Elimination in polynomial state-space space models

Using the SVD to compute output difference equations

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Problem statement

Given a time invariant single output autonomous discrete-time polynomial state-space models of the form

$$\Sigma = \begin{cases} \hat{\boldsymbol{x}}[k+1] = \boldsymbol{f}(\hat{\boldsymbol{x}}[k], \boldsymbol{\theta}) \\ \hat{y}[k] = g(\hat{\boldsymbol{x}}[k], \boldsymbol{\theta}) \end{cases}$$

- where $\hat{\boldsymbol{x}}[k] = \begin{bmatrix} \hat{x}_1[k] & \dots & \hat{x}_n[k] \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$ are the state variables at instant $k \in \mathbb{Z}_{\geq 0}$
- $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 & \dots & \theta_\ell \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^\ell$ are the time invariant model parameters
- $\boldsymbol{f} = \begin{bmatrix} f_1 & \dots & f_n \end{bmatrix}^{\mathrm{T}}$ s.t. $f_i, g \in \mathbb{R}\begin{bmatrix} \hat{x}_1[k] & \dots & \hat{x}_n[k] & \theta_1 & \dots & \theta_\ell \end{bmatrix}$

Find the output difference equation which describes the algebraic relation between consecutive model outputs. An output difference equation of order n is

$$\phi(\boldsymbol{\theta}, \hat{\boldsymbol{y}}) = \boldsymbol{w}^{\mathrm{T}} \hat{\boldsymbol{y}}_{\mathcal{A}} = 0$$

- Here, $\mathcal{A} = \{\hat{y}[k], \dots, \hat{y}[k+n]\}$ and $\hat{\boldsymbol{y}}_{\mathcal{A}}$ is a vector including monomials of the form $\hat{\boldsymbol{y}}^{\boldsymbol{\alpha}} = (\hat{y}[k])^{\alpha_0} (\hat{y}[k+1])^{\alpha_1} \dots (\hat{y}[k+n])^{\alpha_n}$ where $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n+1}$
- $\boldsymbol{w}$ contains entries which are polynomial functions of the model parameters

(1)

The predator-prey model

Given the example of the discretized predator-prey model [1]

$$\Sigma_{\mathrm{LV}} = \begin{cases} \hat{x}_1[k+1] = \hat{x}_1[k](1 + b - p\hat{x}_2[k]) \\ \hat{x}_2[k+1] = \hat{x}_2[k](1 - d + p\hat{x}_1[k]) \\ \hat{y}[k] = \hat{x}_2[k] \end{cases}$$

- where $\hat{\boldsymbol{x}}[k] = \begin{bmatrix} \hat{x}_1[k] & \hat{x}_2[k] \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^2$ are the state variables $k \in \mathbb{Z}_{\geq 0}$
- $\boldsymbol{\theta} = \begin{bmatrix} b & p & d \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^3$ are the time invariant model parameters

Find the output difference equation of order two

$$\boldsymbol{w} = \begin{bmatrix} (pd-p) & p & (bd-b) & -(b+1) & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\hat{\boldsymbol{y}}_{\mathcal{A}} = \begin{bmatrix} \hat{y}[k]^2\hat{y}[k+1] & \hat{y}[k]\hat{y}[k+1]^2 & \hat{y}[k]\hat{y}[k+1] & \hat{y}[k+1]^2 & \hat{y}[k]\hat{y}[k+2] \end{bmatrix}^{\mathrm{T}}$$

$$\phi = \boldsymbol{w}^{\mathrm{T}} \hat{\boldsymbol{y}}_{\mathcal{A}} = 0$$

Example environment

(2)

Model output polynomials

The model outputs at future time instants $\mathcal{A} = \{\hat{y}[k], \dots, \hat{y}[k+n]\}$ are multivariate polynomial functions in the model parameters and the components of the state variable,

$$\hat{\boldsymbol{y}}_{\mathcal{A}} = \boldsymbol{\mathcal{M}}_{\mathcal{A}} \hat{\boldsymbol{x}}_{\mathcal{A}}^{m}$$

for the predator-prey model

$$\begin{bmatrix} \hat{y}[k] \\ \hat{y}[k+1] \\ \hat{y}[k+2] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ (1-d) & p & 0 & 0 & 0 \\ (1-d)^2 & p(2+b)(1-d) & p^2(b+1) & p^2(d-1) & -p^3 \end{bmatrix} \hat{\boldsymbol{x}}_{\mathcal{A}}^{4}$$

$\hat{x}_2[k]$ $\quad$ $\hat{x}_1[k]\hat{x}_2[k]$ $\quad$ $(\hat{x}_1[k])^2\hat{x}_2[k]$ $\quad$ $(\hat{x}_1[k])^2\hat{x}_2[k]$ $\quad$ $(\hat{x}_1[k])^2(\hat{x}_2[k])^2$

Where,

- $\hat{\boldsymbol{y}}_{\mathcal{A}}$: contains the model output variables in $\mathcal{A}$
- $\boldsymbol{\mathcal{M}}_{\mathcal{A}}$:coefficient matrix, entries polynomials of $\boldsymbol{\theta} = \{b, p, d\}$
- $\hat{\boldsymbol{x}}_{\mathcal{A}}^{4}$: monomials of the components of the state variable upto degree $m = \max(\deg_{\boldsymbol{x}_{\mathcal{A}}}(\text{polynomial}(\hat{\boldsymbol{y}}_{\mathcal{A}}))) = 4$

Zero columns in the coefficient matrix are not shown

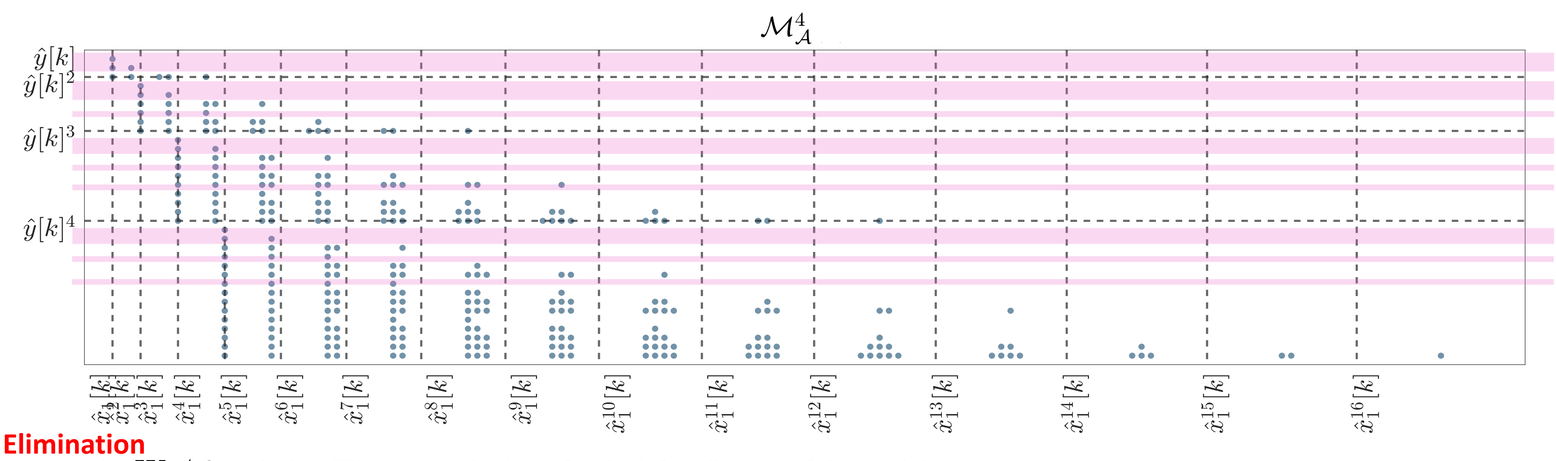
(3)

Model matrix

- The model matrix extends the coefficient matrix to include coefficients of $\hat{\boldsymbol{y}}^{\boldsymbol{\alpha}} = (\hat{y}[k])^{\alpha_0} (\hat{y}[k+1])^{\alpha_1} \dots (\hat{y}[k+n])^{\alpha_n}$
- For $|\boldsymbol{\alpha}| \leq \nu$

$$\hat{\boldsymbol{y}}_{\mathcal{A}}^{\nu} = \boldsymbol{\mathcal{M}}_{\mathcal{A}}^{\nu} \hat{\boldsymbol{x}}_{\mathcal{A}}^{m\nu}$$

- Where $\hat{\boldsymbol{y}}_{\mathcal{A}}^{\nu}$: includes model output variables and their monomials upto degree ν, $\boldsymbol{\mathcal{M}}_{\mathcal{A}}^{\nu}$: the corresponding coefficients and $\hat{\boldsymbol{x}}_{\mathcal{A}}^{mv}$ the state monomials. For the predator prey model, the model matrix where $\nu = 4$



Elimination

- Consider, $\boldsymbol{W} \neq \boldsymbol{0}$ such that $\boldsymbol{W}$ contains the basis for the left null space of the model matrix,

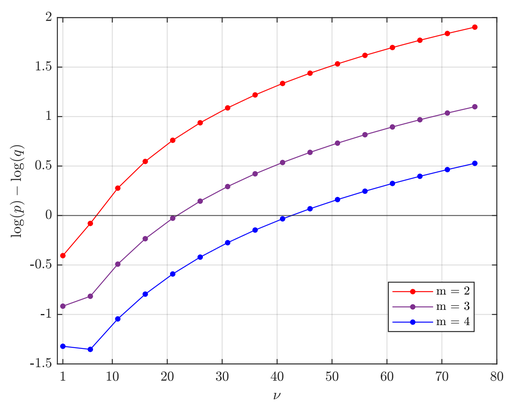
$$\phi = \boldsymbol{W}^{\mathrm{T}} \hat{\boldsymbol{y}}_{\mathcal{A}}^{4} = \left(\boldsymbol{W}^{\mathrm{T}} \boldsymbol{\mathcal{M}}_{\mathcal{A}}^{4}\right) \hat{\boldsymbol{x}}_{\mathcal{A}}^{16} = 0$$

- State monomials eliminated and output difference equation(s) found!

(4)

Existence

The size of the model matrix $\boldsymbol{\mathcal{M}}_{\mathcal{A}}^{\nu}$ where $\mathcal{A} = \{\hat{y}[k], \dots, \hat{y}[k+n]\}$
For a n^{th} order polynomial state space model is $p \times q$



$$p = \binom{\nu + n + 1}{n + 1} \quad \text{vs} \quad q = \binom{\nu m + n}{n}$$

Monomials of model output variables $\qquad$ State monomials

For some degree ν, $p \geq q$: there exists a non-trivial left null space of the model matrix $\Longrightarrow$ the states are eliminated
$\Longrightarrow$ output difference equation exists

(5)

Uniqueness

When the multivariate polynomials of the model outputs are linearly independent and their reduced row echelon form polynomials are irreducible with respect to each other,

- The output difference equation ϕ is of model order n, i.e., it must contain a monomial including a non-zero coefficient of $\hat{y}[k+n]$ (here $\hat{y}[k+2]$)
- All model output monomials of $\mathcal{B} = \{\hat{y}[k], \dots, \hat{y}[k+n-1]\}$ are linearly independent (see rows in Box 4 here $\mathcal{B} = \{\hat{y}[k], \hat{y}[k+1]\}$)
- The output difference equation is unique upto multiplication with monomials of the model output variables.
- $\boldsymbol{W}$ is a Macaulay matrix [2]. Each row is a shifted version of the first row

Here,

$$\boldsymbol{W} = \begin{bmatrix} \phi & \hat{y}[k]\phi & \hat{y}[k+1]\phi & \hat{y}[k+2]\phi \end{bmatrix}^{\mathrm{T}}$$

$\hat{y}[k]\hat{y}[k+2]$ $\quad$ $(\hat{y}[k])^2\hat{y}[k+2]$ $\quad$ $\hat{y}[k]\hat{y}[k+1]\hat{y}[k+2]$ $\quad$ $\hat{y}[k](\hat{y}[k+2])^2$

$$\begin{bmatrix} -b(d-1) & 1 & -(b+1) & 0 & 0 & 0 & p(d-1) & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b(d-1) & 1 & -(b+1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p(d-1) & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b(d-1) & 1 & 0 & -(b+1) & 0 & 0 & 0 & 0 & 0 & 0 & p(d-1) & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b(d-1) & 1 & 0 & -(b+1) & 0 & 0 & 0 & 0 & 0 & p(d-1) & 0 & p \end{bmatrix}$$

- Here, $\boldsymbol{W}$ is computed symbolically
- *Zero columns on either side of the band are omitted*
- Possible to construct higher order output difference equations, by considering monomials with $\hat{y}[k+n+i]$ in the model matrix, but they are not unique.

(6)

Numerical elimination

Observe, the model matrix is symbolic. Not yet possible to compute the left null space using a numerical procedure. Write the model matrix as a multivariate polynomial matrix pencil,

$$\mathcal{M}_{\mathcal{A}}^{4} = \boldsymbol{A}_{000} + d\boldsymbol{A}_{010} + \dots + p^{12}\boldsymbol{A}_{0012}$$

Where, $\boldsymbol{A}_{\omega} \in \mathbb{R}^{p \times q}$. Shift the matrix pencil with the model parameters, to construct $\boldsymbol{M}$: a block Macaulay matrix [3],

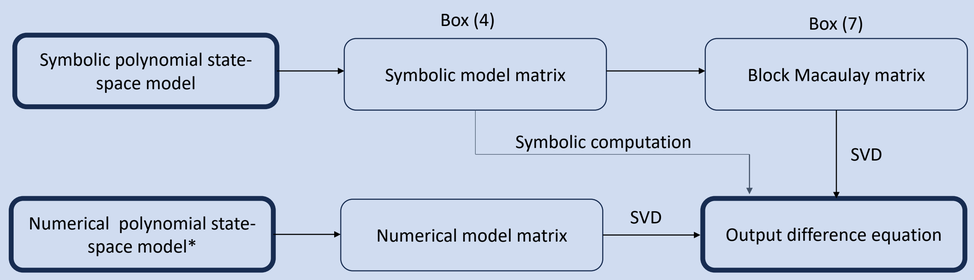
$$\underbrace{\begin{bmatrix} \hat{\boldsymbol{y}}_{\mathcal{A}}^{4} \\ \hline b\hat{\boldsymbol{y}}_{\mathcal{A}}^{4} \\ p\hat{\boldsymbol{y}}_{\mathcal{A}}^{4} \\ d\hat{\boldsymbol{y}}_{\mathcal{A}}^{4} \\ \hline b^2\hat{\boldsymbol{y}}_{\mathcal{A}}^{4} \\ bp\hat{\boldsymbol{y}}_{\mathcal{A}}^{4} \\ \vdots \end{bmatrix}}_{\hat{\boldsymbol{y}}_{\mathrm{b}}} = \underbrace{\begin{bmatrix} \boldsymbol{A}_{000} & \boldsymbol{0} & \boldsymbol{A}_{010} & \boldsymbol{A}_{001} & \boldsymbol{0} & \boldsymbol{0} & \dots \\ \boldsymbol{0} & \boldsymbol{A}_{000} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_{010} & \dots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_{000} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\boldsymbol{M}} \underbrace{\begin{bmatrix} \hat{\boldsymbol{x}}_{\mathcal{A}}^{16} \\ \hline b\hat{\boldsymbol{x}}_{\mathcal{A}}^{16} \\ p\hat{\boldsymbol{x}}_{\mathcal{A}}^{16} \\ d\hat{\boldsymbol{x}}_{\mathcal{A}}^{16} \\ \hline b^2\hat{\boldsymbol{x}}_{\mathcal{A}}^{16} \\ bp\hat{\boldsymbol{x}}_{\mathcal{A}}^{16} \\ \vdots \end{bmatrix}}_{\hat{\boldsymbol{x}}_{\mathrm{b}}}$$

$\boldsymbol{W}_{\mathrm{b}}$ basis of the left null space of $\boldsymbol{M}$, computed numerically using the SVD such that $\boldsymbol{\phi} = \boldsymbol{W}_{\mathrm{b}}^{\mathrm{T}} \hat{\boldsymbol{y}}_{b} = \boldsymbol{0}$

(7)

Elimination: summary

Given: a polynomial state-space model , ***find***: the output difference equation.



*Coefficients of the state-monomials in the model are numerical parameter values.

(8)

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