About higher-order Löwner tensors

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In this report, we give a definition of a Löwner tensor and prove that a Löwner tensor satisfies similar low-rank properties as Löwner matrices.

Matrix case

We have the following definition of a Löwner matrix:

Definition 1. *Given a function $f(t)$ sampled on points $T = \{t_1, t_2, \ldots, t_N\}$. We partition the point set T into two point sets $X = \{x_1, x_2, \ldots, x_I\}$ and $Y = \{y_1, y_2, \ldots, y_J\}$ with $I + J = N$, and define the entries of the Löwner matrix $\mathbf{L} \in \mathbb{K}^{I \times J}$ as*

$$\forall i, j: \quad l_{ij} = \frac{f(x_i) - f(y_j)}{x_i - y_j}. \tag{1}$$

The following has been proven in [1, 2, 5]:

Theorem 1. *Given a Löwner matrix $\mathbf{L}$ of size $I_1 \times I_2$ constructed from a function $f(t)$ sampled in a point set $T = \{t_1, \ldots, t_N\}$ with $N = I_1 + I_2$. If $f(t)$ is a rational function of degree δ and if $I_1, I_2 \geq \delta$, then $\mathbf{L}$ has rank δ:*

$$\operatorname{rank}(\mathbf{L}) = \delta = \deg(f).$$

Tensor case

We now define the Löwner tensor of arbitrary order K as a generalization of the Löwner matrix:

Definition 2. *Given a function $f(t)$ sampled on points $T = \{t_1, t_2, \ldots, t_N\}$. We partition the point set T into K point sets $T^{(k)} = \{t_1^{(k)}, t_2^{(k)}, \ldots, t_{I_k}^{(k)}\}$ with $\sum_{k=1}^{K} I_k = N$. We define the entries of the Löwner tensor $\mathcal{L} \in \mathbb{K}^{I_1 \times \cdots \times I_K}$ as*

$$\forall i_1, \ldots, i_K: \quad l_{i_1 i_2 \cdots i_K} = \sum_{k=1}^{K} \left[\frac{f(t_{i_k}^{(k)})}{\prod\limits_{\substack{m=1 \\ m \neq k}}^{K} \left(t_{i_k}^{(k)} - t_{i_m}^{(m)} \right)} \right].$$

Let us consider two special cases:

- For $K = 2$, we obtain the Löwner matrix $\mathbf{L}$ from (1):

$$\forall i, j: \quad l_{ij} = \frac{f(x_i)}{x_i - y_j} + \frac{f(y_j)}{y_j - x_i} = \frac{f(x_i) - f(y_j)}{x_i - y_j}.$$

- For $K = 3$, we partition a point set T into three point sets X, Y and Z with I, J and K points, respectively. We then obtain a Löwner tensor $\mathcal{L} \in \mathbb{K}^{I \times J \times K}$ with the following entries:

$$\forall i, j, k: \quad l_{ijk} = \frac{f(x_i)}{(x_i - y_j)\,(x_i - z_k)} + \frac{f(y_j)}{(y_j - x_i)\,(y_j - z_k)} + \frac{f(z_k)}{(z_k - x_i)\,(z_k - y_j)}.$$

Interestingly enough, as we will prove, the low-rank property from the matrix case partly holds in the tensor case.

Theorem 2. *If $f(t)$ is a degree-1 rational function, then the Löwner tensor $\mathcal{L} \in \mathbb{K}^{I_1 \times \cdots \times I_K}$ constructed from evaluations of $f(t)$ in the point set $T = \{t_1, \ldots, t_N\}$ with $\sum_{k=1}^{K} = N$ has rank 1.*

Proof. A degree-1 rational function can be parametrized as $f(t) = a\frac{1}{t-p} + b$. It can easily be seen that the entries of the Löwner tensor are invariant to the constant term b. Furthermore, if we show that $\mathcal{L}$ has rank 1 for $a = 1$, $\mathcal{L}$ also has rank 1 for nonzero $a \neq 1$. Hence, it remains to show that $\mathcal{L}$ has rank 1 if $f(t) = \frac{1}{t-p}$.

Let us consider Lagrange interpolation with corresponding Lagrange basis polynomials:

$$l_k(u) = \prod_{\substack{m=1 \\ m \neq k}}^{K} \frac{(u - u_m)}{(u_k - u_m)} \quad \text{for } k = 1, \ldots, K. \tag{2}$$

Given a set of K points (u_k, v_k) for $k = 1, \ldots, K$, let us define the polynomial

$$g(u) = \sum_{k=1}^{K} v_k l_k(u). \tag{3}$$

The function $g(u)$ is the interpolating polynomial of degree at most $K - 1$ interpolating the points (u_k, v_k). It is well-known that $g(u)$ is the unique interpolating polynomial of degree at most $K - 1$.

Let us consider the special case with K points $(u_k, 1)$ for $k = 1, \ldots, K$. We find that the constant function $g(u) = 1$ interpolates these points. Hence, this function is the only interpolating polynomial of degree at most $K - 1$ and we obtain the following equality by substituting (2) in (3):

$$g(u) = \sum_{k=1}^{K} \prod_{\substack{m=1 \\ m \neq k}}^{K} \frac{(u - u_m)}{(u_k - u_m)} = 1.$$

Dividing both sides with $\prod_{m=1}^{K} (u - u_m)$ gives

$$\sum_{k=1}^{K} \frac{1}{(u - u_k)} \prod_{\substack{m=1 \\ m \neq k}}^{K} \frac{1}{(u_k - u_m)} = \prod_{m=1}^{K} \frac{1}{(u - u_m)}.$$

We can safely replace the index m in the right-hand side with k, and replace each $(u - u_k)$ with $-(u_k - u)$:

$$\sum_{k=1}^{K} \frac{1}{(u_k - u)} \prod_{\substack{m=1 \\ m \neq k}}^{K} \frac{1}{(u_k - u_m)} = (-1)^{(K+1)} \prod_{k=1}^{K} \frac{1}{(u_k - u)}.$$

Instead of the set $\{u_1, \ldots, u_K\}$, let us consider the set $\left\{t_{i_1}^{(1)}, \ldots, t_{i_K}^{(K)}\right\}$. Let us also replace u with p. Substitution gives:

$$\sum_{k=1}^{K} \frac{1}{(t_{i_k}^{(k)} - p)} \prod_{\substack{m=1 \\ m \neq k}}^{K} \frac{1}{(t_{i_k}^{(k)} - t_{i_m}^{(m)})} = (-1)^{K+1} \prod_{k=1}^{K} \frac{1}{(t_{i_k}^{(k)} - p)}.$$

The term $1/(p - t_{i_k}^{(k)})$ is equal to the function $f(t) = \frac{1}{t-p}$ evaluated in $t_{i_k}^{(k)}$. Hence, we obtain

$$\sum_{k=1}^{K} \left[\frac{f(t_{i_k}^{(k)})}{\prod_{\substack{m=1 \\ m \neq k}}^{K} \left(t_{i_k}^{(k)} - t_{i_m}^{(m)}\right)} \right] = (-1)^{K+1} \prod_{k=1}^{K} \frac{1}{(t_{i_k}^{(k)} - p)}.$$

The left-hand side is equal to the entry $l_{i_1 \cdots i_K}$. Hence, the tensor $\mathcal{L}$ can be written as the outer product

$$\mathcal{L} = \mathbf{a}^{(1)} \otimes \cdots \otimes \mathbf{a}^{(K)}$$

with nonzero vectors $\mathbf{a}^{(k)} \in \mathbb{K}^{I_k}$ with entries $a_{i_k}^{(k)} = \frac{1}{(t_{i_k}^{(k)} - p)}$. Hence, we have shown that $\mathcal{L}$ has rank 1. □

Theorem 3. *If $f(t)$ is a degree-R rational function with distinct poles, then the Löwner tensor $\mathcal{L} \in \mathbb{K}^{I_1 \times \cdots \times I_K}$ constructed from evaluations of $f(t)$ in the point set $T = \{t_1, \ldots, t_N\}$ with $\sum_{k=1}^{K} = N$ and with $I_k \geq \delta$ for $1 \leq k \leq K$ has rank δ.*

Proof. Based on Theorem 2, it is easy to show that $\mathcal{L}$ can be written as a sum of δ rank-1 terms. To show that $\mathcal{L}$ has rank δ, we have to show that it is not possible to write $\mathcal{L}$ as a sum of less rank-1 terms. This can be done by showing that the decomposition is unique. A sufficient condition for uniqueness is to show that the factor matrices of the CPD have full column rank [4][1].

The columns of each factor matrix of $\mathcal{L}$ are evaluated degree-1 rational functions of the form $\frac{1}{t-p_r}$ as shown in Theorem 2. As the poles p_r are assumed to be distinct, it follows that the factor matrices have full column rank. □

Note that the condition in Theorem 3 can be significantly relaxed. Unlike the matrix case, the rank of a Löwner tensor can be equal to the degree of the rational function even if the latter is larger than one or more dimensions of the tensor [3].

The rank properties of Löwner tensors in the situation of coinciding poles remain to be shown. It is expected that the Löwner tensor has a low multilinear rank, similar to the Hankel case.

[1] This result is actually attributed to by Robert Jennrich.

References

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