ρ on W, a behavior $\mathcal{B} \subseteq W^T$ is ρ-symmetric if

$$\rho\mathcal{B} \subseteq \mathcal{B}.$$

Note that, since ρ_g are isomorphisms, with $\rho_g^{-1} = \rho_{g^{-1}}$, a behavior is ρ-symmetric if and only if $\rho\mathcal{B} = \mathcal{B}$

Remark 3.5. The class of symmetries we are dealing with is not the most general. There are many simple symmetries that are group actions [5] but not representations. One example is given by time-reversal symmetries (if $w(t) \in \mathcal{B}$ then also $w(-t) \in \mathcal{B}$), studied for 1D systems in [2, 11]. Also shift-invariance is a symmetry. Indeed, the behaviors we are considering are symmetric with respect to the action of the group $\mathbb{Z}$ given by the shift operator σ^τ.

4. A canonical form for 2D behaviors

Our aim is to provide a canonical structure for kernel representations of 2D behaviors, that will be useful in the analysis of their symmetries. This canonical form was first introduced as *forward computational representation* in [9] for the analysis of 2D linear discrete systems.

In this paper we illustrate some of the already investigated properties along with new ones that are necessary to our purposes.

The idea is to obtain a representation of a behavior $\mathcal{B}$ as a product of two factors which depend only on one indeterminate. As we will see, this permits to derive properties of $\mathcal{B}$ using theorems about 1D behaviors (namely, Corollary 2.4).

The factors of canonical representations mentioned above are constructed recursively by computing minimal representations of properly defined 1D behaviors. However, just to clarify the whole process and to give some first important definitions, we begin by assuming that a kernel representation of $\mathcal{B}$ is already given.

Let $\mathcal{B} = \ker R(\sigma_1, \sigma_2)$, where $R(s_1, s_2) \in \mathbb{K}[s_1, s_2]^{p\times q}$ is a polynomial matrix in two indeterminates, and let $N-1$ be its degree in s_2. We may write

$$R(s_1, s_2) = \sum_{j=0}^{N-1} R_j(s_1)s_2^j = \begin{bmatrix} R_0(s_1) & R_1(s_1) & \cdots & R_{N-1}(s_1) \end{bmatrix} \begin{bmatrix} I \\ Is_2 \\ \vdots \\ Is_2^{N-1} \end{bmatrix}, \quad (4.1)$$

where I are $q \times q$ identity matrices. If we call $R^N(s) = \begin{bmatrix} R_0(s) & \cdots & R_{N-1}(s) \end{bmatrix}$ and $\Phi^N(s) = \begin{bmatrix} I & \cdots & Is^{N-1} \end{bmatrix}^\top$, then (4.1) is a factorization of $R(s_1, s_2)$ into polynomial factors in one indeterminate

$$R(s_1, s_2) = R^N(s_1)\Phi^N(s_2), \quad (4.2)$$

where $R^N(s) \in \mathbb{K}[s]^{p\times Nq}$ is uniquely determined by $R(s_1, s_2)$.

We will show how to build a canonical representation $C^N(s_1)\Phi^N(s_2)$ of $\mathcal{B}$ where the matrix $C^N(s)$, that is equivalent to $R^N(s)$, has a special nested structure and therefore can be defined recursively. In order to do this, it is necessary to shed some light on the relation between $\mathcal{B}$ and the 1D behavior $\ker R^N(\sigma)$.

First of all note that $w \in \mathcal{B}$ if and only if

$$R(\sigma_1, \sigma_2)w(t_1, t_2) = R^N(\sigma_1)\Phi^N(\sigma_2)w(t_1, t_2) = R^N(\sigma_1) \begin{bmatrix} w(t_1, t_2) \\ w(t_1, t_2 + 1) \\ \vdots \\ w(t_1, t_2 + N - 1) \end{bmatrix} = 0.$$

So, if we partition the trajectories of $\ker R^N(\sigma)$ into N blocks, $w^j : \mathbb{Z} \to \mathbb{K}^q$, $j = 0, \ldots, N-1$, then by the shift invariance of $\mathcal{B}$ we may write that

$$w \in \mathcal{B} \Rightarrow R^N(\sigma) \begin{bmatrix} w^0 \\ \vdots \\ w^{N-1} \end{bmatrix} = 0 \text{ where } w^j(t) = w(t, j)\, \forall j = 0, \ldots, N-1. \quad (4.3)$$

This fact tells us that, roughly speaking, trajectories in the restriction of $\mathcal{B}$ to the domain $\mathbb{Z} \times \{0, 1, \ldots, N-1\}$ (i.e. , made up of N consecutive *horizontal lines* of $\mathcal{B}$), belong to $\ker R^N(\sigma)$. This shows the opportunity of the following definition.

Definition 4.1. Given a 2D behavior $\mathcal{B}$, we define

$$\mathcal{B}^i = \left\{ \begin{bmatrix} w^0 \\ \vdots \\ w^{i-1} \end{bmatrix} : \exists w \in \mathcal{B} \text{ such that } w^j(t) = w(t,j)\, \forall j = 0, \ldots, i-1 \right\}.$$

Theorem 4.2. *The sets $\mathcal{B}^i$ are 1D behaviors* [7].

Remark 4.3. Note that in general $\mathcal{B}^N \subseteq \ker R^N(\sigma)$ is not an equality. Indeed, let $R(s_1, s_2) = s_2$. So, $\mathcal{B} = \{0\}$, hence $\mathcal{B}^2 = \{0\}$ too, but $\ker R^2(s) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} \neq \{0\}$.

In Section 4.1 we show how it is possible to take advantage of kernel representations of $\mathcal{B}^i$ in the construction of a particular kernel representation of $\mathcal{B}^{i+1}$. These *canonical* representations of the behaviors $\mathcal{B}^i$ will be the main tool for defining canonical representations of $\mathcal{B}$ in Section 4.2.

4.1. Iterative construction of representations of $\mathcal{B}^i$

In this section we show how to compute minimal representations of $\mathcal{B}^i$ in an efficient way. More precisely, the structure of a canonical form relative to $\mathcal{B}^i$ will be defined by induction. Actually, in the following proposition, we begin by showing how to construct a kernel representations of $\mathcal{B}^{i+1}$ that is based on a representation of $\mathcal{B}^i$.

Proposition 4.4. *$\mathcal{B}^i = \ker C^i(\sigma)$ if and only if there exist a full row rank matrix $C_{i+1}(s)$ and a matrix $T_{i+1}(s)$ such that $\mathcal{B}^{i+1} = \ker C^{i+1}(\sigma)$ with*

$$C^{i+1}(s) = \begin{bmatrix} C^i(s) & 0 \\ -T_{i+1}(s) & C_{i+1}(s) \end{bmatrix}. \tag{4.4}$$

Proof. Assume first that $\mathcal{B}^i = \ker C^i(\sigma)$. Given any kernel representation $\tilde{C}^{i+1}(s)$ of $\mathcal{B}^{i+1}$, it is always possible to put its last q columns in Hermite form just using elementary operations on the rows. In other words, there exists a unimodular matrix $U(s)$ such that

$$U(s)\tilde{C}^{i+1}(s) = \begin{bmatrix} \tilde{C}^i(s) & 0 \\ -T_{i+1}(s) & C_{i+1}(s) \end{bmatrix}, \tag{4.5}$$

where $C_{i+1}(s)$ has q columns and full row rank.

Moreover, it follows from Definition 4.1 that the trajectories of $\mathcal{B}^i$ are restrictions (to the first iq components) of some trajectory of $\mathcal{B}^{i+1}$. As a consequence, $\mathcal{B}^{i+1} \subseteq \mathcal{B}^i \times (\mathbb{K}^q)^{\mathbb{Z}}$ and thus $\mathcal{B}^{i+1} \subseteq \ker \begin{bmatrix} C^i(\sigma) & 0 \end{bmatrix}$. So, since (4.5) is still a kernel representation of $\mathcal{B}^{i+1}$, we can write

$$\mathcal{B}^{i+1} = \ker \begin{bmatrix} C^i(\sigma) & 0 \\ \tilde{C}^i(\sigma) & 0 \\ -T_{i+1}(\sigma) & C_{i+1}(\sigma) \end{bmatrix}.$$

However, since $\tilde{C}^i(\sigma)w = 0$ for every $w \in \mathcal{B}^i$, the operator $\tilde{C}^i(\sigma)$ does not impose any further restriction on the trajectories of $\mathcal{B}^{i+1}$. Hence it can be deleted from the kernel representation to obtain the equivalent representation (4.4).

Let us suppose now that (4.4) holds, with $C_{i+1}(s)$ having full row rank. Clearly $\mathcal{B}^i \subseteq \ker C^i(\sigma)$. Indeed,

$$w \in \mathcal{B}^i \;\Leftrightarrow\; \exists w^* : \begin{bmatrix} w \\ w^* \end{bmatrix} \in \mathcal{B}^{i+1} \;\Rightarrow\; C^i(\sigma)w = 0.$$

On the other hand, let $w \in \ker C^i(\sigma)$. Since $C_{i+1}(s)$ has full row rank, $C_{i+1}(\sigma)$ is a surjective operator and there exists a w^* such that $T_{i+1}(\sigma)w = C_{i+1}(\sigma)w^*$. Hence $\left[\begin{smallmatrix} w \\ w^* \end{smallmatrix}\right] \in \mathcal{B}^{i+1}$ and so $w \in \mathcal{B}^i$. Thus $\mathcal{B}^i = \ker C^i(\sigma)$. □

In Proposition 4.4 nothing is said about the minimality of $C^i(s)$. However, the statement assures that, once $C^i(s)$ is minimal, $C^{i+1}(s)$ is minimal too. So, the recursive scheme assures that $C^i(s)$ is minimal for any i if $C_1(s) = C^1(s)$ is a minimal representation of $\mathcal{B}^1$, the restriction of $\mathcal{B}$ to one horizontal line (the axis, for example).

Definition 4.5. We call canonical representation of $\mathcal{B}^i$ a lower triangular block matrix (each block having q columns)

$$C^i(s) = \begin{bmatrix} C_1(s) & & 0 \\ & \ddots & \\ * & & C_i(s) \end{bmatrix} \tag{4.6}$$

such that $\mathcal{B}^i = \ker C^i(\sigma)$ is a minimal representation.

To state the properties of this canonical form, a very important role will be played by the family of behaviors defined as follows:

$$\mathcal{B}_i = \left\{ w \in (\mathbb{K}^q)^{\mathbb{Z}} : \begin{bmatrix} 0 \\ w \end{bmatrix} \in \mathcal{B}^i \right\}. \tag{4.7}$$

These can be thought of as the sets of 'line-values' that in $\mathcal{B}$ follow $i-1$ null 'lines'.

Note that $\mathcal{B}_1 = \ker C_1(\sigma)$ by definition. However, this is a general fact: given a canonical representation (4.6) of $\mathcal{B}^i$, the behavior $\mathcal{B}_j$ has minimal representation $C_j(s)$ for any $j = 1, \ldots, i$. Indeed, by the form of $C^j(s)$ given in (4.4),

$$w \in \mathcal{B}_j \;\Leftrightarrow\; C^j(\sigma)\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ C_j(\sigma)w \end{bmatrix} = 0 \;\Leftrightarrow\; w \in \ker C_j(\sigma). \tag{4.8}$$

Also the converse result holds, as the following statement shows.

Lemma 4.6. *The matrix $C^i(s)$ defined by* (4.6) *is a canonical representation of $\mathcal{B}^i$ if and only if $C_j(s)$ is a minimal representation of $\mathcal{B}_j$ for any $j = 1, \ldots, i$.*

Proof. We already showed that the sufficiency holds. To prove the necessary condition we proceed by induction.

If we let $C^1(s) = C_1(s)$, we only have to prove that if $\mathcal{B}^j = \ker C^j(\sigma)$ and $\mathcal{B}_{j+1} = \ker C_{j+1}(\sigma)$, then there exists $T_{j+1}(s)$ such that $\mathcal{B}^{j+1} = \ker C^{j+1}(\sigma)$, with the matrix $C^{j+1}(s)$ as in (4.4).

By Proposition 4.4, there exist matrices $\tilde{T}_{j+1}(s)$ and $\tilde{C}_{j+1}(s)$ with full row rank such that

$$\mathcal{B}^{j+1} = \ker \begin{bmatrix} C^j(\sigma) & 0 \\ -\tilde{T}_{j+1}(\sigma) & \tilde{C}_{j+1}(\sigma) \end{bmatrix}.$$

By (4.8), $\mathcal{B}_{j+1} = \ker \tilde{C}_{j+1}(\sigma)$. Hence there must exist a unimodular $U(s)$ such that $C_{j+1}(s) = U(s)\tilde{C}_{j+1}(s)$. So, if we put $T_{j+1}(s) = U(s)\tilde{T}_{j+1}(s)$, we get that

$$C^{j+1}(s) = \begin{bmatrix} C^j(s) & 0 \\ -T_{j+1}(s) & C_{j+1}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & U(s) \end{bmatrix} \begin{bmatrix} C^j(\sigma) & 0 \\ -\tilde{T}_{j+1}(\sigma) & \tilde{C}_{j+1}(\sigma) \end{bmatrix},$$

which provides the claimed canonical representation of $\mathcal{B}^{j+1}$, $j = 1, \ldots, i$. □

This fact permits us to state the following important theorem that, restricting to a particular class of unimodular matrices, specializes Corollary 2.4 to the case of canonical representations.

Definition 4.7. Given $\mathcal{B}^i$ or, equivalently, a representation (4.6), the set of lower triangular unimodular $i \times i$ block matrices with block sizes $n_l \times n_k$, where n_j is the number of rows of any minimal representations of $\mathcal{B}_j$, is denoted by $\mathcal{U}^i$.

Theorem 4.8. *Let $C^i(s)$ be one canonical representation of $\mathcal{B}^i$. Then $\tilde{C}^i(s)$ is a canonical representation of $\mathcal{B}^i$ if and only if $\tilde{C}^i(s) = U^i(s)C^i(s)$ for some $U^i(s) \in \mathcal{U}^i$.*

Proof. On the diagonal of any matrix $U^i(s) \in \mathcal{U}^i$ there are necessarily unimodular matrices $U_j(s)$, $j = 1, \ldots, i$. Therefore, if $C^i(s)$ has the form (4.6), the diagonal blocks of $U^i(s)C^i(s)$, i.e., $U_j(s)C_j(s)$, still have full row rank and so $U^i(s)C^i(s)$ is a canonical representation of $\mathcal{B}^i$, by Definition 4.5.

Suppose now that $\tilde{C}^i(s)$ is a canonical representation of $\mathcal{B}^i$, with diagonal blocks $\tilde{C}_j(s)$, $j = 1, \ldots, i$. By Lemma 4.6, both the matrices $C_j(s)$ and $\tilde{C}_j(s)$ are minimal representations of the behavior $\mathcal{B}_j$, hence they have the same dimensions and, by Corollary 2.4, there exist unimodular matrices $U_j(s)$ such that $\tilde{C}_j(s) = U_j(s)C_j(s)$. The question is whether there exists a matrix $U^i(s) \in \mathcal{U}^i$ with diagonal blocks $U_j(s)$ such that $\tilde{C}^i(s) = U^i(s)C^i(s)$. We prove it by induction on j. The fact is trivially true for $j = 1$. So let us suppose that $\tilde{C}^j(s) = U^j(s)C^j(s)$ with $U^j(s) \in \mathcal{U}^j$. We want to find $V_{j+1}(s)$ such that $U^{j+1}(s) = \left[\begin{smallmatrix} U^j(s) & 0 \\ -V_{j+1}(s) & U_{j+1}(s) \end{smallmatrix}\right]$. Explicitly,

$$\tilde{C}^{j+1}(s) = \begin{bmatrix} \tilde{C}^j(s) & 0 \\ -\tilde{T}_{j+1}(s) & \tilde{C}_{j+1}(s) \end{bmatrix} = \begin{bmatrix} U^j(s) & 0 \\ -V_{j+1}(s) & U_{j+1}(s) \end{bmatrix} \begin{bmatrix} C^j(s) & 0 \\ -T_{j+1}(s) & C_{j+1}(s) \end{bmatrix}.$$

This condition, since all the other equations are satisfied, is equivalent to $\tilde{T}_{j+1}(s) = V_{j+1}(s)C^j(s) + U_{j+1}(s)T_{j+1}(s)$. Now, for every $w \in \mathcal{B}^j$ there exist w^* such that $\left[\begin{smallmatrix} w \\ w^* \end{smallmatrix}\right] \in \mathcal{B}^{j+1}$. Therefore,

$$\tilde{T}_{j+1}(\sigma)w = \tilde{C}_{j+1}(\sigma)w^* = U_{j+1}(\sigma)C_{j+1}(\sigma)w^* = U_{j+1}(\sigma)T_{j+1}(\sigma)w.$$

This means that $(\tilde{T}_{j+1}(\sigma) - U_{j+1}(\sigma)T_{j+1}(\sigma))w = 0$, i.e., that

$$\ker C^j(\sigma) = \mathcal{B}^j \subseteq \ker(\tilde{T}_{j+1}(\sigma) - U_{j+1}(\sigma)T_{j+1}(\sigma)),$$

which, by Theorem 2.3, implies that there exists a matrix $V_{j+1}(s)$ such that $\tilde{T}_{j+1}(s) - U_{j+1}(s)T_{j+1}(s) = V_{j+1}(s)C^j(s)$, thus proving the theorem. □

4.2. Canonical representations of a 2D behavior

We use now the canonical form introduced in Definition 4.5 to give a canonical form for 2D systems.

Definition 4.9. The representation $\mathcal{B} = \ker C(\sigma_1, \sigma_2)$ of a 2D behavior is canonical (with respect to the second variable) if the degree N in s_2 of $C(s_1, s_2)$ is minimal over all kernel representations of $\mathcal{B}$ and in the factorization (4.2),

$$C(s_1, s_2) = C^N(s_1)\Phi^N(s_2), \tag{4.9}$$

the matrix $C^N(s)$ is a canonical representation (4.6).

Remark 4.10. A similar construction would lead to the definition of a canonical form with respect to the first variable but we will not use it in this paper.

Note that, by its definition, when $\mathcal{B}$ is given by a canonical representation $C^N(s_1)\Phi^N(s_2)$, then $\mathcal{B}^N = \ker C^N(\sigma)$. The canonical representation of $\ker \sigma_2$, given as example in Remark 4.3, is simply $C(s_1, s_2) = 1$.

Theorem 4.11. *Every 2D behavior $\mathcal{B}$ admits a canonical representation.*

Proof. By Theorem 2.2, there exist a kernel representations of $\mathcal{B}$. Let $R(s_1, s_2)$ have minimal degree N in the second variable and factorize it as in Equation (4.1), $R(s_1, s_2) = R^N(s_1)\Phi^N(s_2)$. Then, given minimal representations of the 1D behaviors $\mathcal{B}_i$, Proposition 4.4 and Lemma 4.6 show how to construct canonical representations $C^i(s)$ of $\mathcal{B}^i$ for any $i = 1, \ldots, N$.

We now show that $C(s_1, s_2) = C^N(s_1)\Phi^N(s_2)$ is a canonical representation of $\mathcal{B}$. Indeed, as we saw in Remark 4.3, $\ker C^N(\sigma) = \mathcal{B}^N \subseteq \ker R^N(\sigma)$. So,

$$\ker C^N(\sigma_1)\Phi^N(\sigma_2) \subseteq \ker R^N(\sigma_1)\Phi^N(\sigma_2) = \mathcal{B}.$$

On the other hand, if $w \in \mathcal{B}$ then $\tilde{w}_k(h) = \Phi^N(\sigma_2)w(h, k)$ belongs to $\mathcal{B}^N$ for any $k \in \mathbb{Z}$ and thus

$$\mathcal{B} \subseteq \ker C(\sigma_1, \sigma_2) = \ker C^N(\sigma_1)\Phi^N(\sigma_2).$$

Hence, since $\mathcal{B} = \ker C(\sigma_1, \sigma_2)$, the proof is concluded. □

Another fundamental fact regards the relation between two canonical representations, which is clarified by the following theorem.

Theorem 4.12. *Given a canonical representation $C(s_1, s_2) = C^N(s_1)\Phi^N(s_2)$ of the 2D behavior $\mathcal{B}$, any other canonical representation is equal to $U(s_1)C(s_1, s_2)$ for some $U(s) \in \mathcal{U}^N$, where $\mathcal{U}^N$ depends on $C^N(s)$ as in Definition 4.7.*

Note that using this class of canonical representations, we overcome the two problems occurring in the equivalence of 2D behaviors. Indeed, two matrices that are representations of the same behavior are generally not unimodularly equivalent. However, by Theorem 4.12, this holds always true for canonical representations of 2D systems and, moreover, the unimodular matrix is a polynomial matrix in one indeterminate.

These facts will be very useful in the proof of the main result about symmetries in Section 5.

Example. Let the 2D behavior $\mathcal{B}$ be defined by the kernel representation

$$R(s_1, s_2) = \begin{bmatrix} s_2 - s_1 & s_2 - 1 \\ 1 - s_1 & s_2 - s_1 \end{bmatrix}. \tag{4.10}$$

Since $R(s_1, s_2)$ is a first order polynomial matrix in s_2, i.e., $N = 2$, it is possible to write $R(s_1, s_2) = R_0(s_1) + R_1(s_1)s_2$ where

$$R_0(s) = \begin{bmatrix} -s & -1 \\ 1 - s & -s \end{bmatrix} \text{ and } R_1(s) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Note that both $R_0(s)$ and $R_1(s)$ are full row rank matrices and therefore $R_0(\sigma)$ and $R_1(\sigma)$ are surjective operators.

First of all, we show that $\mathcal{B}^1 = (\mathbb{R}^2)^{\mathbb{Z}}$, i.e., the restriction of $\mathcal{B}$ to one (horizontal) line is *free*. In other words, we prove that any sequence $w(t, 0)$ can be extended to a 2D sequence $w(t, \tau) \in \mathcal{B}$.

Let us fix $w(t, 0)$. By using the kernel representation of $\mathcal{B}$ we have that

$$R_0(\sigma)w(t, 0) + R_1(\sigma)w(t, 1) = 0. \tag{4.11}$$

Since $R_1(\sigma)$ is surjective, there exists a sequence $w(t, 1)$ such that (4.11) holds true. This shows that any $w(t, 0)$ can be recursively extended to a 2D sequence $w(t, \tau)$ which satisfies the defining equation of $\mathcal{B}$ for any positive τ.

The same reasoning can be done for negative τ, in this case by using the equation $R_0(\sigma)w(t, -1) + R_1(\sigma)w(t, 0) = 0$ and the fact that $R_0(\sigma)$ is surjective. We conclude that $\mathcal{B}^1 = \ker C^1(\sigma)$, with $C^1(s) = 0$.

To find a kernel representation of $\mathcal{B}^2$ we use Proposition 4.4 and Lemma 4.6, i.e., we look for a kernel representation of $\mathcal{B}_2$. By definition, $w \in \mathcal{B}_2$ if and only if $\left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right] \in \mathcal{B}^2$. So, by shift-invariance, $\mathcal{B}_2 = \{w(t, 1) : w \in \mathcal{B} \text{ and } w(t, 0) = 0\}$. However, by equation (4.11), if $w(t, 0) = 0$ then $R_1(\sigma)w(t, 1) = 0$ and, being $R_1(s)$ clearly invertible, $w(t, 1) = 0$. Hence, $\mathcal{B}_2 = \ker C_2(\sigma)$ with $C_2(s) = I$.

To find $T_2(s)$ in equation (4.4), observe that $-T_2(\sigma)w(t, 0) + C_2(\sigma)w(t, 1) = 0$ must hold, i.e., $w(t, 1) = T_2(\sigma)w(t, 0)$. By substituting it into equation (4.11), we get $[R_0(\sigma) + R_1(\sigma)T_2(\sigma)]w(t, 0) = 0$. By the freeness of $w(t, 0)$ and the invertibility

of $R_1(\sigma)$, it follows that

$$-T_2(s) = R_1(s)^{-1}R_0(s) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -s & -1 \\ 1-s & -s \end{bmatrix} = \begin{bmatrix} -1 & s-1 \\ 1-s & -s \end{bmatrix}.$$

Finally, since $C^1(s)$ is null, the canonical representation of $\mathcal{B}^2$ is

$$C^2(s) = \begin{bmatrix} -T_2(s) & C_2(s) \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & s-1 & 1 & 0 \\ 1-s & -s & 0 & 1 \end{array}\right] \tag{4.12}$$

and a canonical representation of $\mathcal{B}$ is

$$C(s_1, s_2) = C^2(s_1)\Phi^2(s_2) = \begin{bmatrix} s_2 - 1 & s_1 - 1 \\ 1 - s_1 & s_2 - s_1 \end{bmatrix}.$$

Notice that $U(s_1)C(s_1, s_2) = R(s_1, s_2)$ where $U(s) = R_1(s)$. This means, by Theorem 4.12, that $R(s_1, s_2)$ was already a 2D canonical form.

5. Static symmetries of dynamical systems

The aim of this section is to show the effect of static symmetries of $\mathcal{B}$, corresponding to the condition $\rho\mathcal{B} \subseteq \mathcal{B}$, on its kernel representations.

For 1D systems the following theorem holds [2].

Theorem 5.1. *Given a representation ρ of G on $W = \mathbb{K}^q$, the behavior $\mathcal{B} \subseteq W^{\mathbb{Z}}$ is ρ-symmetric if and only if it admits a minimal representation provided by $R(s) \in \mathbb{K}[s]^{p\times q}$ such that*

$$R(s)\rho = \rho' R(s)$$

where ρ' is a subrepresentation of ρ.

This theorem has been extended to nD systems [1] but under the restrictive assumption of behavior regularity. Moreover, in the real case $\mathbb{K} = \mathbb{R}$, another hypothesis has to be added: the representation cannot have irreducible quaternionic components (see Section 5.2). However, we will prove that for 2D systems a complete extension of Theorem 5.1 is possible.

First of all, note that if $\mathcal{B} = \ker R(s_1, s_2)$ with $R(s_1, s_2) \in \mathbb{K}[s_1, s_2]^{p\times q}$, then for any representation ρ' of G on $\mathbb{K}^p$, the condition

$$R(s_1, s_2)\rho = \rho' R(s_1, s_2) \tag{5.1}$$

is sufficient for $\mathcal{B}$ to be ρ-symmetric. Indeed, for any $w \in \mathcal{B}$ we have

$$R(s_1, s_2)\rho w = \rho' R(s_1, s_2)w = 0 \;\Rightarrow\; \rho w \in \mathcal{B}.$$

We are going to show that this condition is also necessary in the main theorem of this paper.

Theorem 5.2. *Given a representation ρ of G on $W = \mathbb{K}^q$, the behavior $\mathcal{B} \subseteq W^{\mathbb{Z}^2}$ is ρ-symmetric if and only if it admits a kernel representation $R(s_1, s_2) \in \mathbb{K}[s_1, s_2]^{p\times q}$ such that for a suitable representation ρ' of G*

$$R(s_1, s_2)\rho = \rho' R(s_1, s_2). \tag{5.2}$$

Moreover, if $\rho = m_1\eta^1 \oplus \cdots \oplus m_r\eta^r$*, then* $\rho' = m'_1\eta^1 \oplus \cdots \oplus m'_r\eta^r$ *for some* m'_i*,* $i = 1 \ldots, r$*.*

Proof. We already showed that if equation (5.2) holds, $\mathcal{B}$ is ρ-symmetric. We prove now the converse and the condition on ρ'.

Let $C(s_1, s_2)$ be a canonical representation of $\mathcal{B}$. The behavior is ρ-symmetric if and only if $\rho\mathcal{B} = \mathcal{B}$, which is equivalent to $\ker C(\sigma_1, \sigma_2) = \ker C(\sigma_1, \sigma_2)\rho$. Note that $\Phi^N(s)\rho = \rho^N\Phi(s)^N$ where

$$\rho^N = \operatorname{diag}(\overbrace{\rho, \ldots, \rho}^{N}). \tag{5.3}$$

Therefore, if we decompose $C(s_1, s_2)$ as in the factorization (4.9), we obtain that

$$C(s_1, s_2)\rho = C^N(s_1)\Phi^N(s_2)\rho = C^N(s_1)\rho^N\Phi^N(s_2). \tag{5.4}$$

The matrix $C^N(s)\rho^N$ is still a canonical representation of $\mathcal{B}^N$. Indeed it is a lower triangular matrix with diagonal blocks $C_j(s)\rho$, where $C_j(s)$ are the corresponding blocks of $C^N(s)$. These blocks are clearly full row rank matrices, hence the conditions of Definition 4.5 are met and $C(s_1, s_2)\rho$ is a canonical representation of $\mathcal{B}$. Therefore, by Theorem 4.12, a family of unimodular matrices $U_g(s_1) \in \mathcal{U}^N$ is uniquely determined such that

$$C(s_1, s_2)\rho_g = U_g(s_1)C(s_1, s_2),\ \forall g \in G. \tag{5.5}$$

The rest of the proof is analogous to the 1D case. We just give a sketch without the details that can be found in [2].

It is possible to prove that $U_g(s)$ is a continuous *polynomial* representation of G. However, by a theorem proved in [6], it is also isomorphic to a constant representation ρ', i.e., there exists a unimodular matrix $V(s)$ such that $V(s)U_g(s) = \rho'_g V(s)$ for any $g \in G$. This means that, if we let $R(s_1, s_2) = V(s_1)C(s_1, s_2)$, then $\mathcal{B} = \ker R(\sigma_1, \sigma_2)$ and, by (5.5),

$$\rho'_g R(s_1, s_2) = \rho'_g V(s_1)C(s_1, s_2) = V(s_1)C(s_1, s_2)\rho_g = R(s_1, s_2)\rho_g,\ \forall g \in G.$$

To prove that the decomposition of ρ' uses the irreducible subrepresentations of ρ, note that, as we already said, matrix (5.4) is a canonical representation, $\mathcal{B}^N = \ker C^N(\sigma)\rho^N$, and thus $\rho^N\mathcal{B}^N \subseteq \mathcal{B}^N$. Using the matrix $V(s)$ that we have just found, we see that $\rho' V(s)C^N(s) = V(s)C^N(s)\rho^N$ and so, by Theorem 5.1, ρ' is a subrepresentation of ρ^N. This shows that $m'_i \leq Nm_i$. □

One could wonder whether it is possible to take a canonical representation $R(s_1, s_2)$ in formula (5.2) of Theorem 5.2. The answer, in general, is negative. The only result which is possible to state is the following.

Corollary 5.3. *If* $\mathcal{B}$ *is* ρ*-symmetric, then also the* 1*D behaviors* $\mathcal{B}_j$ *are* ρ*-symmetric. Moreover, let* $C_j(s)$ *be the minimal kernel representations of these behaviors such that* $\rho'^j C_j(s) = C_j(s)\rho$*, where* ρ'^j *are subrepresentations of* ρ *(as stated by Theorem 5.1). Then there exist* $U_g(s) \in \mathcal{U}^N$ *such that*

$$U_g(s_1)C(s_1, s_2) = C(s_1, s_2)\rho_g,\ \forall g \in G,$$

where $U_g(s) \in \mathcal{U}^N$ has diagonal blocks ρ'^j_g.

Proof. This is a straightforward consequence of the proof of Theorem 5.2. Indeed, by the properties of $U_g(s)$ therein stated, by the structure (4.6) of $C^N(s)$, and by minimality of $C_j(s)$, the statement follows. □

If we take into account also the decomposition into irreducible components of the representation ρ inducing the symmetry, it is possible to characterize $R(s_1, s_2)$ in a more detailed way. This leads to the definition of a canonical structure for kernel representations of symmetric systems.

However, the decomposition of representations depends highly on the base field, and so we have to distinguish the real and complex cases. We will only expose the main ideas, referring the reader to [2] for further details.

5.1. Symmetric representations of complex behaviors.

Suppose that the basis of W is such that the decomposition (3.1), corresponds to orthogonal subspaces $U_1, \ldots, U_r$. If $n_i = \deg \eta^i$, then $\dim U_i = m_i n_i$. The matrices ρ_g have therefore a block diagonal structure. We will suppose without loss of generality that also each ρ'_g is a block diagonal matrix. We can then partition $R(s_1, s_2)$ into blocks $R_{ij}(s_1, s_2)$ according to ρ_g as regards the columns and to ρ'_g for the rows — so that equation (5.2) can be written blockwise

$$m'_i \eta^i R_{ij}(s_1, s_2) = R_{ij}(s_1, s_2) m_j \eta^j.$$

As a consequence of a fundamental result in representation theory, the Schur Lemma [10], we obtain that $R_{ij}(s_1, s_2) = 0$ for $i \neq j$ and that $R_{ii}(s_1, s_2)$, a block of dimension $m'_i n_i \times m_i n_i$, can be expressed as

$$R_{ii}(s_1, s_2) = \Lambda_i(s_1, s_2) \otimes I_{n_i}, \tag{5.6}$$

where $\Lambda_i(s_1, s_2) \in \mathbb{C}[s_1, s_2]^{m'_i \times m_i}$ is a suitable polynomial matrix, I_{n_i} is the $n_i \times n_i$ identity matrix and $\otimes$ is the Kronecker product, such that $[a_{ij}] \otimes B = [a_{ij} B]$.

Therefore, by partitioning also trajectories $w \in \mathcal{B}$ into r block vectors according to ρ, the kernel representation of $\mathcal{B}$ can be written as

$$\mathcal{B} = \ker(\Lambda_1(s_1, s_2) \otimes I_{n_1}) \oplus \cdots \oplus \ker(\Lambda_r(s_1, s_2) \otimes I_{n_r}).$$

5.2. Symmetric representations of real behaviors.

When $\mathbb{K}$ is the field of real numbers, the general structure presented in the previous section for a complex representation still holds, but the matrices $\Lambda_i(s_1, s_2)$ in equation 5.6 are not so simple anymore. Indeed, three different types of irreducible real representations (*real*, *complex* and *quaternionic*, as defined in [5, §3.5]) give rise to different kinds of blocks $R_{ii}(s_1, s_2)$. If ρ_i is a real irreducible representation, the corresponding block looks like

$$A_i(s_1, s_2) \otimes I_{n_i}$$

where $A_i(s_1, s_2) \in \mathbb{R}[s_1, s_2]^{m'_i \times m_i}$.

For a complex irreducible representation ρ_i the typical block looks like

$$\begin{bmatrix} A_i(s_1,s_2) & B_i(s_1,s_2) \\ -B_i(s_1,s_2) & A_i(s_1,s_2) \end{bmatrix} \otimes I_{n_i}, \tag{5.7}$$

where both $A_i(s_1,s_2), B_i(s_1,s_2) \in \mathbb{R}[s_1,s_2]^{m_i' \times m_i}$.

Finally, if ρ_i is a quaternionic irreducible representation, the block $R_{ii}(s_1,s_2)$ is equal to

$$\begin{bmatrix} A_i(s_1,s_2) & B_i(s_1,s_2) & C_i(s_1,s_2) & D_i(s_1,s_2) \\ -B_i(s_1,s_2) & A_i(s_1,s_2) & -D_i(s_1,s_2) & C_i(s_1,s_2) \\ -C_i(s_1,s_2) & D_i(s_1,s_2) & A_i(s_1,s_2) & -B_i(s_1,s_2) \\ -D_i(s_1,s_2) & -C_i(s_1,s_2) & B_i(s_1,s_2) & A_i(s_1,s_2) \end{bmatrix} \otimes I_{n_i},$$

where $A_i(s_1,s_2), B_i(s_1,s_2), C_i(s_1,s_2), D_i(s_1,s_2) \in \mathbb{R}[s_1,s_2]^{m_i' \times m_i}$.

Note that the size of $R_{ii}(s_1,s_2)$ is divisible by 2 or by 4 when the corresponding subrepresentation is complex or quaternionic.

Example. Consider again the 2D behavior $\mathcal{B}$ defined by the kernel representation (4.10). We want to show that it is ρ-symmetric, where ρ is the representation of the group of permutations $G = \{e, (123), (132)\}$ defined in (3.2), i.e.,

$$\rho_{(123)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \text{ and } \rho_{(132)} = \rho_{(123)}^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

By equations (5.4) and (5.5), the behavior is symmetric if and only if there exist a family of unimodular matrices $U_g(s)$ such that

$$C^2(s)\rho_g^2 = U_g(s)C^2(s),\ \forall g \in G,$$

where $C^2(s) = \begin{bmatrix} -T_2(s) & C_2(s) \end{bmatrix}$ was found in equation (4.12). However, this means that, in particular, $C_2(s)\rho_g = U_g(s)C_2(s)$ and, being $C_2(s) = I$, it follows that $\rho_g = U_g(s)$.

So, we can affirm that $\mathcal{B}$ is ρ-symmetric once we check that also $T_2(s)\rho_g = U_g(s)T_2(s)$ holds true, i.e., if and only if

$$\begin{bmatrix} -1 & s-1 \\ 1-s & -s \end{bmatrix} \rho_g = \rho_g \begin{bmatrix} -1 & s-1 \\ 1-s & -s \end{bmatrix},\ \forall g \in G.$$

As a last remark, note that ρ is a complex irreducible representation: in complex form it is just a cubic root of unity (e.g, $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$). So, by changing the base of $\mathcal{B}$, its kernel representation and the representation of G can be written

$$R(s_1,s_2) = \begin{bmatrix} 2s_2 - s_1 - 1 & \sqrt{3}(s_1-1) \\ -\sqrt{3}(s_1-1) & 2s_2 - s_1 - 1 \end{bmatrix} \text{ and } \rho_{(123)} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix},$$

respectively, according to the structure showed in equation (5.7).

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Paula Rocha
Departamento de Matemática, Universidade de Aveiro,
Campus de Santiago, 3810-193, Aveiro, Portugal
Tel: +351234370359, Fax: +351234382014

E-mail: procha@mat.ua.pt

Paolo Vettori
Departamento de Matemática, Universidade de Aveiro,
Campus de Santiago, 3810-193, Aveiro, Portugal
Tel: +351234370359, Fax: +351234382014

E-mail: pvettori@mat.ua.pt

Jan C. Willems
University of Leuven, ESAT/SCD (SISTA),
Kasteelpark Arenberg 10, B-3001 Leuven-Heverlee, Belgium
Tel: +3216321805, Fax: +3216321970
E-mail: Jan.Willems@esat.kuleuven.ac.be