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Sensing Wave Fields in a Finite Element Framework

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Outline

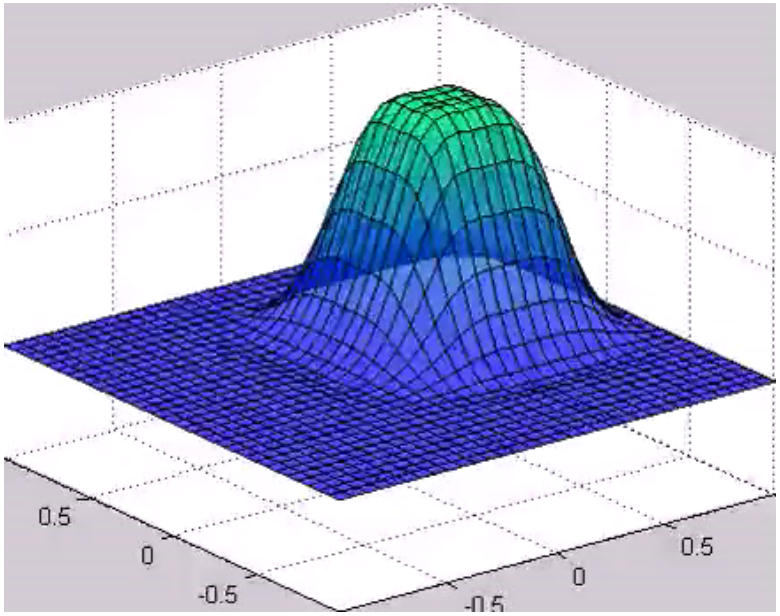
- Introduction
  - Problem statement
  - Finite element method (FEM)
- Wave field sensing
  - Sensor deployment in the FEM grid
  - Application 1: joint wave field estimation and source recovery
  - Application 2: identification of wave propagation model
- Distributed implementation in Wireless Sensor Network
  - FEM grid clustering
  - Separation of FEM system of equations
- Simulation results
- Conclusion

Introduction (1)

Problem statement (1)

- Goal: field estimation in wireless sensor networks using a physical model

- **Field:** physical phenomenon that varies over space/time
  - Electromagnetic wave propagation
  - Acoustic wave propagation
  - Heat diffusion
  - Fluid flow
  - …

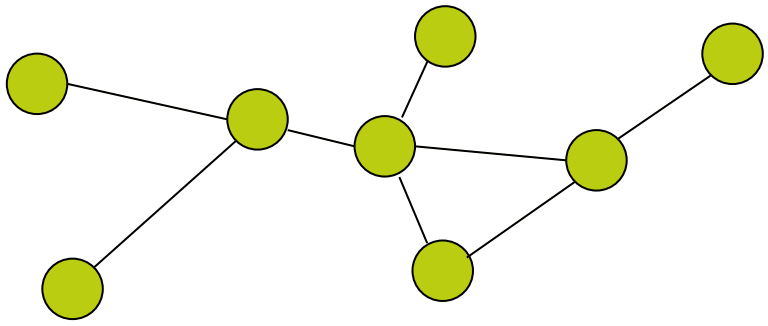


Introduction (2)

Problem statement (2)

- Goal: field estimation in wireless sensor networks using a physical model

- **Field:** physical phenomenon that varies over space/time
- **Wireless Sensor Network (WSN):** collection of spatially distributed sensor nodes capable of measuring, sampling, processing, and communicating



Introduction (3)

Problem statement (3)

- Goal: field estimation in wireless sensor networks using a physical model

- **Field:** physical phenomenon that varies over space/time
- **Wireless Sensor Network (WSN):** collection of spatially distributed sensor nodes capable of measuring, sampling, processing, and communicating
- **Physical model:** partial differential equation (PDE) subject to boundary/initial conditions
  - First-order PDE
  - Second-order PDE: elliptic/parabolic/hyperbolic

Introduction (4)

Problem statement (4)

- Goal: field estimation in wireless sensor networks using a physical model
- Motivation: to combine the strengths of data-driven and model-based approach
- **Data-driven:** WSN = spatiotemporal sampling device, but subject to aliasing, measurement noise, …
- **Model-based:** PDE = spatiotemporal "glue" between samples, but subject to modeling errors and uncertainty

Introduction (5)

Finite Element Method (1)

- **FEM**: 4-step procedure to discretize boundary value problem

1. Weak formulation of boundary value problem

$$-\nabla^2 u(x,y) = s(x,y) \Rightarrow \int_\Omega \left[\nabla^2 u(x,y) + s(x,y)\right] g(x,y)dxdy = 0$$

2. Integration by parts to relax differentiability requirements

$$\int_\Omega \nabla u(x,y) \cdot \nabla g(x,y)dxdy = \int_\Omega s(x,y)g(x,y)dxdy$$

3. Subspace approximation of field and source functions

$$\tilde{u}(x,y) = \sum_{k=1}^{K} u_k \phi_k(x,y), \quad \tilde{s}(x,y) = \sum_{k=1}^{K} s_k \phi_k(x,y)$$

4. Enforce orthogonality of approximation error to subspace

$$\mathbf{Au} = \mathbf{Bs}$$

Stiffness matrix $[\mathbf{A}]_{ij} = \int_\Omega \nabla\phi_j(x,y) \cdot \nabla\phi_i(x,y)dxdy$

Mass matrix $[\mathbf{B}]_{ij} = \int_\Omega \phi_j(x,y)\phi_i(x,y)dxdy$

Introduction (6)

Finite Element Method (2)

- Two particular boundary value problems considered here:

1. 2-D Poisson equation (static field)

   (example: temperature distribution in plate)

$$\begin{aligned} -\nabla^2 u(x,y) &= s(x,y) \\ &\Downarrow \\ \mathbf{Au} &= \mathbf{Bs} \end{aligned}$$

2. 3-D wave equation (dynamic field)

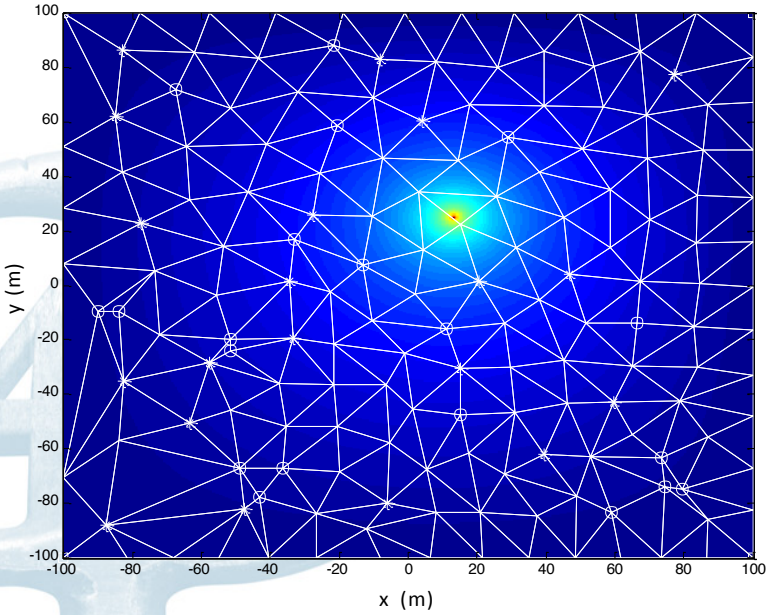
   (example: acoustic wave propagation in air)

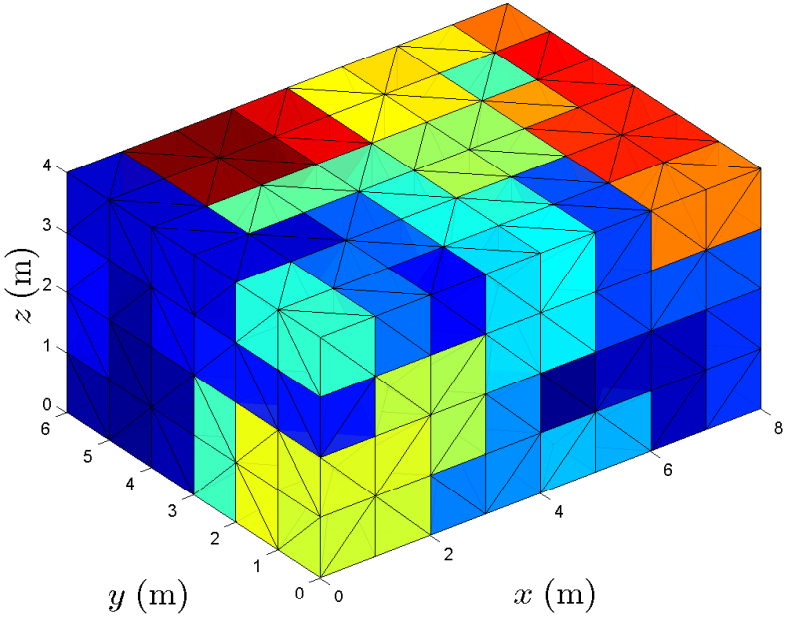
$$\begin{aligned} \nabla^2 u(x,y,z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} u(x,y,z,t) &= s(x,y,z,t) \\ &\Downarrow \\ \nabla^2 U(x,y,z,\omega) + k^2 U(x,y,z,\omega) &= S(x,y,z,\omega) \\ &\Downarrow \\ (\mathbf{K} - k^2\mathbf{L})\mathbf{U}(\omega) &= \mathbf{LS}(\omega) \end{aligned}$$

Introduction (7)

Finite Element Method (3)

- Choice of nodes, elements, and basis functions: discretization must be simple, accurate, well-conditioned, and sparse
  - Choose a “good” mesh (high resolution, well-shaped elements)
  - Choose piecewise linear basis functions with small spatial support
  - Omit boundary elements from FEM system of equations





Wave field sensing (1)

Sensor deployment in the FEM grid

- Choice of basis functions turns FEM into spatial sampling:

$$\phi_i(x_k, y_k) = \delta(i-k) \Rightarrow \begin{cases} \mathbf{u} = [u_1 \ \dots \ u_K]^T = [u(x_1, y_1) \ \dots \ u(x_K, y_K)]^T \\ \mathbf{s} = [s_1 \ \dots \ s_K]^T = [u(x_1, y_1) \ \dots \ u(x_K, y_K)]^T \end{cases}$$

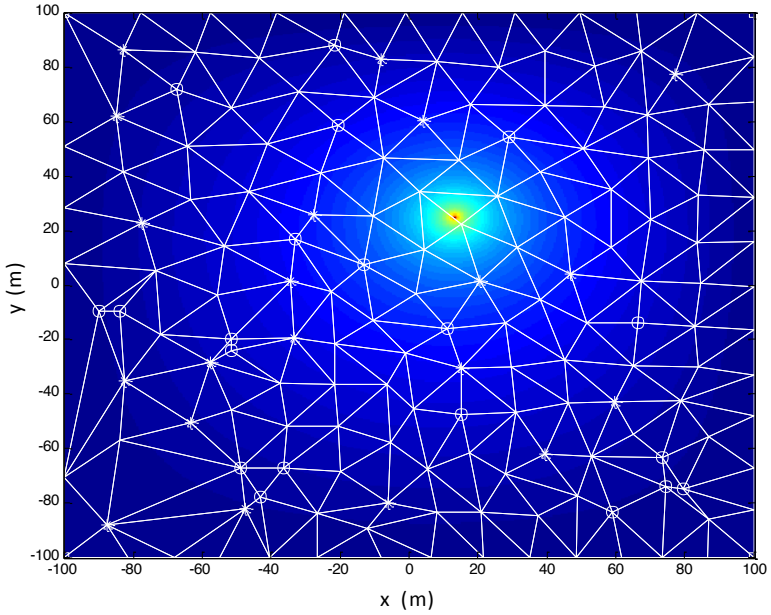
- Deployment of field sensors at subset of sampling points:

$$\underbrace{\mathbf{y}}_{(J\times 1)} = \underbrace{\mathbf{T}}_{(J\times K)} \underbrace{\mathbf{u}}_{(K\times 1)} + \underbrace{\mathbf{v}}_{(J\times 1)}$$

with $J \ll K$

J = number of sensors

K = number of FEM nodes



Wave field sensing (2)

Application 1: joint wave field estimation and source recovery

- Suppose each sensor takes N field measurements

$$\mathbf{y}_n = \mathbf{T}\mathbf{u} + \mathbf{v}_n, \;\; n = 1, \ldots, N$$

- Joint estimation of field and source vector by combining sensor measurements, FEM system of equations, and prior knowledge

$$\min_{\mathbf{u},\mathbf{s}} \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{T}\mathbf{u}\|_2^2 + \lambda\|\mathbf{s}\|_1$$

$$\text{s. t.} \quad \mathbf{A}\mathbf{u} = \mathbf{B}\mathbf{s}, \; \mathbf{u} \geq \mathbf{0}_{K\times 1}, \;\; \mathbf{s} \geq \mathbf{0}_{K\times 1}$$

Sparsity of point source

Nonnegativity of field/source

→ wave field estimation

→ source recovery

→ source localization

Wave field sensing (3)

Application 2: identification of wave propagation model (1)

- Green's function related to the wave equation can be modeled by a common-denominator pole-zero model

$$\mathbf{Y}(\omega) = \mathbf{T}\frac{\boldsymbol{\mathcal{B}}(\omega)}{A(\omega)}\mathbf{S}(\omega) + \mathbf{V}(\omega)$$

- The FEM system of equations $(\mathbf{K} - k^2\mathbf{L})\mathbf{U}(\omega) = \mathbf{L}\mathbf{S}(\omega)$ can be written in terms of the pole-zero model parameters:

$$\underbrace{\begin{bmatrix} \mathbf{M}(\omega) & \mathbf{0} & \dots & \mathbf{0} & \boldsymbol{\Psi}\boldsymbol{\chi}_1(\omega)\mathbf{z}_P^H(\omega) \\ \mathbf{0} & \mathbf{M}(\omega) & \dots & \mathbf{0} & \boldsymbol{\Psi}\boldsymbol{\chi}_2(\omega)\mathbf{z}_P^H(\omega) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{M}(\omega) & \boldsymbol{\Psi}\boldsymbol{\chi}_M(\omega)\mathbf{z}_P^H(\omega) \end{bmatrix}}_{\boldsymbol{\Xi}(\omega)} \underbrace{\begin{bmatrix} \underline{\mathbf{b}}_1 \\ \underline{\mathbf{b}}_2 \\ \vdots \\ \underline{\mathbf{b}}_M \\ \mathbf{a} \end{bmatrix}}_{\underline{\boldsymbol{\theta}}} = \mathbf{0}$$

Wave field sensing (4)

Application 2: identification of wave propagation model (2)

- Frequency domain system identification:

$$\min_{\underline{\boldsymbol{\theta}}} \quad \underline{\boldsymbol{\theta}}^T \mathbf{T}^T \left(\sum_{\omega} \boldsymbol{\Phi}(\omega) \right) \mathbf{T} \underline{\boldsymbol{\theta}}$$

$$\text{s. t.} \quad \begin{cases} \boldsymbol{\Xi}(\omega_1)\underline{\boldsymbol{\theta}} = \mathbf{0} \\ \vdots \\ \boldsymbol{\Xi}(\omega_L)\underline{\boldsymbol{\theta}} = \mathbf{0} \\ a_0 = 1 \end{cases}$$

where $\boldsymbol{\Phi}(\omega)$ contains input and output (cross-)PSD estimates

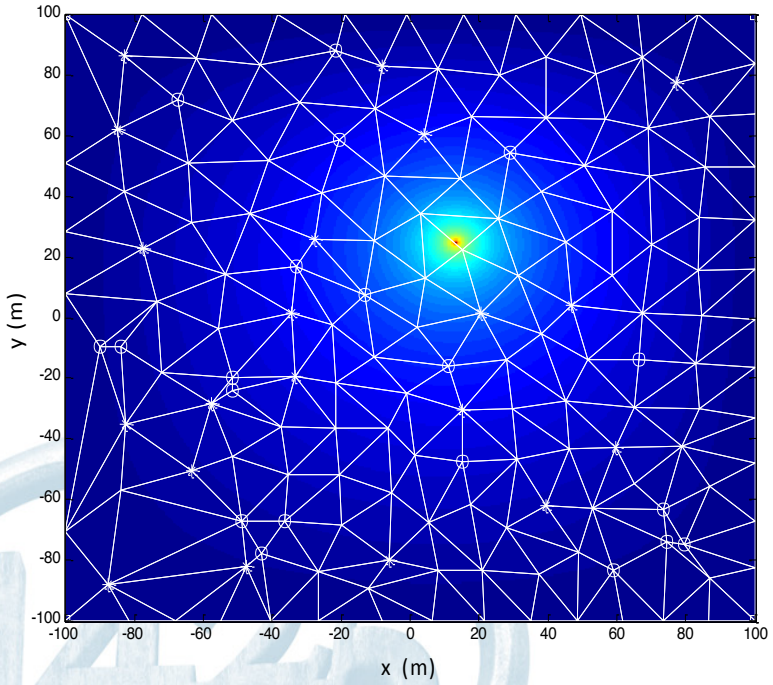
→ system identification

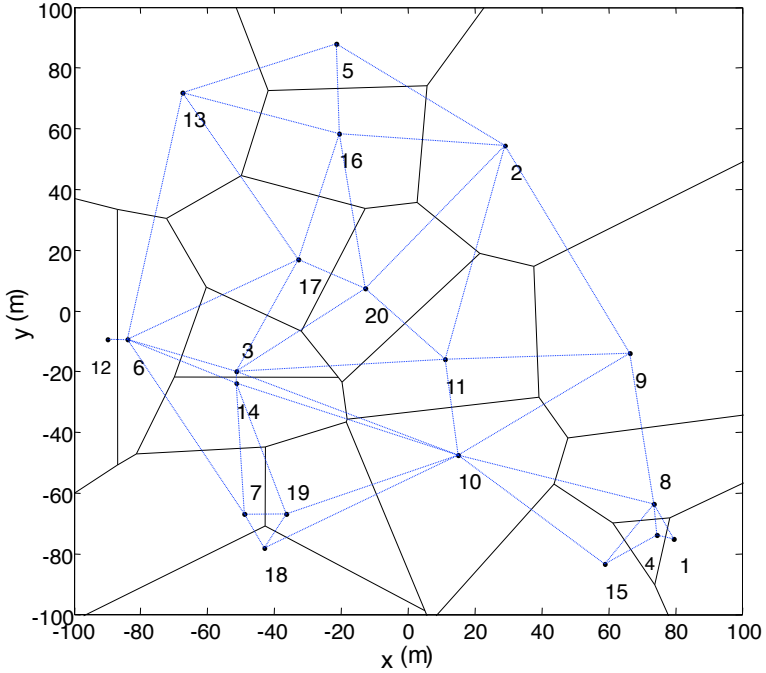
→ source localization (details omitted)

Distributed implementation in WSN (1)

FEM grid clustering

- Clustering of FEM nodes = partitioning of field and source vector





$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_J \end{bmatrix}$$

$$\mathbf{s} = \begin{bmatrix} \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_J \end{bmatrix}$$

- FEM stiffness and mass matrices are permuted and partitioned accordingly

Distributed implementation in WSN (2)

Separation of FEM system of equations

- Separation of optimization problem into J subproblems

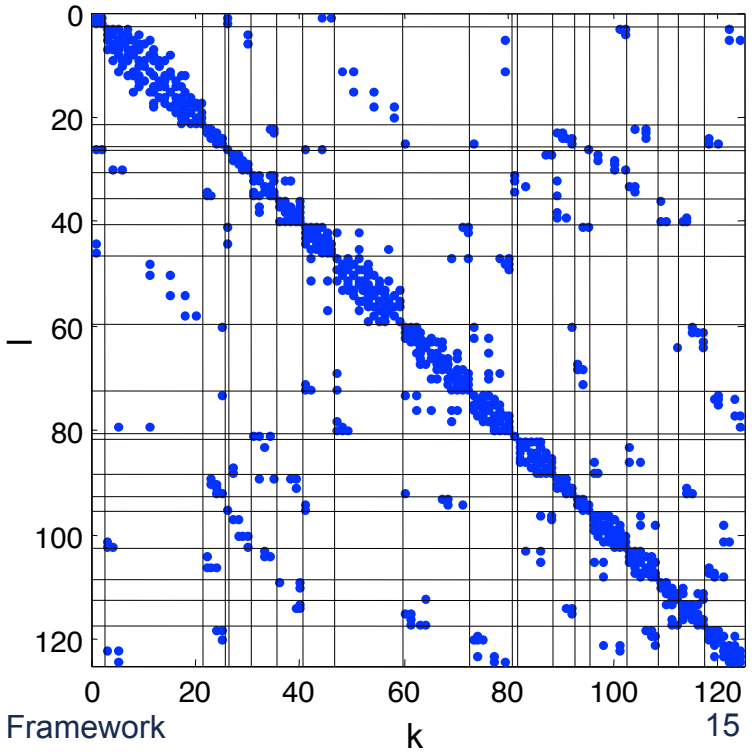
Fully separable

Partially separable

$$\min_{\mathbf{u},\mathbf{s}} \quad \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{T}\mathbf{u}\|_2^2 + \lambda\|\mathbf{s}\|_1$$

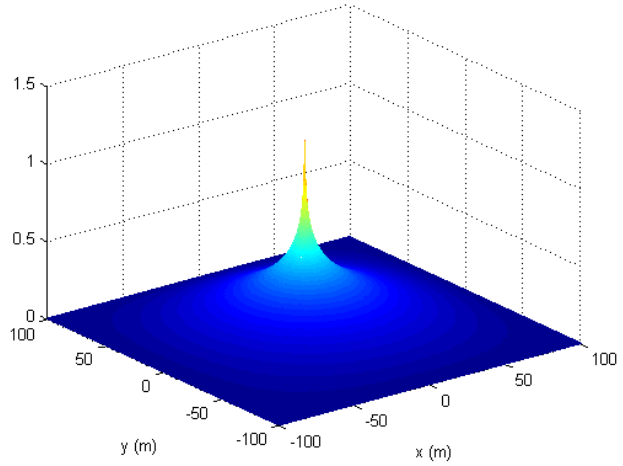
$$\text{s. t.} \quad \mathbf{A}\mathbf{u} = \mathbf{B}\mathbf{s}, \quad \mathbf{u} \geq \mathbf{0}_{K\times 1}, \quad \mathbf{s} \geq \mathbf{0}_{K\times 1}$$

- **Approximate separation** of equality constraints:
  - in j-th subproblem, consider only equality constraints corresponding to j-th block row of $\mathbf{A}, \mathbf{B}$
  - sparsity of $\mathbf{A}, \mathbf{B}$ can be exploited to reduce communication cost



Simulation results (1)

Wave field estimation (1)



- 2-D Poisson PDE on square domain with single point source

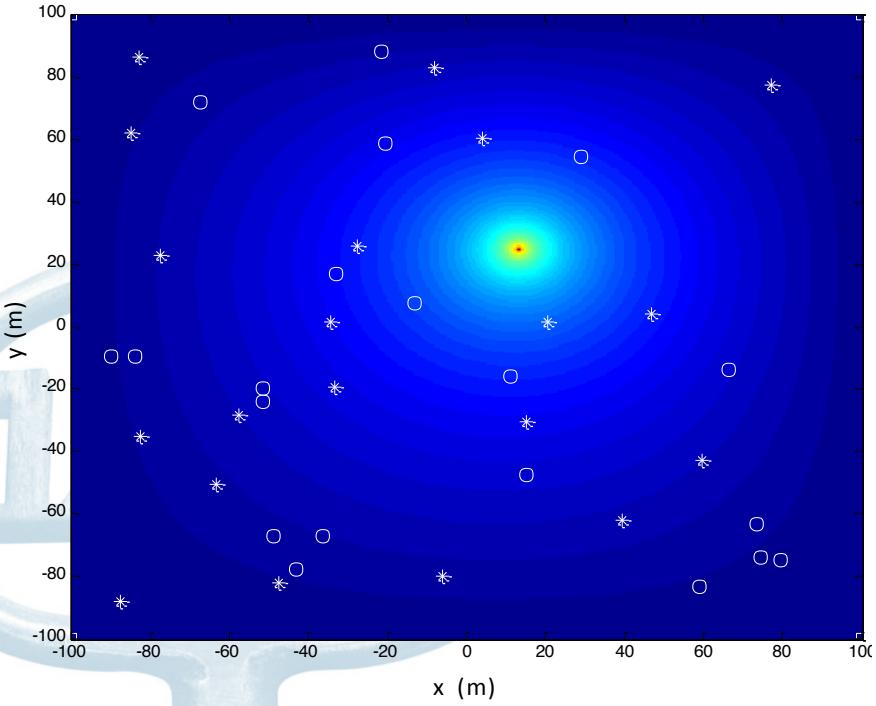
- Benchmark algorithms:
  - Model-based: FEM with known source vector
  - Data-driven: Measurement averaging + interpolation (MAI)

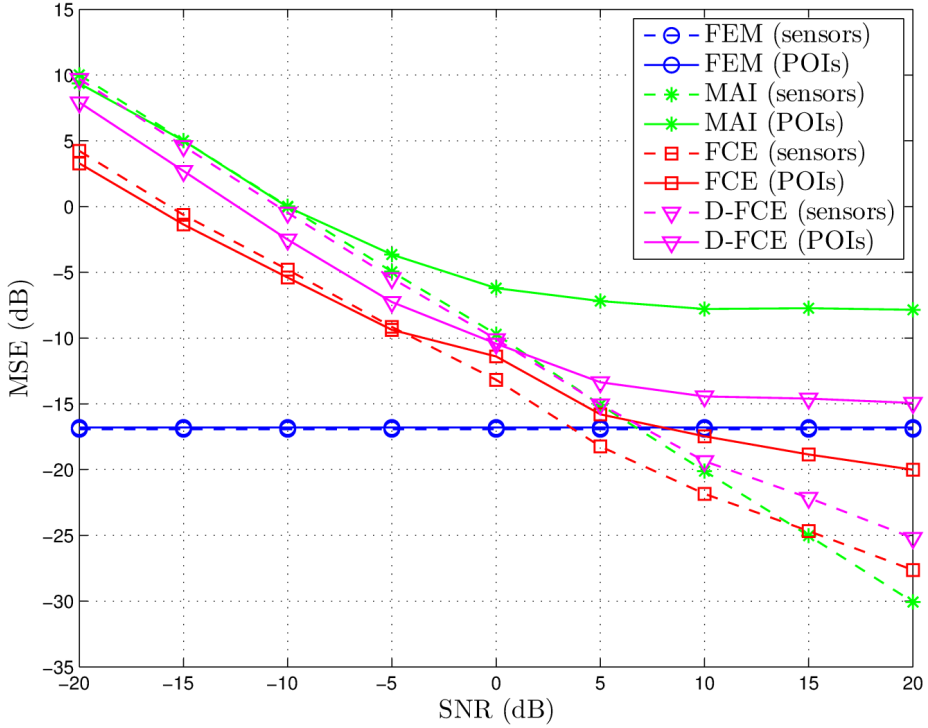
- Proposed algorithms: model-based + data-driven
  - FCE: FEM-constrained cooperative estimation with sparsity and nonnegativity prior
  - D-FCE: FEM-constrained distributed estimation with sparsity and nonnegativity prior + approximately separated equality constraints

Simulation results (2)

Wave field estimation (2)

- Performance measure:
  - mean squared relative field estimation error (MSE) at sensor node (--) and POI (–) locations
- $K = 166, J = 20, N = 20$

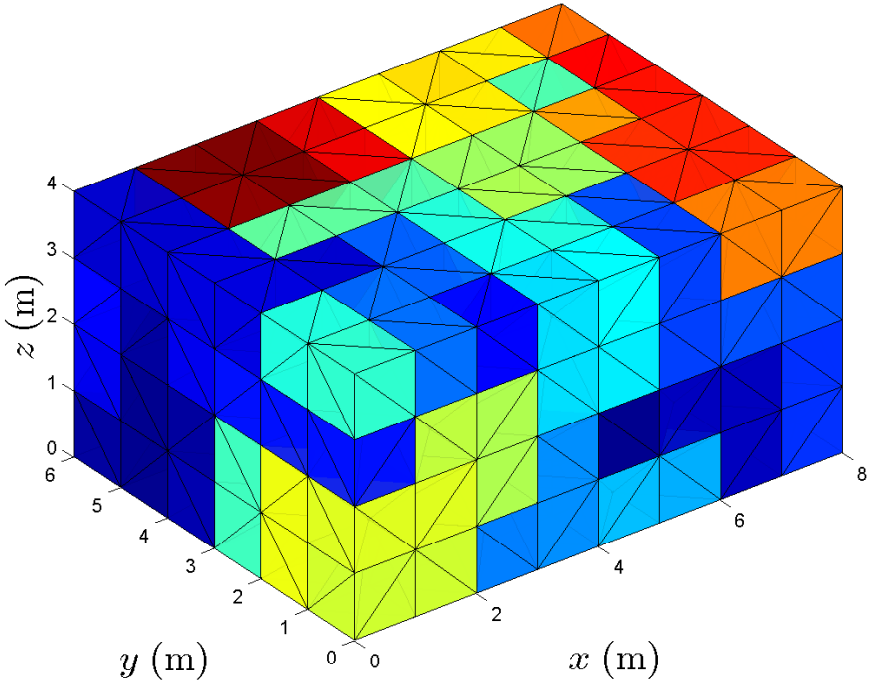


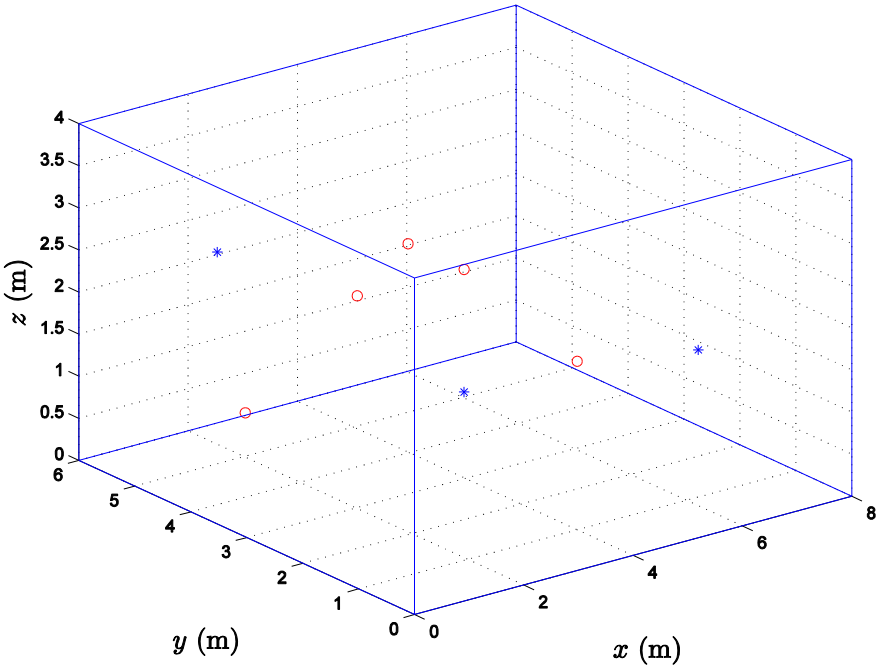


Simulation results (3)

Identification of wave propagation model (1)

- 3-D Helmholtz PDE in rectangular enclosure
- $M = 3, K = 315, J = 5, N = 1000$

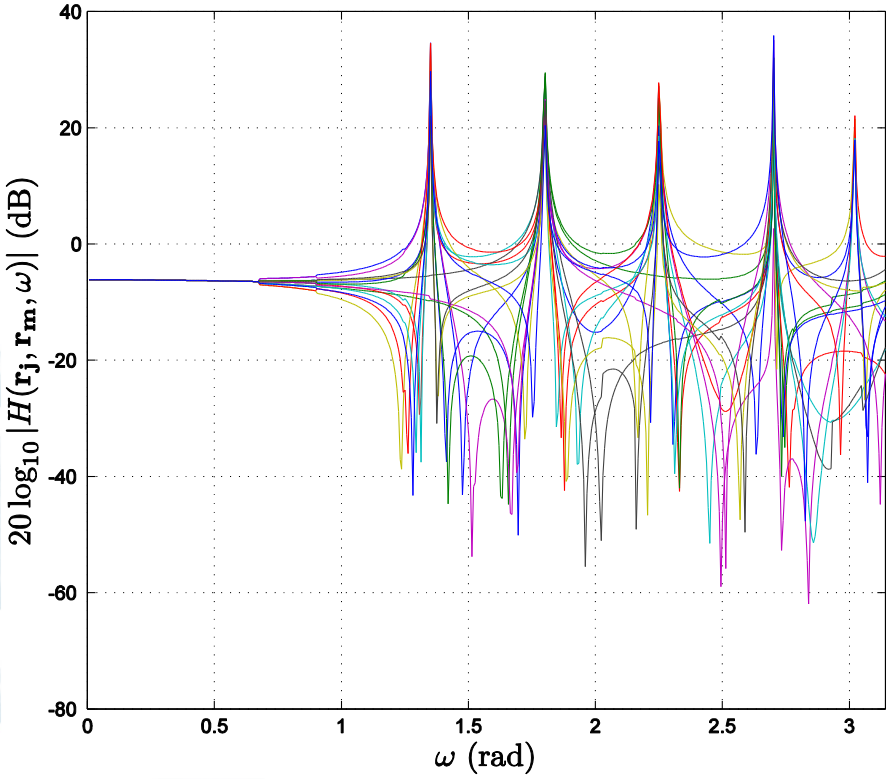


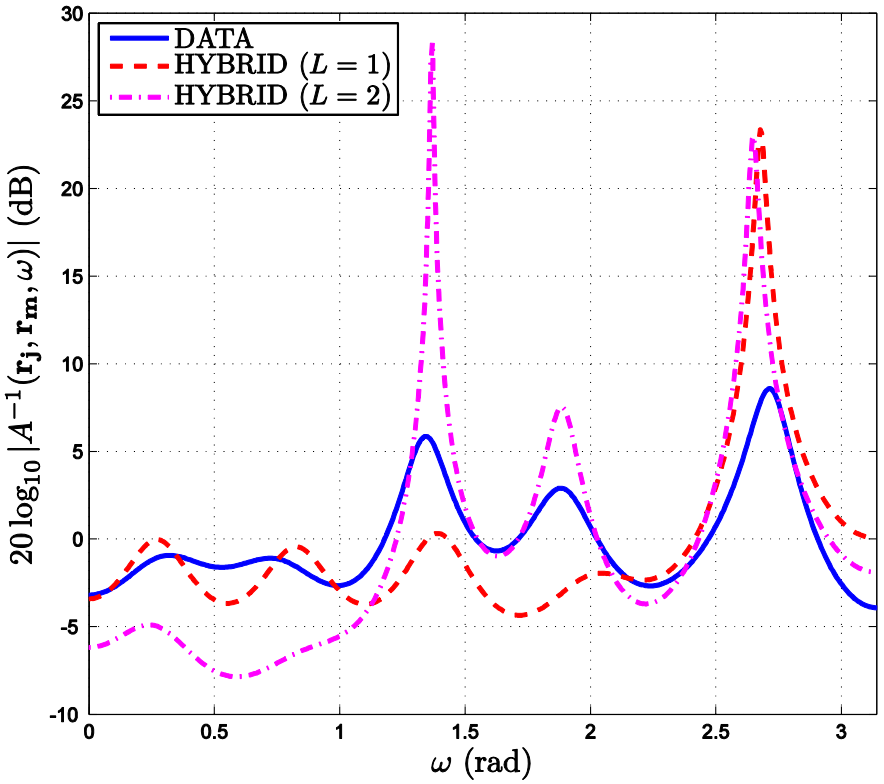


Simulation results (4)

Identification of wave propagation model (2)

- Imposing FEM system of equations at resonance frequencies increases local modeling accuracy





Conclusion

Conclusion

- Novel framework for combined data-driven and model-based field estimation
- Applications: wave field estimation, source recovery, source localization, and system identification
- Sparsity of FEM matrices can be exploited in distributed implementation for wireless sensor networks

References

- Toon van Waterschoot and Geert Leus, "Static field estimation using a wireless sensor network based on the finite element method", in *Proc. Int. Workshop Comput. Adv. Multi-Sensor Adaptive Process. (CAMSAP '11)*, San Juan, PR, USA, Dec. 2011.
- Toon van Waterschoot and Geert Leus, "Distributed estimation of static fields in wireless sensor networks using the finite element method", in *Proc. 2012 IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP '12),* Kyoto, Japan, Mar. 2012.
- Toon van Waterschoot, Moritz Diehl, Marc Moonen, and Geert Leus, "Identification of black-box wave propagation models using large-scale convex optimization", to appear in *Proc. 16th IFAC Symp. System Identification (SYSID '12)*, Brussels, Belgium, Nov. 2011 (invited paper).