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Highlights

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- Introduces a block Macaulay model matrix for state elimination in parametric SISO polynomial state-space models.
- Computes input-output difference equations using numerical linear algebra.
- Provides diagnostics for algebraic global parameter identifiability.
- Demonstrates robustness of the approach under model perturbations.

A Numerical Linear Algebraic Approach to State Elimination in Parametric Single-Input Single-Output Polynomial Models*

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ABSTRACT

State elimination converts a state-space model into an input-output difference equation involving only measurable inputs, outputs, and model parameters. In previous work, we introduced the model matrix to perform this conversion for autonomous, single-output polynomial models using numerical linear algebra. In this paper, we extend that framework to discrete-time, time-invariant, parametric, single-input, single-output polynomial models. Since the resulting model matrix depends polynomially on the parameters and input sequence, we expand it into coefficient blocks and collect their monomial shifts in a block Macaulay (model) matrix. For strongly locally observable models, we prove that the left null space of this matrix coincides with the row space of a block Macaulay matrix generated by the unique irreducible parametric input-output difference polynomial of minimal order. This result leads to an SVD-based algorithm that determines the monomial support and numerically recovers the coefficients of the difference equation. For strongly locally observable and identifiable models, we show that a selected parameter can be isolated in an input-output difference equation, whose degree in that parameter determines whether it is algebraically globally or locally identifiable. Numerical examples recover the corresponding input-output equations under coefficient perturbations and yield the expected identifiability conclusions using only numerical linear-algebraic computations.

1. Introduction

We consider the problem of eliminating the state variables for models in Σ, the class of discrete-time, time-invariant, parametric, single-input, single-output polynomial state-space models described by,

$$\Sigma : \begin{cases} \boldsymbol{x}[k+1] = \boldsymbol{f}(\boldsymbol{\theta}, \boldsymbol{x}[k], u[k]) \\ \quad\; y[k] = g(\boldsymbol{\theta}, \boldsymbol{x}[k], u[k]) \end{cases}, \tag{1}$$

where $\boldsymbol{x}[k] = \begin{bmatrix} x_1[k] & \cdots & x_{n_f}[k] \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n_f}$ is the state variable, $u[k] \in \mathbb{R}$ is the input at time $k \in \mathbb{N}$, and $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 & \cdots & \theta_\ell \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{\ell}$ denotes the model parameters. We write $\mathbb{R}(\begin{bmatrix} \boldsymbol{\theta} & u[k] \end{bmatrix})$ for the field of rational functions [4] in the components of $\boldsymbol{\theta}$ and $u[k]$, subsequently $\mathbb{R}(\begin{bmatrix} \boldsymbol{\theta} & u[k] \end{bmatrix})[\boldsymbol{x}[k]]$ denotes the polynomial ring in the components of $\boldsymbol{x}[k]$ with coefficients in $\mathbb{R}(\begin{bmatrix} \boldsymbol{\theta} & u[k] \end{bmatrix})$. Although the notation $\mathbb{R}(\begin{bmatrix}\cdot\end{bmatrix})$ allows coefficients to be rational functions in the components of $\begin{bmatrix}\cdot\end{bmatrix}$, throughout this paper we restrict attention to coefficients that are polynomial in those variables[1]. The state update and observation functions satisfy $\boldsymbol{f} = \begin{bmatrix} f_1 & \cdots & f_{n_f} \end{bmatrix}^{\mathrm{T}}$ with $f_i \in \mathbb{R}(\begin{bmatrix} \boldsymbol{\theta} & u[k] \end{bmatrix})[\boldsymbol{x}[k]]$ and $g \in \mathbb{R}(\begin{bmatrix} \boldsymbol{\theta} & u[k] \end{bmatrix})[\boldsymbol{x}[k]]$.

Example 1: Consider the following model, which we will use as a running example throughout this paper as an instance of the model class described in Σ (1),

$$\Sigma_{\mathrm{R}} : \begin{cases} x_1[k+1] = \theta_1 x_1[k] + \theta_2 x_2[k] u[k] \\ x_2[k+1] = 3\theta_2^2 x_1[k] x_2[k] \\ \quad\; y[k] = x_2[k] \end{cases}, \tag{2}$$

containing $n_f = 2$ components in the state variable and two unknown model parameters $\boldsymbol{\theta}_{\mathrm{R}} = \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^2$. △

We denote the length $n_y + 1$ output (input) window starting at time k by $\boldsymbol{y}[\boldsymbol{k}_{n_y}] = \begin{bmatrix} y[k] & \cdots & y[k+n_y] \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n_y+1}$, where $n_y \in \mathbb{N}$ is the *order bound* (note that an input window of length at most n_y denoted by $\boldsymbol{u}[\boldsymbol{k}_{n_y}] = \begin{bmatrix} u[k] & \cdots & u[k+n_y] \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n_y+1}$ is needed to describe output of $\boldsymbol{y}[\boldsymbol{k}_{n_y}]$ corresponding to Σ). Output (input) monomials are $\boldsymbol{y}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\gamma}} = \prod_{i=0}^{n_y} y[k+i]^{\gamma_i}$ for $\boldsymbol{\gamma} \in \mathbb{N}^{n_y+1}$ ($\boldsymbol{u}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\delta}} = \prod_{i=0}^{n_y} u[k+i]^{\delta_i}$ for $\boldsymbol{\delta} \in \mathbb{N}^{n_y+1}$). We will refer to input-output monomials as monomials of the form, $\boldsymbol{w}[\boldsymbol{k}_{n_y}]^{(\boldsymbol{\gamma},\boldsymbol{\delta})} = \prod_{i=0}^{n_y} y[k+i]^{\gamma_i} \prod_{i=0}^{n_y} u[k+i]^{\delta_i}$ where $\boldsymbol{w}[\boldsymbol{k}_{n_y}] =$

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[1]Models with rational coefficients in $\mathbb{R}(\begin{bmatrix}\cdot\end{bmatrix})$ can be converted to a model with polynomial coefficients by multiplying throughout with a common denominator. However we will not pursue such translation in this paper.

$\begin{bmatrix} \boldsymbol{y}[\boldsymbol{k}_{n_y}]^{\mathrm{T}} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{2n_y+2}$ contains the behavior of the model [18] for some model parameter in the window of length n_y.

An input-output difference equation is an algebraic relation ϕ between the model parameters, model inputs and the model outputs that vanishes over an $n_y + 1$ window of the output sequence $\boldsymbol{y} = \begin{bmatrix} y[0] & \cdots & y[\mathrm{N}] \end{bmatrix} \in \mathbb{R}^{\mathrm{N}+1}$ generated by some model in Σ through the input $\boldsymbol{u} = \begin{bmatrix} u[0] & \cdots & u[\mathrm{N}] \end{bmatrix} \in \mathbb{R}^{\mathrm{N}+1}$ at an instance of $\boldsymbol{\theta} \in \mathbb{R}^{\ell}$. More precisely,

$$\phi\left(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}], \boldsymbol{y}[\boldsymbol{k}_{n_y}]\right) = 0, \quad (\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}], \boldsymbol{y}[\boldsymbol{k}_{n_y}]) \in \mathcal{B}_{\Sigma}, \tag{3}$$

where $\mathcal{B}_{\Sigma}$ denotes the set of all parameters in $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^{\ell}$, admissible input sequences $\boldsymbol{u}[\boldsymbol{k}_{n_y}] \in \mathcal{U} \subseteq \mathbb{R}^{n_y+1}$ and the corresponding output sequence $\boldsymbol{y}[\boldsymbol{k}_{n_y}] \in \mathbb{R}^{n_y+1}$ generated by models belonging to Σ at some initial state $\boldsymbol{x}[k] \in \mathbb{R}^{n_f}$. Here Θ and $\mathcal{U}$ are open and dense subsets of the ambient space. We can describe such a relationship in different coefficient fields. First, as a polynomial relation between the model outputs with coefficients in $\mathbb{R}\left[\boldsymbol{\theta} \quad \boldsymbol{u}[\boldsymbol{k}_{n_y}]\right]$,

$$\phi = \sum_{\boldsymbol{\gamma} \in \Gamma} c_{\boldsymbol{\gamma}}(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])\, \boldsymbol{y}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\gamma}}, \tag{4}$$

here, $\boldsymbol{\gamma} \in \Gamma$ is a multi-index, where $\Gamma = \{\boldsymbol{\gamma} \in \mathbb{N}^{n_y+1} : c_{\boldsymbol{\gamma}}(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}]) \in \mathbb{R}\left[\boldsymbol{\theta} \quad \boldsymbol{u}[\boldsymbol{k}_{n_y}]\right]) \setminus \{0\}\}$ indexes the output monomials with nonzero coefficients. Second, as a polynomial relation with coefficients in $\mathbb{R}(\boldsymbol{\theta})$, in the input-output variables $\boldsymbol{w}[\boldsymbol{k}_{n_y}] \in \mathbb{R}^{2n_y+2}$ given by

$$\phi = \sum_{\boldsymbol{\omega} \in \Omega} c_{\boldsymbol{\omega}}(\boldsymbol{\theta})\, \boldsymbol{w}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\omega}}, \tag{5}$$

such that $\phi \in \mathbb{R}(\boldsymbol{\theta})[\boldsymbol{w}[\boldsymbol{k}_{n_y}]]$ is the input-output difference polynomial. Here, $\boldsymbol{\omega} = (\boldsymbol{\gamma}, \boldsymbol{\delta}) \in \Omega$ is a multi-index, where $\Omega = \{\boldsymbol{\omega} \in \mathbb{N}^{2n_y+2} : c_{\boldsymbol{\omega}}(\boldsymbol{\theta}) \in \mathbb{R}(\boldsymbol{\theta}) \setminus \{0\}\}$ indexes the input–output monomials with nonzero coefficients. Finally, we can also express (4) as a linear combination of parameter-input-output monomials,

$$\phi = \sum_{\bar{\boldsymbol{\omega}} \in \bar{\Omega}} c_{\bar{\boldsymbol{\omega}}} \boldsymbol{\theta}^{\boldsymbol{\eta}} \boldsymbol{w}[k_{n_y}]^{\boldsymbol{\omega}}, \tag{6}$$

where $\boldsymbol{\theta}^{\boldsymbol{\eta}} = \prod_{i=1}^{\ell} \theta^{\eta_i}$ are the parameter monomials and $\bar{\Omega} = \{\bar{\boldsymbol{\omega}} = (\boldsymbol{\eta}, \boldsymbol{\omega}) \in \mathbb{N}^{2n_y+2+\ell} : c_{\bar{\boldsymbol{\omega}}} \in \mathbb{R} \setminus \{0\}\}$ indexes the parameter-input-output monomials. The order of the difference equation (denoted $\mathrm{ord}(\phi)$) is the largest difference between the shifts of the outputs in (4) and is at most n_y and the degree of the difference equation is given by the maximum degree of the output monomials $\deg(\phi) = \max_{\boldsymbol{\gamma} \in \Gamma} \|\boldsymbol{\gamma}\|_1$.

Example 1 (continued): The difference equation of minimal order satisfied by $(\boldsymbol{\theta}_{\mathrm{R}}, \boldsymbol{w}[\boldsymbol{k}_2]) \in \mathcal{B}_{\Sigma_{\mathrm{R}}}$ is $\phi_{\mathrm{R}}(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_2]) = 0$ where the input-output difference polynomial ϕ_{R} is,

$$\phi_{\mathrm{R}} = 3\theta_2^3 u[k] y[k]^2 y[k+1] + \theta_1 y[k+1]^2 - y[k] y[k+2],$$

which is of order two, degree three and degree four in the parameter-input monomials. △

Problem Definition We study *state elimination*: *given* discrete-time, time-invariant, parametric, single-input, single-output (SISO) polynomial state-space models Σ with n_f state components, *find* a nontrivial polynomial relation among consecutive outputs that holds for all $(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) \in \mathcal{B}_{\Sigma}$. Equivalently, we convert the model from its *image* representation, the state space form in (1), where the model outputs are polynomial functions of the state, model parameters and the input to its *kernel* representation, the difference equation in (3), which specifies a vanishing polynomial relation (4) for $(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) \in \mathcal{B}_{\Sigma}$.

For models in Σ, we can obtain the first $n_y + 1$ model outputs from time instant k and generate $\mathcal{B}_{\Sigma}$ by forward propagation of the initial state $\boldsymbol{x}[k]$ through $\boldsymbol{f}$ driven by the inputs in the window $\boldsymbol{u}[\boldsymbol{k}_{n_y}]$, followed by evaluation of g. To express this compactly, we write the n-step state transition recursively as

$$\begin{aligned} &\boldsymbol{f}^{n_y+1}\left(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{n_y}]\right) \\ &\quad = \boldsymbol{f}\Big(\boldsymbol{\theta}, \boldsymbol{f}^{n_y}\left(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{n_y-1}]\right), \boldsymbol{u}[k+n_y]\Big). \end{aligned}$$

Where $\boldsymbol{f}^0(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{-1}]) = \boldsymbol{x}[k]$, then the outputs are evaluated by the multivariate polynomial functions,

$$\begin{aligned} y[k+n_y] = g\Big(&\boldsymbol{\theta}, \boldsymbol{f}^{n_y}\left(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{n_y-1}]\right), \\ &\ldots, \boldsymbol{u}[k+n_y]\Big), \quad n_y \geq 0. \end{aligned} \tag{7}$$

Example 1 (continued): The first three model outputs of Σ_{R} at time instant k, as multivariate polynomial functions of the components of $\boldsymbol{x}[k]$, $\boldsymbol{u}[k_2]$, and $\boldsymbol{\theta}_{\mathrm{R}}$, are

$$\begin{cases} y[k] = x_2[k], \\ y[k+1] = 3\theta_2^2 x_1[k] x_2[k], \\ y[k+2] = 9u[k]\theta_2^5 x_1[k] x_2[k]^2 + 9\theta_1 \theta_2^4 x_1[k]^2 x_2[k] \end{cases}. \tag{8}$$

From (8) we have $x_2[k] = y[k]$, and subsequently $x_1[k] = \frac{y[k+1]}{3\theta_2^2 y[k]}$. Substitute these expressions for the state variables into $y[k+2]$ and clear denominators. The states are then eliminated, and we recover the difference equation of minimal-order $\phi_{\mathrm{R}}(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_2]) = 0$ where $(\boldsymbol{\theta}, \boldsymbol{w}[k_2]) \in \mathcal{B}_{\Sigma_{\mathrm{R}}}$. Additionally, ϕ_{R} is an irreducible[2] difference polynomial in $\mathbb{R}(\begin{bmatrix} \boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_2] \end{bmatrix})[\boldsymbol{y}[\boldsymbol{k}_2]]$, $\mathbb{R}(\boldsymbol{\theta})[\boldsymbol{w}[\boldsymbol{k}_2]]$ and $\mathbb{R}\left[\boldsymbol{\theta} \quad \boldsymbol{w}[\boldsymbol{k}_2]\right]$. Next, we show that we can obtain higher-order difference equations by eliminating a parameter such that its coefficients are monomials in only one parameter, either θ_1 or θ_2. Consider the model output

$$y[k+3] = 27\theta_1^3 \theta_2^6 x_1[k]^3 x_2[k]$$

[2] Unless stated otherwise, we call a polynomial ϕ irreducible if it is irreducible in every polynomial ring in which it can be represented; that is, every factorization $\phi = rs$ has at least one constant factor. If irreducibility is meant relative to a specific polynomial ring, we state the ring explicitly.

$$
\begin{aligned}
&+ 54\theta_1^2\theta_2^7 u[k]x_1[k]^2x_2[k]^2 \\
&+ 81u[k+1]\theta_1\theta_2^9 x_1[k]^3x_2[k]^2 \\
&+ 27\theta_1\theta_2^8 u[k]^2x_1[k]x_2[k]^3 \\
&\qquad + 81u[k+1]\theta_2^{10}u[k]x_1[k]^2x_2[k]^3 \qquad (9)
\end{aligned}
$$

First substitute $x_2[k] = y[k]$ and $x_1[k] = y[k+1]/(3\theta_2^2 y[k])$ into (9). Then use $\phi_{\mathrm{R}}(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_2]) = 0$ to rewrite θ_2^2 (respectively θ_1) in terms of θ_1 (respectively θ_2^2) and $\boldsymbol{y}[\boldsymbol{k}_2]$. Substituting this into (9) and clearing denominators yields a third-order difference equation $\phi_{\theta_1}(\theta_1, \boldsymbol{w}[\boldsymbol{k}_3]) = 0$ (respectively $\phi_{\theta_2}(\theta_2, \boldsymbol{w}[\boldsymbol{k}_3]) = 0$), where

$$
\begin{aligned}
\phi_{\theta_1} = {} & u[k]y[k]^2y[k+1]y[k+3] \\
& - u[k+1]y[k]y[k+1]y[k+2]^2 \\
& + \theta_1 u[k+1]y[k+1]^3y[k+2] \\
& - \theta_1 u[k]y[k]^2y[k+2]^2.
\end{aligned} \qquad (10)
$$

and

$$
\begin{aligned}
\phi_{\theta_2} = {} & y[k]y[k+2]^3 \\
& - y[k+1]^3y[k+3] \\
& + 3u[k+1]\theta_2^2 y[k+1]^4y[k+2] \\
& - 3u[k]\theta_2^2 y[k]^2y[k+1]y[k+2]^2.
\end{aligned} \qquad (11)
$$

Observe that θ_1 appears linearly in (10); hence, this third-order equation can be used to uniquely determine θ_1 from an input-output sequence $\boldsymbol{w}[\boldsymbol{k}_{n_y}] \in \mathcal{B}_{\Sigma_{\mathrm{R}}}$, i.e., θ_1 is globally identifiable [10]. In contrast, ϕ_{θ_2} is a quadratic equation in θ_2, hence for some $\boldsymbol{w}[\boldsymbol{k}_{n_y}] \in \mathcal{B}_{\Sigma_{\mathrm{R}}}$, ϕ_{θ_2} vanishes over two model parameters making it only locally identifiable [10]. △

While in Example 1 elimination by hand is transparent, it becomes impractical as the state dimension n_f grows, the required window length n_y increases, the polynomial degree rises, or the dynamics exhibit dense cross-term coupling.

Contributions Observe, that the output monomials (input-output monomials and parameter-input-output monomials) can be evaluated as polynomial functions in $\boldsymbol{x}[k]$ (see (7)), the existence of a difference equation (3) implies a nontrivial linear dependence among these polynomials in $\mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})$ ($\mathbb{R}(\begin{bmatrix}\boldsymbol{\theta}\end{bmatrix})$ and $\mathbb{R}$ respectively). We can track the linear relations between the output monomials via the model matrix introduced in [5] which shows that the problem of state elimination in models in the class Σ also reduces to computing the left null space of this model matrix. The final input-output difference polynomial is of the form (4) (equivalently (5) and (6)), but the vectors in the left null space are polynomial functions of the input sequence, the model parameters and are therefore computed symbolically (Section 2). In order to compute the difference equation relying only on numerical linear algebraic computations we rewrite the model matrix as a multiparameter polynomial matrix pencil in the model parameters and the input sequence, and then construct an associated block Macaulay matrix. This block Macaulay matrix allows us to numerically track linear relations among parameter-input-output monomials (6). We extend the orthogonal complementarity property between the Macaulay and model matrices [5] for strongly locally observable (and identifiable) (SLO-I) models to the corresponding block constructions, namely, the block Macaulay (Macaulay) matrix and the block Macaulay (model) matrix (Section 3). Using this extension, we show how the unique irreducible difference equation of minimal order can be computed numerically by means of a practical state elimination algorithm 2. The algorithm proceeds by determining the monomial support over a sequence of successively enlarged polynomial rings and, for the class of strongly locally observable and identifiable (SLO-I) models, provides an efficient test for distinguishing algebraic global from algebraic local parameter identifiability (Section 4). The resulting procedure is then demonstrated on numerical examples in Section 5. The notation used throughout the paper is summarized in Table 1.

Historical perspective The derivation of input–output (I/O) representations from state-space models through state elimination is a classical problem in system analysis and control. For nonlinear continuous-time systems, differential algebra provides a systematic framework for eliminating unobserved states to obtain differential equations in the inputs, outputs, and model parameters [9, 10]. The resulting I/O equations underpin parameter estimation and structural identifiability analysis: their coefficients are functions of the unknown parameters, and identifiability reduces to whether these functions determine the parameters uniquely [6].

Nonlinear discrete-time models have received comparatively less attention. Following the early algebraic-geometric treatment of [14], constructive approaches relied on Gröbner basis computations [7, 8, 13]. In [5] we revisited state elimination for autonomous, time-invariant, single-output polynomial state-space models and showed that the output difference equation lies in the left null space of a model matrix. That construction, however, does not accommodate inputs or unknown parameters. The present paper extends the linear-algebraic framework to parametric SISO polynomial models, yielding parametric I/O difference equations suitable for identifiability testing [12] that are computed via numerical linear algebra that remains robust to coefficient perturbations.

2. State Elimination

In this section we revisit the definitions of the model matrix [5] for the model class Σ. We show that state elimination and the derivation of a difference equation (3) where $\phi \in \mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})[\boldsymbol{y}[\boldsymbol{k}_{n_y}]]$ boils down to computing the left null space of the model matrix. The existence results established for the autonomous case over the coefficient field $\mathbb{R}$ extend directly to parametric SISO models after replacing $\mathbb{R}$ by the field $\mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})$. Finally, we introduce the

Table 1
Summary of the most important symbols used

Notation	Description
a, $\boldsymbol{a}$, a_i, $\boldsymbol{A}$,A_{ij}	Scalar (lowercase), tuple/vector (bold lowercase), ith component of $\boldsymbol{a}$, matrix (bold uppercase), ijth component of $\boldsymbol{A}$.
$\dim(\cdot)$	Dimension of the vector space.
$\mathcal{L}_{\text{null}}(\cdot)$	Left null space.
$\text{row}(\cdot)$, $\text{col}(\cdot)$	Row space, column space.
$\mathcal{K}(\boldsymbol{a})$	Field of rational functions in the components of $\boldsymbol{a}$ with coefficients in $\mathcal{K}$ (e.g., $\mathbb{R}$ or $\mathbb{C}$)
$\mathcal{K}[\boldsymbol{a}]$	Polynomial ring over field $\mathcal{K}$ in the components of $\boldsymbol{a}$.
$\boldsymbol{a}^{\boldsymbol{\omega}}$	Monomial $\prod_{i=0}^{n} a_i^{\omega_i}$, $\boldsymbol{a} \in \mathbb{C}^{n+1}$, $\boldsymbol{\omega} \in \mathbb{N}^{n+1}$. The degree is $\deg(\boldsymbol{a}^{\boldsymbol{\omega}}) = \|\boldsymbol{\omega}\|_1$.
$\boldsymbol{a}^{d}$	Column vector with components that are monomials in $\boldsymbol{a}$ up to degree d (a multivariate Vandermonde vector).
$\boldsymbol{a}^{d_1,d_2}$	block multivariate Vandermonde vector of degree d_1 in the components of $\boldsymbol{a}$, with block rows containing its parameter-input monomials of degree up to d_2.
$\mathcal{C}_r^d$	Space of r-variate polynomials of degree $\leq d$
$\boldsymbol{\gamma}, \boldsymbol{\eta}, \boldsymbol{\delta}$	Multi-indices for output monomials, parameter monomials and input monomials
$\text{LM}(\cdot)$	Leading monomial
n_y	The order bound.
$\boldsymbol{y}[\boldsymbol{k}_{n_y}]$, $\boldsymbol{u}[\boldsymbol{k}_{n_y}]$, $\boldsymbol{w}[\boldsymbol{k}_{n_y}]$	Model outputs $\begin{bmatrix} y[k] & \cdots & y[k+n_y] \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{n_y+1}$, Model inputs, $\begin{bmatrix} u[k] & \cdots & u[k+n_y] \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{n_y+1}$, $\begin{bmatrix} \boldsymbol{y}[\boldsymbol{k}_{n_y}]^{\mathsf{T}} & \boldsymbol{u}[\boldsymbol{k}_{n_y}] \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{2n_y+2}$.
$\mathcal{B}_\Sigma$	Set of all $(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}])$, where $\boldsymbol{y}[\boldsymbol{k}_{n_y}] \in \mathbb{R}^{n_y+1}$ are all output sequences generated by Σ for some $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^{\ell}$ and $\boldsymbol{u}[\boldsymbol{k}_{n_y}] \in \mathcal{U} \subseteq \mathbb{R}^{n_y+1}$ (see remark 3)
n_f	The number of state variables in Σ.
$\boldsymbol{\mathcal{M}}_{n_y}^{d}$	Model matrix, where $\boldsymbol{\mathcal{M}}_{n_y}^{d} \in \mathbb{R}\left[\boldsymbol{\theta} \quad \boldsymbol{u}[\boldsymbol{k}_{n_y}]\right]^{p\times q}$
m	Max. degree of the state monomials in (7).
$\boldsymbol{W}_{n_y}^{d}$	Basis matrix for the left null space of $\boldsymbol{\mathcal{M}}_{n_y}^{d}$.
d_{suff}	Degree at which the number of rows (p) of $\boldsymbol{\mathcal{M}}_{n_y}^{d}$ exceeds the number of columns (q).
$d_{\min}$	Smallest degree for which $\dim(\mathcal{L}_{\text{null}}(\boldsymbol{\mathcal{M}}_{n_y}^{d})) > 0$.
$\text{ord}(\phi)$, $\deg(\phi)$	Max. difference between the shifts of the outputs in ϕ, max. degree of the output monomials in ϕ.
Mac_{ϕ}^{d}	Macaulay matrix of degree d generated by ϕ where $\text{Mac}_{\phi_{(i,j)}}^{d} \in \mathbb{R}(\left[\boldsymbol{\theta} \quad \boldsymbol{u}[\boldsymbol{k}_{n_y}]\right])$
d_{shift},m_{shift}	shift degree of the block Macaulay (model/Macaulay) matrix
$\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}}$, $\text{D}_{\phi}^{d,m_{\text{shift}}}$, $\boldsymbol{H}_{n_y}^{d,d_{\text{shift}}}$	block Macaulay (model/Macaulay) matrix corresponding to the matrix pencils of $\boldsymbol{\mathcal{M}}_{n_y}^{d}(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])$ and $\text{Mac}_{\phi}^{d}(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])$, basis matrix for the left null space of $\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}}$
ζ	Smallest $d_{\text{shift}} = \zeta$, s.t., $\mathcal{L}_{\text{null}}(\dim(\boldsymbol{M}_{n_y}^{d,\zeta})) > 0$

extended state vector constructed by appending to it all but one parameter and show how the same elimination framework can be used to eliminate all but one parameter, yielding a (generally non-minimal order) difference equation whose coefficients are monomials of the remaining parameter. This construction is useful because it enables us to determine whether that parameter is algebraically globally or locally identifiable (Theorem 8).

2.1. Model matrix

To tackle the problem of state elimination in Σ using a linear algebraic framework, we will consider the vector representation of multivariate polynomials in their respective rings. To represent and manipulate polynomial vectors effectively, we fix a monomial order over the ring variables that determines the arrangement of monomials within the coefficient vector. We use the graded negative lexicographic order (*grnlex*) to order monomials [2]. However, the remainder of the paper is valid for any graded multivariate monomial ordering. The monomials in the components of $\boldsymbol{y}[\boldsymbol{k}_2]$ are arranged in the *grnlex* order as,

$$1 < y[k] < y[k+1] < y[k+2] < y[k]^2 < y[k]y[k+1] < y[k]y[k+2] < y[k+1]^2 < \cdots.$$

We can now represent polynomial functions of the output monomials using coefficient row vectors in their corresponding state-monomial basis whose components are ordered according to the *grnlex* monomial order ascending from left to right.

Example 1 (continued): The vector representation of the polynomial $y[k+2] = \boldsymbol{g}(\boldsymbol{\theta}, \boldsymbol{f}^2(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_1]), \boldsymbol{u}[k+2]) \in \mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_2]\end{bmatrix})[\boldsymbol{x}[k]]$ that evaluates $y[k+2]$ in $\mathcal{C}_2^3$ is

$$\begin{array}{c|ccc|ccc} 1 & \cdots & x_2[k]^2 & x_1[k]^3 & x_1[k]^2x_2[k] & x_1[k]x_2[k]^2 & x_2[k]^3 \\ \hline 0 & \boldsymbol{0} & 0 & 0 & 9\theta_1\theta_2^4 & 9\theta_2^5 u[k] & 0 \end{array},$$

where the vertical line separates the coefficients corresponding to the state monomials of subsequent degrees. △

The leading monomial (LM) of a polynomial is the highest-ranked monomial (in the chosen order) with nonzero coefficient. Here, $\text{LM}(y[k+2]) = x_1[k]x_2[k]^2$.

Remark 1: Depending on the context, we will use the same non-bold lower case letter for both the polynomial and its vector representation. In (8) we refer to the polynomial $y[k+2] \in \mathbb{R}(\begin{bmatrix}\boldsymbol{\theta}_{\text{R}} & \boldsymbol{u}[\boldsymbol{k}_2]\end{bmatrix})\begin{bmatrix}\boldsymbol{x}[k]\end{bmatrix}$ whereas in $y[k+2] \in \mathcal{C}_2^3$ we refer to its vector representation (*cf.* Example 1).

Definition 1 (*model matrix*): Given an order bound n_y, consider the set of n_y+1 polynomials $g(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[k]), \ldots, g(\boldsymbol{\theta}, \boldsymbol{f}^{n_y}(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{n_y-1}]), \boldsymbol{u}[k+n_y]) \in \mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})[\boldsymbol{x}[k]]$ which evaluate the model outputs in $\boldsymbol{y}[k_{n_y}] \in \mathbb{R}^{n_y+1}$ at some state $\boldsymbol{x}[k] \in \mathbb{R}^{n_f}$ for a model which belongs to Σ. The *model matrix* of degree $d \in \mathbb{N}$ denoted by $\boldsymbol{\mathcal{M}}_{n_y}^{d} \in \mathbb{R}\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix}^{p\times q}$, is defined as the matrix whose rows are the vector representations of the polynomial functions $g(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[k])^{\gamma_0} \cdots g(\boldsymbol{\theta}, \boldsymbol{f}^{n_y}(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{n_y-1}]), \boldsymbol{u}[k+n_y])^{\gamma_{n_y}}$ in their state monomial basis, that evaluate the output monomials $\boldsymbol{y}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\gamma}} = y[k]^{\gamma_0} \cdots y[k+n_y]^{\gamma_{n_y}} = g(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[k])^{\gamma_0} \cdots g(\boldsymbol{\theta}, \boldsymbol{f}^{n_y}(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{n_y-1}]), \boldsymbol{u}[k+$

	1	$x_1[k]$	$x_2[k]$	$x_1[k]^2$	$x_1[k]x_2[k]$	$x_2[k]^2$	$x_1[k]^3$	$x_1[k]^2x_2[k]$	$x_1[k]x_2[k]^2$	$x_2[k]^3$	...
1	1	0	0	0	0	0	0	0	0	0	⋯
$y[k]$	0	0	1	0	0	0	0	0	0	0	⋯
$y[k+1]$	0	0	0	0	$3\theta_2^2$	0	0	0	0	0	⋯
$y[k+2]$	0	0	0	0	0	0	0	$9\theta_1\theta_2^4$	$9\theta_2^5u[k]$	0	⋯
$y[k]^2$	0	0	0	0	0	1	0	0	0	0	⋯
$y[k]y[k+1]$	0	0	0	0	0	0	0	0	$3\theta_2^2$	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Figure 1: A part of the model matrix $\mathcal{M}_2^2$. Each row corresponds to an output monomial, expressed as its polynomial function in the components of $\boldsymbol{x}[k]$. The vertical and horizontal lines partition monomials by degree.

$n_y]\big)^{\gamma_{n_y}}$, for $\boldsymbol{\gamma} \in \Gamma$, with $\Gamma = \left\{\boldsymbol{\gamma} \in \mathbb{N}^{n_y+1} : \|\boldsymbol{\gamma}\|_1 \leq d\right\}$. In compact form, the model matrix is expressed as

$$\boldsymbol{\mathcal{M}}_{n_y}^d = \begin{bmatrix} 1 \\ g(\boldsymbol{\theta}, \boldsymbol{x}[k], u[k]) \\ \vdots \\ g(\boldsymbol{\theta}, \boldsymbol{f}^{n_y}(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{n_y-1}]), u[k+n_y]) \\ \vdots \\ g(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[k])^d \\ \vdots \\ g(\boldsymbol{\theta}, \boldsymbol{f}^{n_y}(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{n_y-1}]), u[k+n_y])^d \end{bmatrix}.$$

The row space of the model matrix satisfies $\text{row}\left(\boldsymbol{\mathcal{M}}_{n_y}^d\right) \subseteq \mathcal{C}_{n_f}^{md}$, where

$$m = \max_{i\in\{0,\ldots,n_y\}} \left(g(\boldsymbol{\theta}, \boldsymbol{f}^{n_y}(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{n_y-1}]), u[k+n_y])\right).$$

The model matrix satisfies the relation

$$\boldsymbol{y}[\boldsymbol{k}_{n_y}]^d = \boldsymbol{\mathcal{M}}_{n_y}^d \boldsymbol{x}[k]^{md}. \tag{12}$$

Example 1 (continued): A part of the model matrix $\boldsymbol{\mathcal{M}}_2^2$ corresponding to Σ_{R} in Fig. 1.

Remark 2: The size of the model matrix corresponding to models in Σ grows in the same manner as in the case for discrete-time, autonomous, time-invariant, single-output polynomial state-space models and is determined by the order bound n_y, the degree d and the number of state-variables n_f. There exists a sufficiently large degree $d = d_{\text{suff}}$ where $\boldsymbol{\mathcal{M}}_{n_y}^d$ has more rows p, than columns, q $\forall$ $d \geq d_{\text{suff}}$, *iff* $n_y \geq n_f$ (Lemma 1, [5]).

2.2. State elimination using the model matrix

The size of the model matrix influences the structure of its fundamental subspaces. We are interested in the left null space of the model matrix $\boldsymbol{\mathcal{M}}_{n_y}^d$, denoted by $\mathcal{L}_{\text{null}}\left(\boldsymbol{\mathcal{M}}_{n_y}^d\right)$. The columns of the basis matrix $\boldsymbol{W}_{n_y}^d \in \mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})^{p\times\nu}$ span $\mathcal{L}_{\text{null}}\left(\boldsymbol{\mathcal{M}}_{n_y}^d\right)$ where the dimension of the left null space is denoted as $\dim(\mathcal{L}_{\text{null}}(\boldsymbol{\mathcal{M}}_{n_y}^d)) = \nu$. The columns $\boldsymbol{W}_{n_y}^d$ provide the vector representations of the input-output difference polynomials which vanish over all $(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) \in \mathcal{B}_\Sigma$ (Theorem 1).

Remark 3: Since the entries of the model matrix depend on the parameter vector $\boldsymbol{\theta}$ and the input sequence $\boldsymbol{u}[\boldsymbol{k}_{n_y}]$, the dimension of its left null space may increase at exceptional parameter values due to additional rank loss not induced by the underlying algebraic relations. To exclude such non-generic degeneracies, all statements are understood for $\boldsymbol{\theta}$ (and an input sequence $\boldsymbol{u}[\boldsymbol{k}_{n_y}]$) belonging to an open dense subset $\Theta \subseteq \mathbb{R}^\ell$ ($\mathcal{U} \subseteq \mathbb{R}^{n_y+1}$) on which the rank of the model matrix is maximal and therefore the dimension of its left null space is minimal and constant. Consequently, only the null vectors corresponding to this generic null space are interpreted as coefficient representations of input-output difference polynomials. Throughout the paper, any reference to $\mathcal{B}_\Sigma$ is understood in this restricted sense.

Example 1 (continued): For the model matrix, we have $\dim\left(\mathcal{L}_{\text{null}}(\boldsymbol{\mathcal{M}}_2^d)\right) = 0$ for $d \leq 2$. However, for $d = 3$, $\dim\left(\mathcal{L}_{\text{null}}(\boldsymbol{\mathcal{M}}_2^3)\right) = 1$. This holds for all $\boldsymbol{\theta}$ in some open, dense subset of $\Theta \subseteq \mathbb{R}^2$, together with any admissible input sequence $\boldsymbol{u}[\boldsymbol{k}_2] \in \mathcal{U} \subseteq \mathbb{R}^3$. Hence, there exists $\boldsymbol{\phi}_{\text{R}}^{\text{T}} \in \mathcal{L}_{\text{null}}\left(\boldsymbol{\mathcal{M}}_2^3\right)$ such that, upon premultiplication with (12), we obtain the difference equation $\boldsymbol{\phi}_{\text{R}}(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_2]) = \boldsymbol{\phi}_{\text{R}}\boldsymbol{y}[\boldsymbol{k}_2]^3 = \left(\boldsymbol{\phi}_{\text{R}}\boldsymbol{\mathcal{M}}_2^3\right)\boldsymbol{x}[k]^{12} = 0$. The expression $\boldsymbol{\phi}_{\text{R}}\boldsymbol{y}[\boldsymbol{k}_2]^3$ is an I/O difference polynomial in $\mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_2]\end{bmatrix})[\boldsymbol{y}[\boldsymbol{k}_2]]$, and $\boldsymbol{\phi}_{\text{R}} \in \mathcal{C}_3^3$ is its vector representation, given by

$$\begin{array}{cccccccc} \cdots & y[k]y[k+2] & y[k+1]^2 & \cdots & y[k]^2y[k+1] & \cdots \\ \left[\begin{array}{cccccc} \boldsymbol{0} & -1 & \theta_1 & \boldsymbol{0} & 3\theta^3u[k] & \boldsymbol{0} \end{array}\right]. \end{array} \tag{13}$$

△

For a fixed order bound n_y and degree d, if $\dim(\mathcal{L}_{\text{null}}(\boldsymbol{\mathcal{M}}_{n_y}^d)) \neq 0$ then consider its basis matrix $\boldsymbol{W}_{n_y}^d \in \mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})^{p\times\nu}$ which satisfies the relation $(\boldsymbol{W}_{n_y}^d)^{\text{T}}\boldsymbol{\mathcal{M}}_{n_y}^d = \boldsymbol{0}$. We write $\boldsymbol{W}_{n_y}^d = \begin{bmatrix}\boldsymbol{\phi}_1 & \cdots & \boldsymbol{\phi}_\nu\end{bmatrix}$, where each $\boldsymbol{\phi}_i$ is interpreted as the coefficient vector (vector representation) of a degree-d input-output difference polynomial in the output monomial basis. We will now provide a sufficient condition for the existence of an input-output difference equation, by extending Theorem 1 of [2] to parametric, input-dependent model matrices.

Theorem 1 (*Existence*): Given a discrete-time, time-invariant, parametric, single-input, single-output polynomial state-space model Σ with n_f state components, and an order bound $n_y \geq n_f$, there exists a nontrivial difference equation $\boldsymbol{\phi}(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) = 0$ satisfied by the output sequence of a model where $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^\ell$ through an admissible output sequence $\boldsymbol{u}[\boldsymbol{k}_{n_y}] \in \mathcal{U} \subseteq \mathbb{R}^{n_y+1}$, where the input-output difference polynomial $\boldsymbol{\phi} \in \mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})\boldsymbol{y}[\boldsymbol{k}_{n_y}]$ and has order at most n_y.

Proof. The claim follows from the model matrix construction (see remark 2). For $n_y \geq n_f$, this construction guarantees the existence of a degree d_{suff} such that, for every $d \geq d_{\text{suff}}$, the model matrix $\boldsymbol{\mathcal{M}}_{n_y}^d$ has more rows than columns. Consequently, $\mathcal{L}_{\text{null}}\left(\boldsymbol{\mathcal{M}}_{n_y}^d\right) \neq \{\boldsymbol{0}\}$, so there exists

a nonzero vector $\phi^{\mathrm{T}} \in \mathcal{L}_{\mathrm{null}}\left(\mathcal{M}_{n_y}^d\right)$, such that there exists, $\phi(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) = \phi \boldsymbol{y}[\boldsymbol{k}_{n_y}]^d = \left(\phi \mathcal{M}_{n_y}^d\right) \boldsymbol{x}[k]^{md} = 0$. Thus, ϕ is the vector representation of a nonzero polynomial in the output monomial basis corresponding to monomials in $\boldsymbol{y}[\boldsymbol{k}_{n_y}]^d$ that vanishes along every admissible model sequence in $\mathcal{B}_\Sigma$. Since this polynomial depends only on the output samples up to $y[k + n_y]$, its order is at most $n_y \geq n_f$. □

Remark 4: We denote by d_{suff} the smallest degree for which the model matrix $\mathcal{M}_{n_y}^d$ has more rows than columns, ensuring the existence of a non–trivial left null space (cf. Theorem 1). While d_{suff} guarantees the existence of a non–trivial left null space for all parameter values, in practice a non–trivial null space may already appear at a smaller degree $d_{\mathrm{min}} < d_{\mathrm{suff}}$. Moreover, the input output difference polynomial derived at d_{min} is irreducible in the ring $\mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})\begin{bmatrix}\boldsymbol{y}[\boldsymbol{k}_{n_y}]\end{bmatrix}$ (Lemma 2, [5]) .

Example 1 (continued): For the model Σ_{R}, $\mathrm{size}\left(\mathcal{M}_2^{22}\right) = 2300 \times 2278$ such that $d_{\mathrm{suff}} = 22$ is the smallest degree for which the number of rows exceeds the number of columns. However, a non–trivial left null space already appears at $d_{\mathrm{min}} = 3$, where $\mathrm{size}\left(\mathcal{M}_2^3\right) = 20 \times 55$. △

Extended state variable. Recall from our introductory example that, by considering the model outputs up to $y[k + 3]$, it is possible to derive an input-output difference polynomial $\phi_{\theta_i} \in \mathbb{R}(\begin{bmatrix}\theta_i & \boldsymbol{u}[\boldsymbol{k}_3]\end{bmatrix})[\boldsymbol{y}[\boldsymbol{k}_3]]$ for $i \in \{1, 2\}$ that vanishes on $\mathcal{B}_{\Sigma_{\mathrm{R}}}$. The parametric I/O difference polynomials (10) and (11), which allow us to determine global or local identifiability of the model parameters, are obtained by eliminating the state variables along with all but one parameter, whose identifiability is to be assessed. Consider the extended state variable $\tilde{\boldsymbol{x}}[k] := \begin{bmatrix}\boldsymbol{x}[k] & \cdots & \theta_{i-1} & \theta_{i+1} & \cdots\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{\tilde{n}_f}$, where $\tilde{n}_f = n_f + \ell - 1$ contains all model parameters except θ_i. The additional states satisfy the dynamics $\theta_j[k + 1] = \theta_j[k]$. Up to now, we have considered multivariate polynomial functions of the model outputs in the ring of the components of the state variable; here, we instead work in the ring of the extended state variable.

Example 1 (continued): The vector representation of $y[k + 2]$ in $\mathcal{C}_3^3$ in the ring of the extended state-variable $\tilde{\boldsymbol{x}}[k] = \begin{bmatrix}x_1[k] & x_2[k] & \theta_2\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^3$ is,

$$\begin{array}{c} 1 \quad \cdots \quad \theta_2^4 x_1[k]^2 x_2[k]^2 \quad \cdots \quad \cdots \quad \theta_2^5 x_1[k]^2 x_2[k] \quad \cdots \\ \left[\begin{array}{c|ccc|c|ccc} 0 & \mathbf{0} & 9\theta_1 & \mathbf{0} & \mathbf{0} & 9u[k] & \mathbf{0} \end{array}\right]. \end{array}$$

The model matrix $\mathcal{M}_3^4$ has coefficients that are polynomial functions of θ_1 and $\boldsymbol{u}[\boldsymbol{k}_3]$, and $\dim\left(\mathcal{L}_{\mathrm{null}}(\mathcal{M}_3^4)\right) = 1$; moreover, this left null space contains the vector representation of the input-output difference polynomial in (10). Theorem 1 provides a sufficient condition for the existence of a difference equation of order at most $n_y = 3$. However, it does not guarantee the existence of a difference equation in which the parameter θ_1 appears. We will show that this is always possible for strongly locally observable and identifiable models (Theorem 7). △

3. Numerical state elimination

3.1. Block Macaulay (model) matrix

As shown in Section 2, for a sufficiently large model matrix $\mathcal{M}_{n_y}^d$ with $n_y \geq n_f$ and $d \geq d_{\mathrm{min}}$, the left null space is nontrivial, i.e., $\dim(\mathcal{L}_{\mathrm{null}}(\mathcal{M}_{n_y}^d)) > 0$. The state elimination algorithm for autonomous state space models based on the model matrix (Algorithm 1,[5]) iteratively enlarges the model matrix by increasing the degree d for a fixed order bound n_y. At each degree, the singular value decomposition (SVD) is used to compute the basis matrix of the left null space of the model matrix until a non-trivial instance of the same is found. For parametric models with inputs, the entries of the model matrix are functions of the parameters $\boldsymbol{\theta}$ and the input sequence $\boldsymbol{u}[\boldsymbol{k}_{n_y}]$. As a result, numerical techniques such as the SVD cannot be directly applied to determine its left null space. In this section, we will first show how the parametric model matrix can be written as a multivariate polynomial pencil in the parameters and the input sequence. Second, use this "*seed pencil*" [15] to construct a block Macaulay matrix, to which we will refer to as the block Macaulay (model) matrix (to be read as the block Macaulay nested model matrix) and demonstrate that vectors in its left null space contain the coefficient representations of the input–output difference polynomials in the parameter-input-output monomial basis.

Definition 2 (*Matrix pencil*): Given a matrix $\boldsymbol{A}$, and $A_{ij} \in \mathbb{R}(\boldsymbol{x})$, where $\boldsymbol{x} = \begin{bmatrix}x_1 & \cdots & x_n\end{bmatrix} \in \mathbb{R}^n$, a matrix pencil is a polynomial matrix representation

$$\boldsymbol{A}(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} \boldsymbol{A}_{\boldsymbol{\alpha}}\, \boldsymbol{x}^{\boldsymbol{\alpha}},$$

where $\mathcal{A} \subset \mathbb{N}^n$ is set of multi-indices where $\boldsymbol{A}_{\boldsymbol{\alpha}}$ are the corresponding real non-zero coefficient matrices. The seed degree of the pencil is defined as $d_{\mathrm{seed}} = \max_{\boldsymbol{\alpha} \in \mathcal{A}} \deg(\boldsymbol{\alpha})$.

Example 1: Consider the matrix pencil corresponding to $\mathcal{M}_2^3$,

$$\mathcal{M}_2^3(\boldsymbol{\theta}_{\mathrm{R}}, \boldsymbol{u}[\boldsymbol{k}_0]) = \sum_{(\boldsymbol{\eta}, \delta) \in \mathcal{A}} \boldsymbol{A}_{(\boldsymbol{\eta}, \delta)} \boldsymbol{\theta}^{\boldsymbol{\eta}} \boldsymbol{u}[\boldsymbol{k}_0]^{\delta},$$

which is a multivariate polynomial matrix pencil in $\boldsymbol{\theta}_{\mathrm{R}}$ and $\boldsymbol{u}[\boldsymbol{k}_0]$ where $\mathcal{A} = \{(\boldsymbol{\eta}, \delta) \in \mathbb{N}^3 : \boldsymbol{A}_{(\boldsymbol{\eta}, \delta)} \in \mathbb{R}^{20 \times 55} \setminus \{\mathbf{0}\}\}$ and the seed degree $d_{\mathrm{seed}} = 18$. Its coefficient matrices are generally sparse. For example the coefficient matrix $\boldsymbol{A}_{(0,2,0)} \in \mathbb{R}^{20 \times 55}$, corresponding to the monomial θ_2^2, has only three nonzero entries, namely $\boldsymbol{A}_{(0,2,0),(3,5)} = \boldsymbol{A}_{(0,2,0),(6,9)} = \boldsymbol{A}_{(0,2,0),(12,14)} = 3$. △

Consider a model matrix pencil, i.e., the matrix pencil corresponding to some model matrix $\mathcal{M}_{n_y}^d$,

$$\mathcal{M}_{n_y}^d(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}]) = \sum_{(\boldsymbol{\eta}, \boldsymbol{\delta}) \in \mathcal{A}} \boldsymbol{A}_{(\boldsymbol{\eta}, \boldsymbol{\delta})} \boldsymbol{\theta}^{\boldsymbol{\eta}} \boldsymbol{u}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\delta}} \tag{14}$$

here $\mathcal{A} = \{(\boldsymbol{\eta}, \boldsymbol{\delta}) \in \mathbb{N}^{\ell+(n_y+1)} : \boldsymbol{A}_{(\boldsymbol{\eta},\boldsymbol{\delta})} \in \mathbb{R}^{p\times q} \setminus \{\boldsymbol{0}\}\}$. We can now rewrite (12) in terms of the coefficient matrices of a model matrix pencil as,

$$\boldsymbol{y}[\boldsymbol{k}_{n_y}]^d = \boldsymbol{M}_{n_y}^d \boldsymbol{z}, \tag{15}$$

where $\boldsymbol{M}_{n_y}^d$ is the block matrix obtained by concatenating the coefficient matrices of the parameter-input monomials of the model matrix pencil $\boldsymbol{\mathcal{M}}_{n_y}^d$ whose seed degree is d_{seed}. Arranging the coefficient matrices of the parameter-input monomials in the model matrix pencil according to the ascending *grnlex* monomial order, we have,

$$\boldsymbol{M}_{n_y}^d = \left[\begin{array}{c|ccc|ccc} \boldsymbol{A_0} & \boldsymbol{A}_{10} & \cdots & \boldsymbol{A}_{01} & \boldsymbol{A}_{d0} & \cdots & \boldsymbol{A}_{0d} \end{array}\right].$$

where $\boldsymbol{z} = \left[\boldsymbol{x}[k]^{md} \;\middle|\; \theta_1\boldsymbol{x}[k]^{md} \cdots u[k+n_y]\boldsymbol{x}[k]^{md} \;\middle|\; \theta_1^{d_{\text{seed}}}\boldsymbol{x}[k]^{md} \cdots u[k+n_y]^{d_{\text{seed}}}\boldsymbol{x}[k]^{md}\right]^{\mathrm{T}}$ is a block multivariate Vandermonde vector [15].

Each row of $\boldsymbol{M}_{n_y}^d$ is the vector representation of an output monomial in the parameter-input-state monomial basis. Consequently, $\boldsymbol{M}_{n_y}^d$ defines a linear map from the parameter-input-state monomials collected in $\boldsymbol{z}$ to the output monomials in $\boldsymbol{y}[\boldsymbol{k}_{n_y}]^d$. This reformulation is the key step that allows the symbolic elimination problem to be recast in terms of numerical linear algebra.

Example 1 (continued): The block matrix associated with the model matrix pencil of $\boldsymbol{\mathcal{M}}_2^3$ is denoted by $\boldsymbol{M}_2^3 \in \mathbb{R}^{20\times 73150}$. However, unlike $\boldsymbol{\mathcal{M}}_2^3$, for which $\dim(\mathcal{L}_{\text{null}}(\boldsymbol{\mathcal{M}}_2^3)) = 1$, the $\dim(\mathcal{L}_{\text{null}}(\boldsymbol{M}_2^3)) = 0$, since the output monomials do not satisfy any nontrivial linear relation with coefficients in $\mathbb{R}$. The coefficient block matrix alone does not yet reveal the desired input-output relations. △

To recover the input-output difference equation numerically, the relation in (15) is extended by considering its parameter-input monomial multiples. Specifically, multiplying both sides by a parameter-input monomial $\boldsymbol{\theta}^{\boldsymbol{\eta}}\boldsymbol{u}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\delta}}$ yields another relation with the same coefficient block matrix, namely $(\boldsymbol{\theta}^{\boldsymbol{\eta}}\boldsymbol{u}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\delta}})\boldsymbol{y}[\boldsymbol{k}_{n_y}]^d = \boldsymbol{M}_{n_y}^d \tilde{\boldsymbol{z}}$, where $\tilde{\boldsymbol{z}} = (\boldsymbol{\theta}^{\boldsymbol{\eta}}\boldsymbol{u}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\delta}})\boldsymbol{z}$ is a block multivariate Vandermonde vector containing higher degree parameter-input monomials than $\boldsymbol{z}$. In this way, each parameter-input monomial multiple of (15) generates another numerical linear relation between parameter-input-state monomials and parameter-input-output monomials. Arranging all such parameter-input monomial multiples up to a prescribed multiple degree d_{shift} in a single block Macaulay (model) matrix $\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}}$ yields,

$$\begin{array}{c|ccccccccc} & \boldsymbol{x}[k]^{md} & \theta_1\boldsymbol{x}[k]^{md} & \cdots & u[k+n_y]\boldsymbol{x}[k]^{md} & \theta_1^2\boldsymbol{x}[k]^{md} & \theta_1\theta_2\boldsymbol{x}[k]^{md} & \cdots & u[k+n_y]^2\boldsymbol{x}[k]^{md} & \cdots \\ \boldsymbol{y}[\boldsymbol{k}_{n_y}]^d & \boldsymbol{A}_1 & \boldsymbol{A}_{\theta_1} & \cdots & \boldsymbol{A}_{u[k+n_y]} & \boldsymbol{A}_{\theta_1^2} & \boldsymbol{A}_{\theta_1\theta_2} & \cdots & \boldsymbol{A}_{u[k+n_y]^2} & \cdots \\ \hline \theta_1\boldsymbol{y}[\boldsymbol{k}_{n_y}]^d & \boldsymbol{0} & \boldsymbol{A}_1 & \cdots & \boldsymbol{0} & \boldsymbol{A}_{\theta_1} & \boldsymbol{A}_{\theta_2} & \cdots & \boldsymbol{0} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ u[k+n_y]\boldsymbol{y}[\boldsymbol{k}_{n_y}]^d & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{A}_1 & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{A}_{u[k+n_y]} & \cdots \\ \theta_1^2\boldsymbol{y}[\boldsymbol{k}_{n_y}]^d & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{A}_1 & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots \\ \theta_1\theta_2\boldsymbol{y}[\boldsymbol{k}_{n_y}]^d & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_1 & \cdots & \boldsymbol{0} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ u[k+n_y]^2\boldsymbol{y}[\boldsymbol{k}_{n_y}]^d & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{A}_1 & \cdots \end{array}.$$

Definition 3 (*Block Macaulay (model) matrix*): Let $\boldsymbol{\mathcal{M}}_{n_y}^d(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])$ be the model matrix pencil of seed degree d_{seed}, and of the form (14). Consider the shift degree $d_{\text{shift}} \in \mathbb{N}$, then the block Macaulay (model) matrix $\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}} \in \mathbb{R}^{p_{\text{B}}\times q_{\text{B}}}$, is the block matrix obtained by arranging the coefficient matrices of all relations generated from (15) by multiplication with parameter-input monomials of total degree at most d_{shift}, according to the chosen graded monomial ordering.

Equivalently, $\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}}$ contains the coefficient matrices of the model matrix pencil together with those of all its parameter-input monomial multiples $\boldsymbol{\theta}^{\boldsymbol{\eta}}\boldsymbol{u}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\delta}}\boldsymbol{\mathcal{M}}_{n_y}^d(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])$, for all multi-indices $\boldsymbol{\alpha} = (\boldsymbol{\eta}, \boldsymbol{\delta})$ satisfying $\deg(\boldsymbol{\alpha}) \leq d_{\text{shift}}$.

The block Macaulay (model) matrix $\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}}$ is of size $p_{\text{B}} \times q_{\text{B}}$ and is given by,

$$p_{\text{B}} = p\binom{d_{\text{shift}} + n_\pi}{n_\pi}, \qquad q_{\text{B}} = q\binom{d_{\text{seed}} + d_{\text{shift}} + n_\pi}{n_\pi},$$

where $n_\pi = \ell + (n_y + 1)$ is the number of parameter and input(s) in the model matrix pencil $\boldsymbol{\mathcal{M}}_{n_y}^d(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])$. Analogously to (12), the block Macaulay (model) matrix satisfies the relation

$$\boldsymbol{y}[\boldsymbol{k}_{n_y}]^{d,d_{\text{shift}}} = \boldsymbol{M}_{n_y}^{d,d_{\text{shift}}} \boldsymbol{x}[k]^{md,d_{\text{shift}}}. \tag{16}$$

Here, $\boldsymbol{y}[\boldsymbol{k}_{n_y}]^{d,d_{\text{shift}}}$ collects all parameter-input monomial multiples, up to total degree d_{shift}, of the output monomials of up to degree d i.e., $\boldsymbol{y}[k]^{d,d_{\text{shift}}} = \left[\boldsymbol{y}[\boldsymbol{k}]^d \;\; \theta_1\boldsymbol{y}[\boldsymbol{k}]^d \;\; \cdots \;\; u[k+n_y]^{d_{\text{shift}}}\boldsymbol{y}[\boldsymbol{k}]^d\right]$, while $\boldsymbol{x}[k]^{md,d_{\text{shift}}}$ collects the parameter-input-state monomials, with $d_{\text{seed}} + d_{\text{shift}}$ degree parameter-input monomial multiples of the state monomial vector $\boldsymbol{x}[k]^{md}$.

Example 1 (continued): For the block Macaulay (model) matrix constructed using the model matrix pencil $\boldsymbol{\mathcal{M}}_2^3(\boldsymbol{\theta}_{\text{R}}, \boldsymbol{u}[k]_0)$ we have $\dim\left(\mathcal{L}_{\text{null}}(\boldsymbol{M}_2^{3,d_{\text{shift}}})\right) = 0$ for $d_{\text{shift}} \leq 3$ (here $d_{\text{seed}} = 18$). However, for $d_{\text{shift}} = 4$, $\dim\left(\mathcal{L}_{\text{null}}(\boldsymbol{M}_2^{3,4})\right) = 1$. Hence, there exists $\boldsymbol{h}_{\text{R}} \in \mathcal{L}_{\text{null}}\left(\boldsymbol{M}_2^{3,4}\right)$ such that, upon pre multiplication with (16), we obtain the difference equation $\phi_{\text{R}}(\boldsymbol{\theta}, \boldsymbol{w}[k_2]) = \boldsymbol{h}_{\text{R}}^{\mathrm{T}}\boldsymbol{y}[\boldsymbol{k}_2]^3 = \left(\boldsymbol{h}_{\text{R}}^{\mathrm{T}}\boldsymbol{M}_2^{3,4}\right)\boldsymbol{x}[k]^{12,4} = 0$. The polynomial $\boldsymbol{h}_{\text{R}}^{\mathrm{T}}\boldsymbol{y}[\boldsymbol{k}_2]^3$ is an irreducible input-output difference polynomial in $\mathbb{R}\left[\boldsymbol{\theta} \;\; \boldsymbol{w}[\boldsymbol{k}_2]\right]$ that vanishes over $\mathcal{B}_{\Sigma_{\text{R}}}$, and $\boldsymbol{h}_{\text{R}}^{\mathrm{T}}$ is its vector representation, given by

$$\begin{array}{cccccccccc} & \cdots & y[k]y[k+2] & \cdots & \theta_1 y[k+1]^2 & \cdots & \theta_2^3 u[k]y[k]^2 y[k+1] \\ \left[\right. & \boldsymbol{0} & -1 & \boldsymbol{0} & 1 & \boldsymbol{0} & 3 & \left.\right]. \end{array}$$

△

For a model matrix $\boldsymbol{\mathcal{M}}_{n_y}^d$, and a given d_{shift} we denote by $\boldsymbol{H}_{n_y}^{d,d_{\text{shift}}} \in \mathbb{R}^{p_{\text{B}}\times v}$ as the basis matrix of the left null space of the block Macaulay (model) matrix corresponding to its model matrix pencil. Consequently, $(\boldsymbol{H}_{n_y}^{d,d_{\text{shift}}})^{\mathrm{T}}\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}} =$

$\mathbf{0}$, where $\boldsymbol{H}_{n_y}^{d,d_{\text{shift}}} = \begin{bmatrix}\boldsymbol{h}_1 & \cdots & \boldsymbol{h}_\nu\end{bmatrix}$, and each $\boldsymbol{h}_i$ is the coefficient vector of a degree-d input-output difference polynomial with parameter-input monomials of degree up to d_{shift} in the parameter-input-output monomial basis.

We will now provide a sufficient condition for the existence of the left null space of the block Macaulay (model) matrix polynomial, allowing us to compute the input-output difference polynomial numerically.

Lemma 1 (*Existence of the left null space of the block Macaulay (model) matrix*): Consider a discrete-time, time-invariant, parametric, single-input, single-output polynomial state-space model Σ, and let $n_y \geq n_f$. Suppose that, for some degree $d \geq d_{\min} \in \mathbb{N}$ of the model matrix, there exists a nontrivial input-output difference polynomial $\phi \in \mathbb{R}\left[\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{w}[\boldsymbol{k}_{n_y}]\end{bmatrix}\right]$ which satisfies the equation $\phi(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) = 0 \; \forall \; (\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) \in \mathcal{B}_\Sigma$. Then there exists a sufficiently large shift degree $\zeta \in \mathbb{N}$ such that, for every $d_{\text{shift}} \geq \zeta$, the associated block Macaulay (model) matrix $\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}}$, has a nontrivial left null space, i.e., $\dim\big(\mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}})\big) > 0$.

Proof. By Theorem 1, for $n_y \geq n_f$ there exists, a nontrivial input-output difference equation $\phi(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) = 0$ where $\phi \in \mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})[\boldsymbol{y}[\boldsymbol{k}_{n_y}]]$ of some degree $d \geq d_{\min}$. Consider that $\deg(\phi) = d_{\min}$ then the coefficients of ϕ in $\mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})$ are polynomial functions of degree ζ. A block Macaulay (model) matrix $\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}}$ where $d \geq d_{\min}$ and $d_{\text{shift}} \geq \zeta$, contains the vector representations of all the parameter-input-output monomials appearing in ϕ. Hence, the rows of $\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}}$ are linearly dependent, resulting in a nonzero vector $\boldsymbol{h} \in \mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}})$. It follows that $\mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}}) \neq \{\mathbf{0}\}$, and $\dim\big(\mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}})\big) > 0$. □

Lemma 2 (*Irreducible input-output difference polynomials*): In the setting of Lem. 1, suppose for some $d = d_{\min}$ and $d_{\text{shift}} = \zeta$, $\dim\big(\mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_y}^{d_{\min},\zeta})\big) = \nu \geq 1$, and $\dim\big(\mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}})\big) = 0$ for $d \leq d_{\min}$ and $d_{\text{shift}} < \zeta$. Each $\boldsymbol{h}_i$ in $\boldsymbol{H}_{n_y}^{d_{\min},\zeta} = \begin{bmatrix}\boldsymbol{h}_1^{\mathrm{T}} & \cdots & \boldsymbol{h}_\nu^{\mathrm{T}}\end{bmatrix}$, is the vector representation of an irreducible input-output difference polynomial $\phi_i \in \mathbb{R}[\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{w}[\boldsymbol{k}_{n_y}]\end{bmatrix}]$, and $\phi(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) = 0$ for $(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) \in \mathcal{B}_\Sigma$.

Proof. Suppose ϕ_i is reducible in the parameter-input-output monomial basis, i.e. $\phi_i = rs$ with $r, s \in \mathbb{R}[\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{w}[\boldsymbol{k}_{n_y}]\end{bmatrix}]$. Then at least one of the factors must vanish, either $r = 0$ or $s = 0$ on $\mathcal{B}_\Sigma$. Since the degrees of r, s are necessarily smaller than that of ϕ_i, either r or s should already be recovered in a basis matrix of $\mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}})$ for some $d_{\text{shift}} < \zeta$. However, this contradicts our assumption that $\dim\big(\mathcal{L}_{\text{null}}(\mathcal{M}_{n_y}^{d,d_{\text{shift}}})\big) = 0$ for $d \leq d_{\min}$ and $d_{\text{shift}} < \zeta \in \mathbb{N}$. Thus, each ϕ_i is irreducible. □

3.2. Strongly locally observable and identifiable model

In this section, we study state elimination in the class of strongly locally observable and identifiable models. For this class, we show that there exists a unique irreducible minimal-order input-output difference equation, with order at most n_f. Furthermore, we show that an input-output difference equation $\phi(\theta_i, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) = 0$ where $\phi \in \mathbb{R}(\begin{bmatrix}\theta_i & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})[\boldsymbol{y}[\boldsymbol{k}_{n_y}]]]$ can be derived for $n_y \geq n_f + (\ell - 1)$, thereby providing an algebraic way to determine whether the model parameter θ_i is globally or locally identifiable.

Definition 4: (*Strongly locally observable models (SLO)*,[1]). For a fixed parameter vector $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^\ell$, a model Σ is strongly locally observable at $\boldsymbol{x}[k]$ under an admissible input sequence $\boldsymbol{u}[\boldsymbol{k}_{n_f-1}] \in \mathcal{U}_{n_f} \subseteq \mathbb{R}^{n_f}$ if there exists a neighborhood $\Psi \subseteq \mathbb{R}^{n_f}$ of $\boldsymbol{x}[k]$ such that, for every $\bar{\boldsymbol{x}}[k] \in \Psi$, the equalities $g(\boldsymbol{\theta}, \boldsymbol{f}^{\,i}(\boldsymbol{\theta}, \bar{\boldsymbol{x}}[k], \boldsymbol{u}[\boldsymbol{k}_{i-1}]), u[k+i]) = g(\boldsymbol{\theta}, \boldsymbol{f}^{\,i}(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{i-1}]), u[k+i])$ for all $i \in \{0, \ldots, n_f - 1\}$ imply $\bar{\boldsymbol{x}}[k] = \boldsymbol{x}[k]$.

Theorem 2: (*Nonlinear observability rank condition (NORC)*,[1]). Given the model Σ, for some $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^\ell$ and an admissible input sequence $\boldsymbol{u}[\boldsymbol{k}_{n_f-1}] \in \mathcal{U}_{n_f} \subseteq \mathbb{R}^{n_f}$. Consider the vector function $\boldsymbol{b}$ whose components are $g(\boldsymbol{\theta}, \boldsymbol{x}[k], u[k])$, $g\big(\boldsymbol{\theta}, \boldsymbol{f}(\boldsymbol{\theta}, \boldsymbol{x}[k], u[k]), u[k+1]\big)$, …, $g\big(\boldsymbol{\theta}, \boldsymbol{f}^{\,n_f-1}(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{n_f-2}]), u[k+n_f-1]\big)$. Let $\text{Jac}_{\boldsymbol{x}[k]}(\boldsymbol{b})$ denote the Jacobian of $\boldsymbol{b}$ with respect to $\boldsymbol{x}[k]$. If $\text{rank}\big(\text{Jac}_{\boldsymbol{x}[k]}(\boldsymbol{b})\big) = n_f$ at some $\boldsymbol{x}[k] \in \mathbb{R}^{n_f}$, then the model is strongly locally observable at $\boldsymbol{x}[k]$.

Theorem 3 (*Converse of the NORC*[1]): Given the model Σ, for a parameter vector $\boldsymbol{\theta} \in \Theta$ and an admissible input sequence $\boldsymbol{u}[\boldsymbol{k}_{n_f}] \in \mathcal{U}_{n_f}$. If the model is strongly locally observable at $\boldsymbol{x}[k]$, then the NORC is satisfied on an open and dense subset of the state space $\mathcal{X} \subseteq \mathbb{R}^{n_f}$.

Example 1 (continued): For Σ_{R}, $\boldsymbol{b} = \begin{bmatrix}x_2[k] & 3\theta_2^2 x_1[k] x_2[k]\end{bmatrix}^{\mathrm{T}}$,

$$\text{Jac}_{\boldsymbol{x}[k]}(\boldsymbol{b}) = \begin{bmatrix} 0 & 1 \\ 3\theta_2^2 x_2[k] & 3\theta_2^2 x_1[k] \end{bmatrix},$$

and $\det\big(\text{Jac}_{\boldsymbol{x}[k]}(\boldsymbol{b})\big) = -3\theta_2^2 x_2[k]$. For all fixed parameter vectors $\boldsymbol{\theta} \in \Theta = \Big\{\begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^2 : \theta_2 \neq 0\Big\}$, and for all admissible inputs $\boldsymbol{u}[\boldsymbol{k}_1] \in \mathcal{U}_1 = \mathbb{R}^2$, one has $\text{rank}(\text{Jac}_{\boldsymbol{x}[k]}(\boldsymbol{b})) = 2$ for all $\boldsymbol{x}[k] \in \mathcal{X} = \Big\{\begin{bmatrix}x_1[k] & x_2[k]\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^2 : x_2[k] \neq 0\Big\}$. Hence, the model (2) at $\boldsymbol{\theta} \in \Theta$ is strongly locally observable on $\mathcal{X}$ through an admissible input sequence in $\mathcal{U}_1$.

Theorem 4 (*Uniqueness of minimal order difference equations*): Given discrete-time, time-invariant, parametric, single-input, single-output polynomial state-space model Σ, with n_f components in the state variable and which is strongly locally observable. For every parameter vector $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^\ell$, there exists an input-output difference equation $\phi(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_f}]) = 0$ where

$\phi \in \mathbb{R}[\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{w}[\boldsymbol{k}_{n_f}]\end{bmatrix}]$ that holds for all admissible input sequences $\boldsymbol{u}[\boldsymbol{k}_{n_f}] \in \mathcal{U}_{n_f} \subseteq \mathbb{R}^{n_f+1}$ and the associated output sequences generated by the model, i.e., $\forall\ (\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_f}]) \in \mathcal{B}_\Sigma$. The irreducible output difference polynomial ϕ is unique (up to a nonzero scalar multiple) and is of order at most n_f.

Proof. We can show that the input-output difference polynomial ϕ of order at most n_f, degree $d_{\min}$ is irreducible in the ring $\mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_f}]\end{bmatrix})\boldsymbol{y}[\boldsymbol{k}_{n_f}]$ and unique up to multiplication by a non-zero element in $\mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_f}]\end{bmatrix})$ by extending the proof of Theorem 5 in [5], from the coefficient field $\mathbb{R}$ to $\mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_f}]\end{bmatrix})$. Furthermore, we have shown in Lemma 1 that the input-output difference polynomial ϕ of degree $d_{\min}$ contains parameter-input monomials of degree up to ζ and that it is irreducible in $\mathbb{R}\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{w}[\boldsymbol{k}_{n_f}]\end{bmatrix}$ (Lemma 2). While all polynomial multiples of the form $r\phi$ also vanish for all $(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_f}]) \in \mathcal{B}_\Sigma$ where $r \in \mathbb{R}\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{w}[\boldsymbol{k}_{n_f}]\end{bmatrix}$ they are no longer irreducible. Hence the irreducible input-output difference polynomial ϕ of $\deg(\phi) = d_{\min}$, degree ζ in the parameter-input monomials and of order at most n_f is unique up to a non-zero scalar multiple. □

Structural identifiability in continuous time models can be reformulated as assessing the observability of the parameter variables by augmenting the parameter vector as a constant state, i.e., with zero dynamics; see [17] and the references therein. Accordingly, we formulate for the models in class Σ the following notion for observability-identifiability and the consequent rank conditions for the extended state variable $\tilde{\boldsymbol{x}}_\ell[k] = \begin{bmatrix}\boldsymbol{x}[k] & \boldsymbol{\theta}\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n_f+\ell}$.

Definition 5: (*Strongly locally observable and identifiable models (SLO-I)*). A model Σ is strongly locally observable and identifiable at $\tilde{\boldsymbol{x}}_\ell[k] = \begin{bmatrix}\boldsymbol{x}[k] & \boldsymbol{\theta}\end{bmatrix}^{\mathrm{T}}$ under an admissible input sequence $\boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}] \in \mathcal{U}_{n_f+\ell} \subseteq \mathbb{R}^{n_f+\ell}$ if there exists a neighborhood Ψ_ℓ of $\tilde{\boldsymbol{x}}_\ell[k]$ such that, for every $\bar{\tilde{\boldsymbol{x}}}_\ell[k] = \begin{bmatrix}\bar{\boldsymbol{x}}[k] & \bar{\boldsymbol{\theta}}\end{bmatrix} \in \Psi_\ell$, the equalities $g(\bar{\boldsymbol{\theta}}, \boldsymbol{f}^i(\bar{\boldsymbol{\theta}}, \bar{\tilde{\boldsymbol{x}}}_\ell[k], \boldsymbol{u}[\boldsymbol{k}_{i-1}]), u[k+i]) = g(\boldsymbol{\theta}, \boldsymbol{f}^i(\boldsymbol{\theta}, \tilde{\boldsymbol{x}}_\ell[k], \boldsymbol{u}[\boldsymbol{k}_{i-1}]), u[k+i])$ for all $i \in \{0, \dots, n_f+\ell-1\}$ imply $\bar{\tilde{\boldsymbol{x}}}_\ell[k] = \tilde{\boldsymbol{x}}_\ell[k]$.

The proof of the following theorem proceeds identically to those of Theorems 2 and 3 after augmenting the state vector with the model parameters. Specifically, we introduce the extended state variable $\tilde{\boldsymbol{x}}_\ell[k]$, in which the parameter states evolve according to $\boldsymbol{\theta}[k+1] = \boldsymbol{\theta}[k]$.

Theorem 5: (*Nonlinear observability-identifiability rank condition (NOIRC)*). Given the model Σ, an admissible input sequence $\boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}] \in \mathcal{U}_{n_f+\ell} \subseteq \mathbb{R}^{n_f+\ell}$. Consider the vector function $\tilde{\boldsymbol{b}}$ whose components are $g(\boldsymbol{\theta}, \boldsymbol{x}[k], u[k])$, $g\big(\boldsymbol{\theta}, \boldsymbol{f}(\boldsymbol{\theta}, \boldsymbol{x}[k], u[k]), u[k+1]\big)$, …, $g\big(\boldsymbol{\theta}, \boldsymbol{f}^{n_f+\ell-1}(\boldsymbol{\theta}, \boldsymbol{x}[k], \boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-2}]), u[k+n_f+\ell-1]\big)$. Let $\mathrm{Jac}_{\tilde{\boldsymbol{x}}_\ell[k]}(\tilde{\boldsymbol{b}})$ denote the Jacobian of $\tilde{\boldsymbol{b}}$ with respect to the extended state variable $\boldsymbol{x}_\ell[k]$. If $\mathrm{rank}\big(\mathrm{Jac}_{\boldsymbol{x}_\ell[k]}(\tilde{\boldsymbol{b}})\big) = n_f+\ell$ at some $\tilde{\boldsymbol{x}}_\ell[k] \in \mathbb{R}^{n_f+\ell}$, then the model is strongly locally observable and identifiable at $\tilde{\boldsymbol{x}}_\ell[k]$.

Theorem 6 (*Converse of the augmented NOIRC*): Given the model Σ, and an admissible input sequence $\boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}] \in \mathcal{U}_{n_f+\ell} \subseteq \mathbb{R}^{n_f+\ell}$. If the model is strongly locally observable and identifiable at $\tilde{\boldsymbol{x}}_\ell[k]$, then the NOIRC is satisfied on an open and dense subset of the extended state space $\mathcal{X}_\ell \subseteq \mathbb{R}^{n_f+\ell}$.

In Example 1, we derived the difference equation (10) in the single variable θ_1 by introducing the extended state variable $\tilde{\boldsymbol{x}}[k] = \begin{bmatrix}x_1[k] & x_2[k] & \theta_2\end{bmatrix}^{\mathrm{T}}$. Theorem 1 guarantees that, for the associated model matrix $\boldsymbol{\mathcal{M}}_3^d$, there exists a nontrivial left null space for some degree $d \geq d_{\mathrm{suff}}$, so that the extended state variable can be eliminated. However, this does not by itself imply that θ_1 enters the resulting difference equation nontrivially. We next show that this is indeed the case for strongly locally observable and identifiable models.

Theorem 7 (*Existence and uniqueness of single-parameter difference equations*): Given a discrete-time, time-invariant, parametric, single-input, single-output polynomial state-space model Σ, with n_f components in the state variable and ℓ parameters, and which is strongly locally observable and identifiable. Then, for $i \in \{1, \dots, \ell\}$, there exists an input-output difference equation $\phi(\theta_i, \boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-1}]) = 0$, in which the parameter θ_i appears nontrivially, and holds for all admissible input sequences $\boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}] \in \mathcal{U}_{n_f+\ell} \subseteq \mathbb{R}^{n_f+\ell}$ and the associated output sequences generated by the model i.e., $\forall\ (\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-1}]) \in \mathcal{B}_\Sigma$. Moreover, the corresponding irreducible input-output difference polynomial $\phi \in \mathbb{R}\begin{bmatrix}\theta_i & \boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-1}]\end{bmatrix}$ is unique up to multiplication by a nonzero scalar, and has order at most $n_f+\ell-1$.

Proof. We will first show the *existence* of such a relation and then its *uniqueness*. Since the model is strongly locally observable and identifiable, the NOIRC is satisfied on an open and dense subset of the extended state space. For a fixed admissible input sequence, consider the vector function $\tilde{\boldsymbol{b}}$ whose components are the multivariate polynomial in $\mathbb{R}(\begin{bmatrix}\boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}]\end{bmatrix})[\tilde{\boldsymbol{x}}_\ell[k]]$ that evaluate $y[k], \dots, y[k+n_f+\ell-1]$. Here $\tilde{\boldsymbol{x}}_\ell = \begin{bmatrix}\boldsymbol{x}[k] & \boldsymbol{\theta}\end{bmatrix}^{\mathrm{T}}$. Since $\mathrm{rank}(\mathrm{Jac}_{\tilde{\boldsymbol{x}}[k]}(\tilde{\boldsymbol{b}})) = n_f+\ell$ for $\tilde{\boldsymbol{x}}_\ell[k] \in \mathcal{X}_\ell \subseteq \mathbb{R}^{n_f+\ell}$ through an admissible input $\boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}] \in \mathcal{U}_{n_f+\ell} \subseteq \mathbb{R}^{n_f+\ell}$, the Jacobian criterion [3] implies that the polynomials functions of $y[k], \dots, y[k+n_f+\ell-1]$ are algebraically independent over $\mathbb{R}[\boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}]]$. In particular, there exists no nontrivial input-output difference equation of order at most $n_f+\ell-1$ with coefficients in $\mathbb{R}[\boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}]]$ alone. Now fix an index $i \in \{1, \dots, \ell\}$ and consider the extended state variable $\tilde{\boldsymbol{x}}[k] = \begin{bmatrix}\cdots & \theta_{i-1} & \theta_{i+1} & \cdots & \boldsymbol{x}[k]\end{bmatrix} \in \mathbb{R}^{n_f+\ell-1}$ obtained by excluding θ_i, so that θ_i is the only parameter not absorbed into the state. For the corresponding model matrix, Theorem 1 guarantees the existence of a nontrivial input-output difference equation of order at most $n_f+\ell-1$, namely

$\phi(\theta_i, \boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-1}]) = 0 \ \forall \ (\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-1}]) \in \mathcal{B}_\Sigma$ where $\phi \in \mathbb{R}(\begin{bmatrix}\theta_i & \boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}]\end{bmatrix})[\boldsymbol{y}[\boldsymbol{k}_{n_f+\ell-1}]]$. We have shown that there does not exist a difference equation of order at most $n_f+\ell-1$ whose coefficients are entirely in $\mathbb{R}[\boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}]]$. Therefore ϕ must depend nontrivially on the remaining parameter θ_i. Hence, there exists an input-output difference equation in which θ_i appears nontrivially. To show that ϕ is unique, we will first show that there exists no nontrivial difference equation of the form $\rho(\theta_i, \boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-2}]) = 0$ and where $\rho \in \mathbb{R}(\begin{bmatrix}\theta_i & \boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-2}]\end{bmatrix})[\boldsymbol{y}[\boldsymbol{k}_{n_f+\ell-2}]]$ is irreducible i.e., we will rule out the possibility of a difference equation that is of an order lower than $n_f+\ell-1$. If such a relation existed, then its one-step shift $\rho' \in \mathbb{R}(\begin{bmatrix}\theta_i & \boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}]\end{bmatrix})[\boldsymbol{y}[\boldsymbol{k}_{n_f+\ell-1}]]$ would yield a second irreducible relation $\rho'(\theta_i, \boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-1}]) = 0$ in the same parameter θ_i. We can then write $\rho, \rho' \in \mathbb{R}(\begin{bmatrix}\boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-1}]\end{bmatrix})[\theta_i]$. Since ρ and its one-step shift ρ' depend essentially on different endpoint variables of their respective time windows their greatest common divisor is one. Eliminating θ_i via a resultant gives a non-zero integer polynomial in the $\mathbb{R}\begin{bmatrix}\boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-1}]\end{bmatrix}$ (Proposition 3, Chapter 3 in [4]) that vanishes whenever $\rho(\theta_i, \boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-2}]) = \rho'(\theta_i, \boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-1}]) = 0$, contradicting the algebraic independence of the polynomial functions of outputs in $\boldsymbol{y}[\boldsymbol{k}_{n_f+\ell-1}]$ over $\mathbb{R}[\boldsymbol{u}[\boldsymbol{k}_{n_f+\ell-1}]]$. Hence no single parameter difference equation of order lower than $n_f+\ell-1$ exists. The uniqueness of the irreducible single parameter difference equation $\phi \in \mathbb{R}\begin{bmatrix}\theta_i & \boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-1}]\end{bmatrix}$ follows exactly as in Theorem 4. Therefore, the irreducible input-output difference polynomial with respect to the output variables is unique up to multiplication by a nonzero scalar and is of order at most $n_f+\ell-1$ by construction. □

Remark 5: Theorem 7 asserts uniqueness of the irreducible input-output difference polynomial of order at most $n_f+\ell-1$ where $\phi \in \mathbb{R}\begin{bmatrix}\theta_i & \boldsymbol{w}[\boldsymbol{k}_{n_f+\ell-1}]\end{bmatrix}$ for a fixed parameter θ_i. This does not imply uniqueness with respect to the order of the difference equation, since an analogous difference equation may be obtained for each $\theta_i, i \in \{1, \ldots, \ell\}$. This is already the case for Σ_{R}, see (10)-(11). By contrast the difference equation of minimal order at most n_f is unique.

3.3. Orthogonal complementarity

We now establish the orthogonal complementarity between the image representation defined by the block Macaulay model matrix and the block Macaulay (Macaulay) matrix its kernel representation generated by the unique minimal-order input-output difference equation. The construction of the block Macaulay (Macaulay) matrix proceeds in two steps. First, the ordinary Macaulay matrix is constructed from the input-output difference polynomial with respect to the output variables, while its entries remain functions of the parameters and inputs. Second, this Macaulay matrix is written as a matrix pencil in the parameter-input monomials and finally this matrix generates the block Macaulay (Macaulay) matrix (to be read as the block Macaulay nested Macaulay matrix) associated with the difference equation.

Definition 6 (*Macaulay matrix*, [2]): Given a set of polynomials $r_1, \ldots, r_s$ where $r_i \in \mathcal{K}[x_1, \ldots, x_n]$, of degree d_i respectively, the Macaulay matrix of degree d, denoted by Mac^d contains in its rows the vector representations of

$$\mathrm{Mac}^d = \begin{bmatrix} r_1 \\ (x_1 r_1) \\ \vdots \\ \left(x_n^{d-d_1} r_1\right) \\ \vdots \\ r_s \\ (x_1 f_s) \\ \vdots \\ \left(x_n^{d-d_s} r_s\right) \end{bmatrix}$$

in the monomial basis of the x_i variables where each polynomial r_i is multiplied with all monomials from degree 0 to degree $d-d_i$.

Example 1 (continued): Consider the difference equation $\phi \in \mathbb{R}(\begin{bmatrix}\theta & \boldsymbol{u}[\boldsymbol{k}_2]\end{bmatrix})[\boldsymbol{y}[\boldsymbol{k}_2]]$ of the minimal order $\mathrm{ord}(\phi) = 2$, its vector representation in (13) is a Macaulay matrix generated by ϕ of $\deg(\phi) = 3$. △

In particular, if $\phi \in \mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_y}]\end{bmatrix})[\boldsymbol{y}[\boldsymbol{k}_{n_y}]]$ is an irreducible input-output difference polynomial of degree $d_{\min}$, then the Macaulay matrix of degree d generated by ϕ is denoted by Mac_ϕ^d, and contains in its rows the coefficient vectors of the form $\boldsymbol{y}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\gamma}}\phi$ where $\|\boldsymbol{\gamma}\|_1 \leq d - d_{\min}$. The entries of Mac_ϕ^d are functions of the parameters and inputs, whereas its columns are indexed only by output monomials. We will denote the size of Mac^ϕ by $a \times b$. Furthermore, we can expand Mac_ϕ^d as a matrix pencil as per Definition 2 resulting in,

$$\mathrm{Mac}_\phi^d(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}]) = \sum_{(\boldsymbol{\eta},\boldsymbol{\delta})\in\mathcal{D}} \boldsymbol{D}_{(\boldsymbol{\eta},\boldsymbol{\delta})} \boldsymbol{\theta}^{\boldsymbol{\eta}} \boldsymbol{u}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\delta}}. \tag{17}$$

where $\mathcal{D} = \{(\boldsymbol{\eta}, \boldsymbol{\delta}) \in \mathbb{N}^{\ell+n_y+1} : \boldsymbol{D}_{\boldsymbol{\eta},\boldsymbol{\delta}} \in \mathbb{R}^{a\times b} \setminus \{\boldsymbol{0}\}\}$ and the seed degree of the pencil is m_{seed}. Note that, input-output difference polynomial corresponding to some Σ vanish over all $(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) \in \mathcal{B}_\Sigma$, by extension, we have the following equation

$$\mathrm{Mac}_\phi^d(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])\boldsymbol{y}[\boldsymbol{k}_{n_y}]^d = \boldsymbol{D}_\phi^d \boldsymbol{y}[\boldsymbol{k}]^{d, m_{\mathrm{seed}}} = \boldsymbol{0}, \tag{18}$$

where $\boldsymbol{D}_\phi^d$ is a block matrix, which is constructed by concatenating in some monomial order (here $grnlex$) the coefficient matrices $\boldsymbol{D}_{(\boldsymbol{\eta},\boldsymbol{\delta})}$ of $\mathrm{Mac}_\phi^d(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])$ of the parameter-input monomials up to the seed degree of the pencil m_{seed}.

Definition 7 (*Block Macaulay (Macaulay) matrix*): Let $\mathrm{Mac}_\phi^d(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])$ be the matrix pencil generated by the Macaulay matrix of degree d corresponding to ϕ which is

of seed degree m_{seed} and of form (17). Consider the shift degree $m_{\text{shift}} \in \mathbb{N}$, then the block Macaulay (Macaulay) matrix $\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}}$, is the block matrix obtained by arranging the coefficient matrices of all relations generated from (18) by multiplication with parameter-input monomials of total degree at most m_{shift}, according to the chosen graded monomial ordering.

The ordinary Macaulay matrix Mac_{ϕ}^{d} is therefore a symbolic matrix whose columns are indexed by output monomials of degree up to d. By contrast, $\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}}$ is a numerical coefficient matrix whose columns are indexed by joint parameter-input-output monomials, where the output monomials are of degree up to d and the parameter-input monomials up to degree m_{shift}.

Lemma 3 (*Orthogonal complementarity of the block representations*): Consider the setting of Theorem 4, and let ϕ be the unique irreducible input-output difference polynomial of minimal order $\text{ord}(\phi) \leq n_f$, up to multiplication by a nonzero element of the coefficient field. For every $d \geq d_{\min}, d_{\text{shift}} \geq \zeta$, define $m_{\text{shift}} = d_{\text{shift}} - \zeta$. then, $\text{row}(\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}}) = \mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_f}^{d,d_{\text{shift}}})$. Consequently, $\text{row}(\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}}) \perp \text{col}(\boldsymbol{M}_{n_f}^{d,d_{\text{shift}}})$, and $\dim(\text{row}(\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}})) + \dim(\text{col}(\boldsymbol{M}_{n_f}^{d,d_{\text{shift}}})) = p_{\text{B}}$.

Proof. In the setting of Theorem 4, we can extend Lemma 3 in [5] to the models in class Σ by following the proof in [5] in the coefficient field $\mathbb{R}(\begin{bmatrix}\boldsymbol{\theta} & \boldsymbol{u}[\boldsymbol{k}_{n_f}]\end{bmatrix})$ to show that, $\mathcal{L}_{\text{null}}(\boldsymbol{\mathcal{M}}_{n_f}^{d}) = \text{row}(\text{Mac}_{\phi}^{d})$. We can then write, $(\text{Mac}_{\phi}^{d}\boldsymbol{\mathcal{M}}_{n_f}^{d})\boldsymbol{x}[k]^{md} = \boldsymbol{0}$. Upon expanding Mac_{ϕ}^{d} in its matrix pencil form (17), we have

$$\left(\sum_{(\boldsymbol{\eta},\boldsymbol{\delta})\in\mathcal{D}} \boldsymbol{D}_{(\boldsymbol{\eta},\boldsymbol{\delta})}\boldsymbol{\theta}^{\boldsymbol{\eta}}\boldsymbol{u}[\boldsymbol{k}_{n_y}]^{\boldsymbol{\delta}}\boldsymbol{\mathcal{M}}_{n_f}^{d}\right)\boldsymbol{x}[k]^{md} = \boldsymbol{0}, \tag{19}$$

where $(\boldsymbol{\eta},\boldsymbol{\delta}) \in \mathcal{D} = \{(\boldsymbol{\eta},\boldsymbol{\delta}) \in \mathbb{N}^{\ell+(n_f+1)} : \|(\boldsymbol{\eta},\boldsymbol{\delta})\|_1 \leq \zeta\}$. Observe that $m_{\text{seed}} = \zeta$. We can rewrite (19) as,

$$\left(\boldsymbol{D}_{\phi}^{d,0}\begin{bmatrix}\boldsymbol{\mathcal{M}}_{n_f}^{d}\\ \theta_1\boldsymbol{\mathcal{M}}_{n_f}^{d}\\ \vdots \\ u[k+n_y]^{\zeta}\boldsymbol{\mathcal{M}}_{n_f}^{d}\end{bmatrix}\right)\boldsymbol{x}[k]^{md} = \boldsymbol{0}, \tag{20}$$

notice that each block row contains parameter-input monomial multiples of the model matrix (or by expansion, its corresponding matrix pencil). Hence, we can rewrite (20) using Definition 3 as,

$$(\boldsymbol{D}_{\phi}^{d,0}\boldsymbol{M}_{n_f}^{d,\zeta})\boldsymbol{x}[k]^{md,\zeta} = \boldsymbol{0}. \tag{21}$$

Since (21) is valid for all $\boldsymbol{x}[k]^{md,\zeta}$ we have $\text{row}(\boldsymbol{D}_{\phi}^{d,0}) \subseteq \mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_f}^{d,\zeta})$. Moreover, every row of $\boldsymbol{D}_{\phi}^{d,0}$ is the coefficient-vector representation of a distinct output-monomial multiple of ϕ in the parameter-input-output monomial basis. Consequently, the rows of $\boldsymbol{D}_{\phi}^{d,0}$ are linearly independent, and $\boldsymbol{D}_{\phi}^{d,0}$ has full row rank. Furthermore, any vector in $\mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_f}^{d,\zeta})$ outside $\text{row}(\boldsymbol{D}_{\phi}^{d,0})$ would induce an extra polynomial relation in $\mathcal{L}_{\text{null}}(\boldsymbol{\mathcal{M}}_{n_f}^{d})$, contradicting $\mathcal{L}_{\text{null}}(\boldsymbol{\mathcal{M}}_{n_f}^{d}) = \text{row}(\text{Mac}_{\phi}^{d})$ since the latter is exactly spanned by the output-monomial multiples of ϕ, i.e., the rows of $\boldsymbol{D}_{\phi}^{d,0}$. Hence, $\mathcal{L}_{\text{null}}\left(\boldsymbol{M}_{n_f}^{d,\zeta}\right) = \text{row}(\boldsymbol{D}_{\phi}^{d,0})$. Now, consider up to degree m_{shift} parameter-input monomial multiples of (19), and stack the blocks rows vertically in the ascending monomial order to obtain,

$$(\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}}\boldsymbol{M}^{d,d_{\text{shift}}})\boldsymbol{x}[k]^{md,d_{\text{shift}}} = \boldsymbol{0},$$

here $d_{\text{shift}} = \zeta + m_{\text{shift}}$ and $\text{row}(\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}}) \subseteq \mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_f}^{d,d_{\text{shift}}})$. The rows of $\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}}$ are obtained by multiplying the relations represented by $\boldsymbol{D}_{\phi}^{d,0}$ with parameter-input monomials of degree at most m_{shift}. These relations have distinct leading monomials, and the rows of $\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}}$ are therefore linearly independent. Moreover, any additional vector in $\mathcal{L}_{\text{null}}\left(\boldsymbol{M}_{n_f}^{d,d_{\text{shift}}}\right)$ would define an additional relation in $\mathcal{L}_{\text{null}}\left(\boldsymbol{\mathcal{M}}_{n_f}^{d}\right)$. Since every such relation is generated by ϕ, and ϕ has parameter-input degree ζ, the corresponding multiplier has degree at most $d_{\text{shift}} - \zeta = m_{\text{shift}}$. All such relations are already represented by the rows of $\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}}$. Hence, $\mathcal{L}_{\text{null}}(\boldsymbol{M}_{n_f}^{d,d_{\text{shift}}}) = \text{row}(\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}})$. It follows from the fundamental theorem of linear algebra that $\text{row}(\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}}) \perp \text{col}(\boldsymbol{M}_{n_f}^{d,d\text{shift}})$, and therefore $\dim(\text{row}(\boldsymbol{D}_{\phi}^{d,m_{\text{shift}}})) + \dim(\text{col}(\boldsymbol{M}_{n_f}^{d,d_{\text{shift}}})) = p_{\text{B}}$. □

Remark 6: We can extend the orthogonal complementarity between the row space of the block Macaulay (Macaulay) matrices generated by the unique, irreducible single–parameter difference equation (Theorem 7) and the column space of the block Macaulay (model) matrix constructed using the model matrix of order bound $n_f + (\ell - 1)$ and extended state-variable with all but one model parameter considered as the state.

4. Numerical algorithms

Before presenting the numerical algorithm, we briefly explain its role and scope. In Algorithm 1, we propose a naive state elimination procedure which, given a state-space model in Σ and an order bound n_y, finds an input-output difference equation of minimal degree $d_{\min}$, and of order at most n_y. For a given state-space model, the minimal degree $d_{\min}$ at which a nontrivial input-output difference equation appears (Lemma 1) is generally not known a priori. For this reason, the algorithm proceeds iteratively with respect to the degree d of the model matrix, and at each degree checks whether a corresponding difference equation exists. By Lemma 1, once $d \geq d_{\min}$, there exists a sufficiently large shift degree ζ such that the associated block Macaulay

(model) matrix has a nontrivial left null space, and hence yields a difference equation. However, when the model matrix degree $d < d_{\min}$, no such relation exists, and the algorithm must still terminate at that degree before proceeding to the next one. In principle, this can be ensured by deriving an upper bound on the required shift degree for a fixed d, under the assumption that a difference equation of degree d exists. Such a bound may be obtained, for example, by determinant-based arguments. In practice, however, this bound is typically highly conservative, and the resulting block Macaulay (model) matrix becomes prohibitively large. For this reason, we do not provide such bounds here.

Algorithm 1 Naive state elimination algorithm

Input: State-space model in Σ, order bound $n_y \geq n_f$, shift bound $\zeta_{\text{bound}} \in \mathbb{N}$.

Output: Difference equation(s) $\phi(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) = 0$, where $\phi \in \mathbb{R}\left[\boldsymbol{\theta} \quad \boldsymbol{w}[\boldsymbol{k}_{n_y}]\right]$ is irreducible, with $\text{ord}(\phi) \leq n_y$ and $\deg(\phi) = d_{\min}$.

Initialize $d \leftarrow 0$ and $\boldsymbol{H}_{n_y}^{d,d_{\text{shift}}} \leftarrow \boldsymbol{0}$.
while $\boldsymbol{H}_{n_y}^{d,d_{\text{shift}}} = \boldsymbol{0}$ **do**
  $d \leftarrow d + 1$ and construct the model matrix $\boldsymbol{\mathcal{M}}_{n_y}^{d}$.
  Initialize $d_{\text{shift}} \leftarrow 0$ and $\boldsymbol{H}_{n_y}^{d,d_{\text{shift}}} \leftarrow \boldsymbol{0}$.
  Split $\boldsymbol{\mathcal{M}}_{n_y}^{d}$ into its matrix pencil.
  while $\boldsymbol{H}_{n_y}^{d,d_{\text{shift}}} = \boldsymbol{0}$ **and** $d_{\text{shift}} < \zeta_{\text{bound}}$ **do**
    Construct $\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}}$.
    Compute the SVD of $\boldsymbol{M}_{n_y}^{d,d_{\text{shift}}}$ and extract a numerical basis matrix of its left null space, denoted by $\boldsymbol{H}_{n_y}^{d,d_{\text{shift}}}$.
    if $\boldsymbol{H}_{n_y}^{d,d_{\text{shift}}} \neq \boldsymbol{0}$ **then**
      $d_{\min} \leftarrow d$.
      return $\boldsymbol{H}_{n_y}^{d_{\min},d_{\text{shift}}} = \begin{bmatrix} h_1^{\mathrm{T}} & \cdots & h_\nu^{\mathrm{T}} \end{bmatrix}$.
    end if
    $d_{\text{shift}} \leftarrow d_{\text{shift}} + 1$.
  end while
end while

We now describe a practical implementation of Algorithm 1, organized in three stages. Each stage performs a numerical SVD on a model/block Macaulay (model) matrix [3]; what changes across stages is which variables are treated as known and which monomial support of the polynomial difference equation is being identified. The reason why this numerical strategy is useful is that the difference equations we seek are valid on an open and dense subset of the model parameters and admissible input sequences. Therefore, in the exact setting, sampling parameters and inputs independently from continuous distributions selects a point in this open and dense subset with probability one. In an actual implementation, these samples are represented by floating-point numbers. Repeating the procedure over several independent trials makes the support detection more robust against exceptional choices, finite-precision effects, and ill-conditioned samples.

Stage 1: estimating $d_{\min}$ and the output-monomial support. We randomly sample a parameter $\bar{\boldsymbol{\theta}} \in \mathbb{R}^{\ell}$ and an admissible input sequence $\bar{\boldsymbol{u}}[\boldsymbol{k}_{n_y}] \in \mathbb{R}^{n_y+1}$. Substituting these into Σ yields a fully numerical instance of the model matrix, to which we apply Algorithm 1 in [5]. We record both the degree of the resulting difference equation(s) and the output monomials appearing in it with nonzero coefficient. Repeating over N trials, the union of the recorded supports gives the output-monomial set $\mathcal{S}_y$ and the maximum degree of the output monomials in $\mathcal{S}_y$ is the $d_{\min}$. The degree of the input-output difference polynomial is at most $d_{\min}$. We have now determined, for an instance of $\boldsymbol{\theta}$ and $\boldsymbol{u}[\boldsymbol{k}_{n_y}]$ the linearly dependent multivariate polynomials that evaluate output monomials corresponding to some Σ.

Example 1 (continued): We will demonstrate each stage of algorithm 2 on Σ_{R}. Consider the extended state variable $\tilde{\boldsymbol{x}}[k] = \begin{bmatrix} x_1[k] & x_2[k] & \theta_2 \end{bmatrix}$ and the order bound $n_y = 3$. We then have at stage 1, for a numerical instance

$$\begin{bmatrix} \bar{\theta}_1 & \bar{u}[k] & \bar{u}[k+1] \end{bmatrix} = \begin{bmatrix} 2 & 7 & 11 \end{bmatrix}.$$

We recover the difference equation at $d_{\min} = 4$ given by,

$$\begin{aligned} &7y[k]^2y[k+1]y[k+3] - 11y[k]y[k+1]y[k+2]^2 \\ &\quad + 22y[k+1]^3y[k+2] - 14y[k]^2y[k+2]^2 = 0. \end{aligned}$$

The output-monomial support recovered in this trial is

$$\begin{aligned} \mathcal{S}_y = \{&y[k]^2y[k+1]y[k+3],\ y[k]y[k+1]y[k+2]^2, \\ &y[k+1]^3y[k+2],\ y[k]^2y[k+2]^2\}. \end{aligned}$$

Repeating this computation over several generic numerical instances gives the same degree and support. In, stage 1 we identify the the output monomials, in the input-output difference polynomial, but not yet the dependence of the coefficients on the inputs or parameters. △

Stage 2: identifying the input-output monomials. We consider the model matrix pencil $\boldsymbol{\mathcal{M}}(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])_{\mathcal{S}_y}$, obtained by retaining the rows of the model matrix $\boldsymbol{\mathcal{M}}_{n_y}^{d_{\min}}$ indexed only by the output-monomials in the support $\mathcal{S}_y$ identified in stage 1. Evaluating the restricted pencil at a randomly sampled $\bar{\boldsymbol{\theta}}$ yields a polynomial pencil in $\boldsymbol{u}[\boldsymbol{k}_{n_y}]$ with numerical coefficient matrices. We then enlarge the associated block Macaulay matrix until it admits a nontrivial left null space that spans the basis matrix $\boldsymbol{H}_{n_y}^{d_{\min},d_{\text{shift}}}$. Each basis vector represents a difference equation in the input-output monomial basis for some fixed model parameter. The monomials corresponding to the nonzero entries in each basis vector determine the input-output monomial support of the difference equation (Lemma 1). Repeating over N trials, and taking the union of the recorded supports gives the input-output monomial set $\mathcal{S}_{uy}$.

[3] Whenever a model matrix or block Macaulay matrix is constructed, we first remove its zero columns. This preprocessing step does not affect the computation of a basis matrix for its left null space

Example 1 (continued): Continuing the example, we now consider the model matrix pencil $\mathcal{M}(\theta_1, \boldsymbol{u}[\boldsymbol{k}_1])_{\mathcal{S}_y}$ obtained by considering the rows of the model matrix $\mathcal{M}_3^4$ that correspond to the output support found in stage 1. At this stage, we sample only the parameters. For instance, we take $\bar{\theta}_1 = 2$, while keeping the input variables $u[k]$ and $u[k+1]$ undetermined. The model matrix pencil $\mathcal{M}(\bar{\theta}_1, \boldsymbol{u}[\boldsymbol{k}_1])_{\mathcal{S}_y}$ is of the form,

$$\mathcal{M}(\bar{\theta}_1, \boldsymbol{u}[\boldsymbol{k}_1])_{\mathcal{S}_y} = \sum_{\boldsymbol{\delta} \in \mathcal{A}} \boldsymbol{A}_{\boldsymbol{\delta}} \boldsymbol{u}[\boldsymbol{k}_1]^{\boldsymbol{\delta}},$$

where $\boldsymbol{A}_{\boldsymbol{\delta}} \in \mathbb{R}^{4\times 6}$ and $\mathcal{A} = \{\boldsymbol{\delta} \in \mathbb{N}^2 : |\boldsymbol{\delta}| \leq 2\}$. At $d_{\text{shift}} = 1$, the associated block Macaulay matrix admits a nontrivial left null space of dimension one where $\boldsymbol{H}_3^{4,1}$ is,

$$\begin{bmatrix} \cdots & \cdots & u[k]y[k]^2y[k+1]y[k+3] & \cdots & u[k+1]y[k]y[k+1]y[k+2]^2 & \cdots & u[k+1]y[k+1]^3y[k+2] & \cdots \\ \mathbf{0} & -2 & \mathbf{0} & 1 & \mathbf{0} & -1 & \mathbf{0} & 2 \quad \mathbf{0} \end{bmatrix}.$$

The recovered input-output relation is

$$\begin{aligned} &u[k]y[k]^2y[k+1]y[k+3] \\ &\quad - u[k+1]y[k]y[k+1]y[k+2]^2 \\ &\quad + 2u[k+1]y[k+1]^3y[k+2] \\ &\quad - 2u[k]y[k]^2y[k+2]^2 = 0. \end{aligned}$$

and the input-output monomial support recovered is

$$\begin{aligned} \mathcal{S}_{uy} = \{&u[k]y[k]^2y[k+1]y[k+3], \\ &u[k+1]y[k]y[k+1]y[k+2]^2, \\ &u[k+1]y[k+1]^3y[k+2], \\ &u[k]y[k]^2y[k+2]^2\}, \end{aligned}$$

Repeating this computation over several generic numerical instances gives the same support. △

Stage 3: recovering the coefficients. We now consider the polynomial pencil $\mathcal{M}(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])_{\mathcal{S}_{uy}}$ corresponding to the model matrix whose rows are restricted to the vector representation of the multivariate polynomial functions in the state monomial basis of the input-output monomials in $\mathcal{S}_{uy}$. While the pencil contains both $\boldsymbol{\theta}$ and $\boldsymbol{u}[\boldsymbol{k}_{n_y}]$, it suffices to enlarge the associated block Macaulay matrix by considering the parameter monomial multiples to reveal the linear dependencies between parameter-input-output monomials. We increase the shift degree until the associated block Macaulay (model) matrix admits a nontrivial left null space whose basis we will denote by $\boldsymbol{H}_{n_f}^{d_{\min}, d_{\text{shift}}}$. The basis vectors of $\boldsymbol{H}_{n_f}^{d_{\min}, d_{\text{shift}}}$ are the vector representations of the input-output difference polynomial ϕ in the parameter-input-output monomial basis.

Example 1 (continued): Continuing the example, we now consider the model matrix pencil $\mathcal{M}(\theta_1, \boldsymbol{u}[\boldsymbol{k}_1])_{\mathcal{S}_{uy}}$ obtained from the output support found in stage 2 where,

$$\mathcal{M}(\theta_1, \boldsymbol{u}[\boldsymbol{k}_1])_{\mathcal{S}_{uy}} = \sum_{(\boldsymbol{\eta}, \boldsymbol{\delta}) \in \mathcal{A}} \boldsymbol{A}_{(\boldsymbol{\eta}, \boldsymbol{\delta})} \boldsymbol{\theta}^{\boldsymbol{\eta}} \boldsymbol{u}[\boldsymbol{k}_1]^{\boldsymbol{\delta}},$$

where $\boldsymbol{A}_{(\boldsymbol{\eta}, \boldsymbol{\delta})} \in \mathbb{R}^{4\times 6}$ and $\mathcal{A} = \{(\boldsymbol{\eta}, \boldsymbol{\delta}) \in \mathbb{N}^3 : \|(\boldsymbol{\eta}, \boldsymbol{\delta})\|_1 \leq 4\}$. At $d_{\text{shift}} = 1$ with the model parameter θ_1, the associated block Macaulay matrix admits a nontrivial left null space of dimension one where $\boldsymbol{H}_3^{4,1}$ is,

$$\begin{bmatrix} \cdots & u[k]y[k]^2y[k+1]y[k+3] & \cdots & u[k+1]y[k]y[k+1]y[k+2]^2 & \cdots & \theta_1 u[k+1]y[k+1]^3y[k+2] & \cdots & \theta_1 u[k]y[k]^2y[k+2]^2 & \cdots \\ \mathbf{0} & 1 & \mathbf{0} & -1 & \mathbf{0} & 1 & \mathbf{0} & -1 & \mathbf{0} \end{bmatrix}$$

which is the vector representation of the difference equation (10) in the parameter-input-output monomial basis. △

Both Algorithm 1 and 2 yield the difference equation of minimal order at most n_f when the algorithms are initialized with strongly locally observable models and the order bound is set to $n_y = n_f$. When the algorithm is initialized with a strongly locally observable (and identifiable) model, the extended state variable $\tilde{\boldsymbol{x}}[k] = \begin{bmatrix} \boldsymbol{x}[k] & \cdots & \theta_{i-1} & \theta_{i+1} & \cdots \end{bmatrix} \in \mathbb{R}^{n_f + (\ell - 1)}$ and the order bound is set to $n_y = n_f + \ell - 1$, then we obtain an unique difference equation in the irreducible single parameter input-output difference polynomial of order at most n_y.

Theorem 8 (*Algebraic global and local identifiability of a model parameter*): Apply Algorithm 2 with $n_y = n_f + \ell - 1$ to a model that is strongly locally observable and identifiable with respect to the extended state $\tilde{\boldsymbol{x}}[k] = \begin{bmatrix} \boldsymbol{x}[k]^{\mathrm{T}} & \theta_1 & \cdots & \theta_{i-1} & \theta_{i+1} & \cdots & \theta_\ell \end{bmatrix}^{\mathrm{T}}$. Suppose that the algorithm returns an input-output difference polynomial of the form $\phi = \phi_1 \theta_i + \phi_0$, where $\phi_0, \phi_1 \in \mathbb{R}[\boldsymbol{w}[\boldsymbol{k}_{n_y}]]$. Then, on the subset of admissible input-output sequences $\boldsymbol{w}[\boldsymbol{k}_{n_y}] \in \mathcal{B}_\Sigma$ for which $\phi_1\big(\boldsymbol{w}[\boldsymbol{k}_{n_y}]\big) \neq 0$, the parameter θ_i is algebraically globally identifiable. More generally, suppose that the algorithm returns $\phi = \sum_{j=0}^{d} \phi_j\big(\boldsymbol{w}[\boldsymbol{k}_{n_y}]\big)\theta_i^j$, where $d > 1$, $\phi_j \in \mathbb{R}[\boldsymbol{w}[\boldsymbol{k}_{n_y}]]$, and $\phi_d\big(\boldsymbol{w}[\boldsymbol{k}_{n_y}]\big) \neq 0$. Then θ_i is algebraically locally identifiable.

Proof. By Lemma 1, every input-output difference polynomial obtained from the left null space of a block Macaulay model matrix vanishes on the behavior $\mathcal{B}_\Sigma$. In the linear case, the relation $\phi_1\big(\boldsymbol{w}[\boldsymbol{k}_{n_y}]\big)\theta_i + \phi_0\big(\boldsymbol{w}[\boldsymbol{k}_{n_y}]\big) = 0$ holds for every compatible parameter and input-output sequence. Whenever $\phi_1\big(\boldsymbol{w}[\boldsymbol{k}_{n_y}]\big) \neq 0$, it uniquely determines the parameter as $\theta_i = -\phi_0\big(\boldsymbol{w}[\boldsymbol{k}_{n_y}]\big)/\phi_1\big(\boldsymbol{w}[\boldsymbol{k}_{n_y}]\big)$. Hence, θ_i is algebraically globally identifiable [10]. In the second case, the relation $\sum_{j=0}^{d} \phi_j\big(\boldsymbol{w}[\boldsymbol{k}_{n_y}]\big)\theta_i^j = 0$ defines, whenever $\phi_d\big(\boldsymbol{w}[\boldsymbol{k}_{n_y}]\big) \neq 0$, a nonzero univariate polynomial of degree d in θ_i. It therefore has at most d distinct roots over $\mathbb{C}$, so only finitely many parameter values are compatible with the corresponding input-output sequence. Thus, θ_i is algebraically locally identifiable. The qualifier "algebraic" emphasizes that these conclusions are based on the solution set of the resulting polynomial equation over $\mathbb{C}$. Additional real-valued or model-specific constraints may reduce the number of admissible parameter values, but uniqueness over $\mathbb{R}$ cannot be inferred from the polynomial degree alone. □

Algorithm 2 Practical state elimination algorithm

Input: State-space model in Σ, order bound $n_y \geq n_f$, number of trials N.

Output: Difference equation $\phi(\boldsymbol{\theta}, \boldsymbol{w}[\boldsymbol{k}_{n_y}]) = 0$, with $\phi \in \mathbb{R}\left[\boldsymbol{\theta} \quad \boldsymbol{w}[\boldsymbol{k}_{n_y}]\right]$ irreducible, $\mathrm{ord}(\phi) \leq n_y$.

Stage 1: Estimate output-monomial support.
for $j = 1, \ldots, \mathrm{N}$ **do**
  Sample random numerical values $\bar{\boldsymbol{\theta}}_{(j)}$ and $\bar{\boldsymbol{u}}[\boldsymbol{k}_{n_y}]_{(j)}$.
  Construct the corresponding numerical instance of Σ.
  Apply Algorithm 1 in [5] to obtain the output-monomial support $\mathcal{S}_{y_{(j)}}$.
end for
$\mathcal{S}_y \leftarrow \bigcup_{j=1}^{\mathrm{N}} \mathcal{S}_{y_{(j)}}$, $d_{\min} \leftarrow$ max. deg. monomial in $\mathcal{S}_y$
Stage 2: Identify input-output monomial support.
Form the restricted pencil $\boldsymbol{\mathcal{M}}(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])_{\mathcal{S}_y}$.
for $j = 1, \ldots, \mathrm{N}$ **do**
  Sample random numerical values $\bar{\boldsymbol{\theta}}_{(j)}$.
  Evaluate $\boldsymbol{\mathcal{M}}(\bar{\boldsymbol{\theta}}_{(j)}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])_{\mathcal{S}_y}$, viewed as a pencil in $\boldsymbol{u}[\boldsymbol{k}_{n_y}]$.
  Initialize $d_{\mathrm{shift}} \leftarrow 0$, $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}} \leftarrow \boldsymbol{0}$.
  while $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}} = \boldsymbol{0}$ **do**
    Construct the associated block Macaulay matrix $\boldsymbol{M}_{n_y}^{d_{\min}, d_{\mathrm{shift}}}$ and extract the basis of its left null space $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}}$ via SVD.
    if $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}} \neq \boldsymbol{0}$; **return** $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}}$; **end if**
    $d_{\mathrm{shift}} \leftarrow d_{\mathrm{shift}} + 1$.
  end while
  Extract the input-output support $\mathcal{S}_{uy_{(j)}}$ from the nonzero coefficients of $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}}$.
end for
$\mathcal{S}_{uy} \leftarrow \bigcup_{j=1}^{\mathrm{N}} \mathcal{S}_{uy_{(j)}}$.
Stage 3: Identify parameter-input-output monomial support.
Form the pencil corresponding to $\boldsymbol{\mathcal{M}}(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])_{\mathcal{S}_{uy}}$.
Initialize $d_{\mathrm{shift}} \leftarrow 0$, $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}} \leftarrow \boldsymbol{0}$.
while $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}} = \boldsymbol{0}$ **do**
  Construct the associated block Macaulay (model) matrix by shifting only with respect to $\boldsymbol{\theta}$ and extract the basis of its left null space $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}}$ via SVD.
  if $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}} \neq \boldsymbol{0}$; **return** $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}}$; **end if**
  $d_{\mathrm{shift}} \leftarrow d_{\mathrm{shift}} + 1$.
end while
return $\boldsymbol{H}_{n_y}^{d_{\min}, d_{\mathrm{shift}}}$, whose vectors are the coefficient representation of ϕ in the parameter-input-output monomial basis.

Note that under the setting of Theorem 8, we can integrate a terminating test for global identifiability. If the left null space of the block Macaulay (model) matrix corresponding to $\boldsymbol{\mathcal{M}}(\boldsymbol{\theta}, \boldsymbol{u}[\boldsymbol{k}_{n_y}])_{\mathcal{S}_{uy}}$ in stage 3 of Algorithm 2 remains empty after considering the block Macaulay (model) matrix with degree one monomial multiple i.e., $d_{\mathrm{shift}} = 1$ then the model parameter θ_i is not algebraically globally identifiable

5. Numerical examples

Example 1 (continued on *False identifiability determinations*): For the extended state vector $\tilde{\boldsymbol{x}}[k] = \begin{bmatrix} x_1[k] & x_2[k] & \theta_2 \end{bmatrix}^{\mathrm{T}}$, we have $\dim\left(\mathcal{L}_{\mathrm{null}}\left(\boldsymbol{\mathcal{M}}_3^4\right)\right) = 1$, and the coefficient representation of ϕ_{θ_1} belongs to this left null space; see (10). Algorithm 1 therefore recovers ϕ_{θ_1} numerically from $\mathcal{L}_{\mathrm{null}}(\boldsymbol{M}_3^{4,2})$, which is also one-dimensional. Here, $d_{\mathrm{shift}} = 2$ because (10) contains coefficients of degree two in the parameter-input variables, such as $\theta_1 u[k]$ and $\theta_1 u[k+1]$. By contrast, Stage 3 of Algorithm 2 requires only $d_{\mathrm{shift}} = 1$. Besides providing a stopping criterion, Algorithm 2 prevents the dimensions of the intermediate matrices from becoming prohibitively large. Indeed, without removing zero columns, the block Macaulay (model) matrix $\boldsymbol{M}_3^{4,2}$ used by Algorithm 1 has size $1470 \times \left(\approx 6 \times 10^9\right)$. In contrast, stage 1 of algorithm 2 determines $\mathcal{S}_y$ from a 69×70 model matrix. Stage 2 determines $\mathcal{S}_{uy}$ from a 12×60 block Macaulay (model) matrix, and Stage 3 computes the difference equation from an 8×336 block Macaulay (model) matrix. Consider now a perturbed state-space model, where we inject degree one model perturbations to Σ_{R} as $x_1[k+1] = \theta_1 x_1[k] + \theta_2 x_2[k] u[k] + (1 \times 10^{-h})\mathtt{rand} x_1[k]$, $x_2[k+1] = 3\theta_2^2 x_1[k] x_2[k] + (1 \times 10^{-h})\mathtt{rand} x_1[k]$ and $y[k] = x_2[k] + (1 \times 10^{-h})\mathtt{rand} x_1[k]$. For $h = 7$, algorithm 2 gives the correct difference equation (10), however a Grobner basis computation yields a polynomial in θ_1 of a higher degree. Resulting in a false algebraic local identifiability determination of the model parameter.

Example 2: We will now assess how Algorithm 2 scales[4] on a family of strongly locally observable Hammerstein-Wiener models [19] given by,

$$\Sigma_{\mathrm{B}} : \left\{ \boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\, u[k]^2, \quad y[k] = x_1[k]^2, \right. \quad (22)$$

where $\boldsymbol{A} \in \mathbb{R}^{n_f \times n_f}$ is a structured binary companion matrix whose entries satisfy $A_{ij} = 1$ if $j = i+1$ or $i = n_f$, and $A_{ij} = 0$ otherwise. Here, $\boldsymbol{B} \in \mathbb{R}^{n_f \times 1}$ is a sparse matrix with only one non-zero entry $B_{n_f} = 1$. For each $n_f = \{1, 2, 3\}$ we run Algorithm 2 with the order bound n_f, to find the difference equation of minimal order. In Table 2 $d_{\min}$ we provide, the size($\boldsymbol{\mathcal{M}}_{n_y}^{d_{\min}}$) (constructed at the end of Stage 1 in Algorithm 2), ζ i.e., the minimum degree d_{shift} where $\boldsymbol{H}_{n_f}^{d_{\min}, \zeta} \neq \boldsymbol{0}$ at the end of stage 2, size($\boldsymbol{M}^{d_{\min}, \zeta}$) and the

[4] All experiments were run on an HP EliteBook 845 G8 laptop with a 6-core AMD Ryzen 7 PRO 5850U CPU (1.90GHz) and 32GB RAM, on a 64-bit operating system.

Table 2
Scaling of Algorithm 2 on (22) as n_f increases. Reported are the degree $d_{\min}$ yielding a nontrivial numerical left null space and the size of the corresponding model matrix (constructed at the end of Stage 1 of Algorithm 2), the sufficient shift degree ζ and the size of the resulting block Macaulay (model) matrix (constructed at the end of Stage 2 of Algorithm 2), and the total computation time.

n_f	$d_{\min}$	size$\left(\mathcal{M}_{n_f}^{d_{\min}}\right)$	ζ	size$\left(\boldsymbol{M}_{n_f}^{d_{\min},\zeta}\right)$	Time (s)
1	2	6×5	8	54×85	1.5073
2	4	35×45	16	595×1485	2.1146
3	8	495×969	32	16335×63985	$\geq 6.5001\times 10^4$

computation time. The algorithm terminates at Stage 2 as the model class (22) is non-parametric. As n_f increases, so does $d_{\min}$, causing the size of $\mathcal{M}_{n_y}^{d}$ to grow rapidly (Remark 2). Already for $n_f = 3$, explicitly constructing the matrix pencil $\mathcal{M}(\boldsymbol{u}[\boldsymbol{k}_2])_{\mathcal{S}_y}$ becomes computationally demanding when the support $\mathcal{S}_y$ obtained after stage 1 is dense. Constructing the corresponding block Macaulay matrix and performing the subsequent rank tests further limits computations, particularly at degrees approaching 32. This motivates the development of recursive methods, such as [16], for computing left-null-space bases in larger-scale problems, as well as the incorporation of additional structural information on the monomial support, for example by extending the results of [11] to discrete-time models. △

6. Conclusion

We extend the linear-algebraic framework for state elimination to discrete-time, time-invariant, parametric, single-input, single-output polynomial state-space models with unknown parameters by introducing the block Macaulay (model) matrix, a block-structured generalization of the model matrix. For strongly locally observable models, we show that the left null space of the block Macaulay (model) matrix coincides with the row space of an associated block Macaulay (Macaulay) matrix generated by the coefficients of the unique, irreducible parametric input–output difference polynomial, preserving the orthogonal complementarity between image and kernel representations in the parametric setting. The proposed SVD-based state elimination algorithm recovers the input–output difference equation numerically and, with simple algorithmic modifications, provides practical diagnostics for algebraic global parameter identifiability. We provide theoretical guarantees for the recovery of a single-parameter input-output difference equation and for its use in distinguishing algebraic global from finite local parameter identifiability. Numerical examples demonstrate the robustness of the approach under model perturbations.

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