Globally Optimal Least-Squares ARMA Model Identification Is an Eigenvalue Problem

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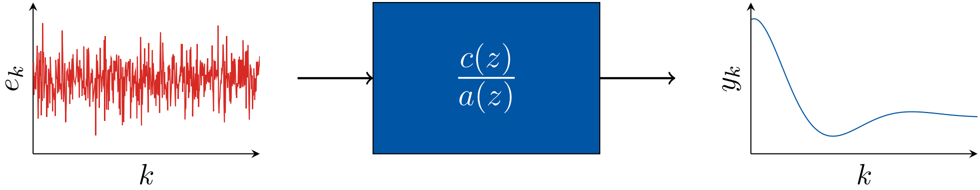
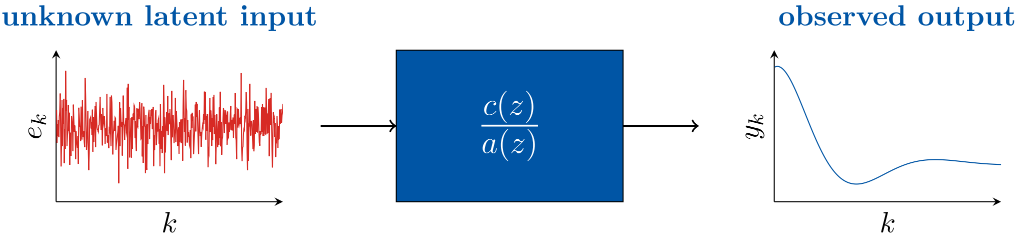
Introduction



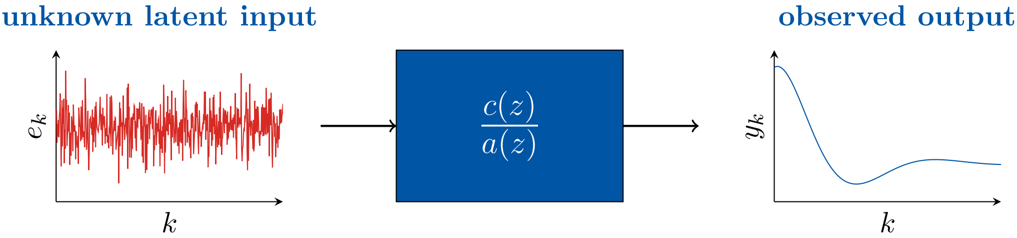
autoregressive moving-average (ARMA) model

Introduction



autoregressive moving-average (ARMA) model

Introduction



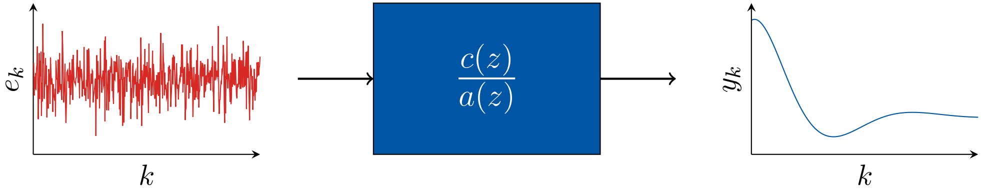
autoregressive moving-average (ARMA) model

globally optimal least-squares ARMA model identification is an eigenvalue problem

Outline

Outline

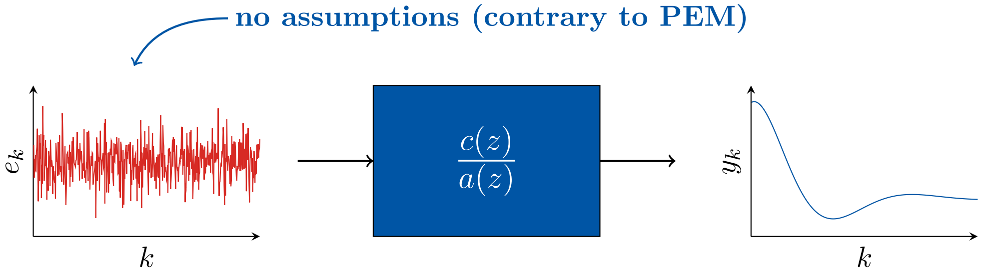
Autoregressive moving-average model



$$a(z)y_k = c(z)e_k$$

$$\sum_{i=0}^{n_a} \alpha_i y_{k-i} = \sum_{j=0}^{n_c} \gamma_j e_{k-j}$$

Autoregressive moving-average model



$$a(z)y_k = c(z)e_k$$

$$\sum_{i=0}^{n_a} \alpha_i y_{k-i} = \sum_{j=0}^{n_c} \gamma_j e_{k-j}$$

Autoregressive moving-average model

$$\sum_{i=0}^{n_a} \alpha_i y_{k-i} = \sum_{j=0}^{n_c} \gamma_j e_{k-j} \qquad k = 1, \dots, N$$

$$\Downarrow$$

$$\begin{bmatrix} \alpha_{n_a} & \cdots & \alpha_1 & 1 & & & \\ & \alpha_{n_a} & \cdots & \alpha_1 & 1 & & \\ & & \ddots & & \ddots & \ddots & \\ & & & \alpha_{n_a} & \cdots & \alpha_1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \gamma_{n_c} & \cdots & \gamma_1 & 1 & & & \\ & \gamma_{n_c} & \cdots & \gamma_1 & 1 & & \\ & & \ddots & & \ddots & \ddots & \\ & & & \gamma_{n_c} & \cdots & \gamma_1 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_M \end{bmatrix}$$

$$\Downarrow$$

$$T_a y = T_c e$$

Autoregressive moving-average model

$$\sum_{i=0}^{n_a} \alpha_i y_{k-i} = \sum_{j=0}^{n_c} \gamma_j e_{k-j}$$

k = 1,. . . ,N

number of data points

$$\Downarrow$$

$$\begin{bmatrix} \alpha_{n_a} & \cdots & \alpha_1 & 1 & & & \\ & \alpha_{n_a} & \cdots & \alpha_1 & 1 & & \\ & & \ddots & & \ddots & \ddots & \\ & & & \alpha_{n_a} & \cdots & \alpha_1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \gamma_{n_c} & \cdots & \gamma_1 & 1 & & & \\ & \gamma_{n_c} & \cdots & \gamma_1 & 1 & & \\ & & \ddots & & \ddots & \ddots & \\ & & & \gamma_{n_c} & \cdots & \gamma_1 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_M \end{bmatrix}$$

$$\Downarrow$$

$$T_a y = T_c e$$

Identification problem

$$\min_{a,c,e} \|e\|_2^2$$
$$\text{subject to } T_a y = T_c e$$

Identification problem

optimality condition: minimize e, while $y = \hat{y}$

$$\min_{a,c,e} \|e\|_2^2$$

$$\text{subject to } T_a y = T_c e$$

Identification problem

optimality condition: minimize e, while $y = \hat{y}$

$$\min_{a,c,e} \|e\|_2^2$$

$$\text{subject to } T_a y = T_c e$$

model condition: $y = T_a^{\dagger} T_c e = \hat{y}$

First order optimality conditions

$$\sigma^2 = \|e\|_2^2 = y^\mathrm{T} T_a^\mathrm{T} \left(T_c T_c^\mathrm{T}\right)^{-1} T_a y$$

First order optimality conditions

$$\begin{aligned}\sigma^2 = \|e\|_2^2 &= y^{\mathrm{T}} T_a^{\mathrm{T}} \left(\mathbf{T_c} \mathbf{T_c}^{\mathrm{T}}\right)^{\mathbf{-1}} T_a y \\ &= y^{\mathrm{T}} T_a^{\mathrm{T}} \mathbf{D_c^{-1}} T_a y\end{aligned}$$

First order optimality conditions

$$
\begin{aligned}
\sigma^2 = \|e\|_2^2 &= y^{\mathrm{T}} T_a^{\mathrm{T}} \left(T_c T_c^{\mathrm{T}}\right)^{-1} T_a y \\
&= y^{\mathrm{T}} T_a^{\mathrm{T}} \mathbf{D_c^{-1} T_a y} \\
&= y^{\mathrm{T}} T_a^{\mathrm{T}} \mathbf{f}
\end{aligned}
$$

First order optimality conditions

$$\sigma^2 = y^{\mathrm{T}} T_a^{\mathrm{T}} D_c^{-1} T_a y = y^{\mathrm{T}} T_a^{\mathrm{T}} f$$

partial derivatives: $\frac{\partial \cdot}{\partial \alpha_i} = 0, \frac{\partial \cdot}{\partial \gamma_j} = 0$

$$\begin{cases} y^{\mathrm{T}} T_a^{\mathrm{T}} f^{\alpha_i} + y^{\mathrm{T}} T_a^{\alpha_i \mathrm{T}} f = 0 & \forall i = 1, \ldots, n_a \\ y^{\mathrm{T}} T_a^{\mathrm{T}} f^{\gamma_j} = 0 & \forall j = 1, \ldots, n_c \\ D_c f^{\alpha_i} - T_a^{\alpha_i} y = 0 & \forall i = 1, \ldots, n_a \\ D_c^{\gamma_j} f + D_c f^{\gamma_j} = 0 & \forall j = 1, \ldots, n_c \\ D_c f - T_a y = 0 & \end{cases}$$

Multiparameter eigenvalue problem

$$\begin{bmatrix} I\otimes y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & \left\{y^{\mathrm{T}}T_a^{\alpha_i\mathrm{T}}\right\}_i & 0 \\ 0 & I\otimes y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & 0 \\ I\otimes D_c & 0 & 0 & \{T_a^{\alpha_i}y\}_i \\ 0 & I\otimes D_c & \left\{D_c^{\gamma_j}\right\}_j & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix} \begin{bmatrix} \{f^{\alpha_i}\}_i \\ \{f^{\gamma_j}\}_j \\ f \\ -1 \end{bmatrix} = 0$$

Multiparameter eigenvalue problem

$$\begin{bmatrix} I \otimes y^{\mathrm{T}} T_a^{\mathrm{T}} & 0 & \left\{ y^{\mathrm{T}} T_a^{\alpha_i \mathrm{T}} \right\}_i & 0 \\ 0 & I \otimes y^{\mathrm{T}} T_a^{\mathrm{T}} & 0 & 0 \\ I \otimes D_c & 0 & 0 & \{ T_a^{\alpha_i} y \}_i \\ 0 & I \otimes D_c & \left\{ D_c^{\gamma_j} \right\}_j & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix} \begin{bmatrix} \{ f^{\alpha_i} \}_i \\ \{ f^{\gamma_j} \}_j \\ f \\ -1 \end{bmatrix} = 0$$

Kronecker product:

$$A \otimes B = \begin{bmatrix} a_{11} B & \cdots & a_{1n} B \\ \vdots & \ddots & \vdots \\ a_{m1} B & \cdots & a_{mn} B \end{bmatrix}$$

Multiparameter eigenvalue problem

$$\begin{bmatrix} I \otimes y^{\mathrm{T}} T_a^{\mathrm{T}} & 0 & \left\{ y^{\mathrm{T}} T_a^{\alpha_i \mathrm{T}} \right\}_i & 0 \\ 0 & I \otimes y^{\mathrm{T}} T_a^{\mathrm{T}} & 0 & 0 \\ I \otimes D_c & 0 & 0 & \{T_a^{\alpha_i} y\}_i \\ 0 & I \otimes D_c & \left\{ D_c^{\gamma_j} \right\}_j & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix} \begin{bmatrix} \{f^{\alpha_i}\}_i \\ \{f^{\gamma_j}\}_j \\ f \\ -1 \end{bmatrix} = 0$$

Kronecker product:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

matrix stack:

$$\{M_i\}_i = \begin{bmatrix} M_1 \\ \vdots \\ M_m \end{bmatrix}$$

Outline

Multiparameter eigenvalue problem

Multiparameter eigenvalue problem:

$$\left(\sum_{\omega} A_\omega \omega\right) z = 0, \qquad z \neq 0$$

- $\omega\left(\lambda_1, \ldots, \lambda_n\right) = \lambda_1^{k_1} \cdots \lambda_n^{k_n}$ is a monomial function of the eigenvalues
- A_ω is the corresponding matrix of coefficients

Multiparameter eigenvalue problem

comparison

standard eigenvalue problem	**multiparameter eigenvalue problem**
$Az = \lambda z$	$A_1 z = \sum_{\omega \neq 1} A_\omega \omega z$
$(A - I\lambda)\, z = 0$	$\left(\sum_\omega A_\omega \omega\right) z = 0$
one eigenvalue	multiple eigenvalues

Multiparameter eigenvalue problem

examples

standard eigenvalue problem: $(A - I\lambda)\, z = 0$

generalized eigenvalue problem: $(A - B\lambda)\, z = 0$

linear two-parameter eigenvalue problem: $(A_1 + A_\alpha \alpha + A_\gamma \gamma)\, z = 0$

multiparameter eigenvalue problem: $\left(A_1 + A_{\lambda_1^2} \lambda_1^2 + A_{\lambda_2 \lambda_3^3} \lambda_2 \lambda_3^3\right) z = 0$

Multiparameter eigenvalue problem

autoregressive moving-average model

$$\underbrace{\begin{bmatrix} I \otimes y^{\mathrm{T}} T_a^{\mathrm{T}} & 0 & \left\{ y^{\mathrm{T}} T_a^{\alpha_i \mathrm{T}} \right\}_i & 0 \\ 0 & I \otimes y^{\mathrm{T}} T_a^{\mathrm{T}} & 0 & 0 \\ I \otimes D_c & 0 & 0 & \{ T_a^{\alpha_i} y \}_i \\ 0 & I \otimes D_c & \left\{ D_c^{\gamma_j} \right\}_j & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix}}_{A(\alpha_i, \gamma_j)} \underbrace{\begin{bmatrix} \{ f^{\alpha_i} \}_i \\ \{ f^{\gamma_j} \}_j \\ f \\ -1 \end{bmatrix}}_{z} = 0$$

Multiparameter eigenvalue problem

autoregressive moving-average model

$$\underbrace{\begin{bmatrix} I \otimes y^{\mathrm{T}} T_a^{\mathrm{T}} & 0 & \left\{ y^{\mathrm{T}} T_a^{\alpha_i \mathrm{T}} \right\}_i & 0 \\ 0 & I \otimes y^{\mathrm{T}} T_a^{\mathrm{T}} & 0 & 0 \\ I \otimes D_c & 0 & 0 & \{T_a^{\alpha_i} y\}_i \\ 0 & I \otimes D_c & \left\{ D_c^{\gamma_j} \right\}_j & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix}}_{A(\alpha_i, \gamma_j)} \underbrace{\begin{bmatrix} \{f^{\alpha_i}\}_i \\ \{f^{\gamma_j}\}_j \\ f \\ -1 \end{bmatrix}}_{z} = 0$$

$$\left(\sum_{\omega} A_{\omega} \omega \right) z = 0$$

Multiparameter eigenvalue problem

autoregressive moving-average model

$$\underbrace{\begin{bmatrix} I \otimes y^{\mathrm{T}} T_a^{\mathrm{T}} & 0 & \left\{ y^{\mathrm{T}} T_a^{\alpha_i \mathrm{T}} \right\}_i & 0 \\ 0 & I \otimes y^{\mathrm{T}} T_a^{\mathrm{T}} & 0 & 0 \\ I \otimes D_c & 0 & 0 & \{ T_a^{\alpha_i} y \}_i \\ 0 & I \otimes D_c & \left\{ D_c^{\gamma_j} \right\}_j & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix}}_{A(\alpha_i, \gamma_j)} \underbrace{\begin{bmatrix} \{ f^{\alpha_i} \}_i \\ \{ f^{\gamma_j} \}_j \\ f \\ -1 \end{bmatrix}}_{z} = 0$$

$$\left(\sum_{\omega} A_{\omega} \omega \right) z = 0$$

eigenvector (z) and $(n_a + n_c)$-tuple of eigenvalues (α_i and γ_j)

Multiparameter eigenvalue problem

example: first order ARMA(1,1) model

$$\begin{bmatrix} y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & y^{\mathrm{T}}T_a^{\alpha_1\mathrm{T}} & 0 \\ 0 & y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & 0 \\ D_c & 0 & 0 & T_a^{\alpha_1}y \\ 0 & D_c & D_c^{\gamma_1} & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix} \begin{bmatrix} f^{\alpha_1} \\ f^{\gamma_1} \\ f \\ -1 \end{bmatrix} = 0$$

Multiparameter eigenvalue problem

example: first order ARMA(1,1) model

$$\begin{bmatrix} y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & y^{\mathrm{T}}T_a^{\alpha_1 \mathrm{T}} & 0 \\ 0 & y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & 0 \\ D_c & 0 & 0 & T_a^{\alpha_1}y \\ 0 & D_c & D_c^{\gamma_1} & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix} \begin{bmatrix} f^{\alpha_1} \\ f^{\gamma_1} \\ f \\ -1 \end{bmatrix} = 0$$

$$\left(A_1 + A_\alpha \alpha + A_\gamma \gamma + A_{\alpha^2}\alpha^2 + A_{\alpha\gamma}\alpha\gamma + A_{\gamma^2}\gamma^2\right) z = 0$$

Multiparameter eigenvalue problem

example: first order ARMA(1,1) model

$$\begin{bmatrix} y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & y^{\mathrm{T}}T_a^{\alpha_1\mathrm{T}} & 0 \\ 0 & y^{\mathrm{T}}T_a^{\mathrm{T}} & 0 & 0 \\ D_c & 0 & 0 & T_a^{\alpha_1}y \\ 0 & D_c & D_c^{\gamma_1} & 0 \\ 0 & 0 & D_c & T_a y \end{bmatrix} \begin{bmatrix} f^{\alpha_1} \\ f^{\gamma_1} \\ f \\ -1 \end{bmatrix} = 0$$

$$\left(A_1 + A_\alpha \alpha + A_\gamma \gamma + A_{\alpha^2}\alpha^2 + A_{\alpha\gamma}\alpha\gamma + A_{\gamma^2}\gamma^2\right) z = 0$$

eigenvector (z) and 2-tuple of eigenvalues (α and γ)

Outline

Quadratic two-parameter eigenvalue problem

for example, a **first order ARMA(1,1) model** with parameters α and γ:

$$\left(A_1 + A_\alpha \alpha + A_\gamma \gamma + A_{\alpha^2} \alpha^2 + A_{\alpha\gamma} \alpha\gamma + A_{\gamma^2} \gamma^2\right) z = 0$$

Block Macaulay matrix

$$\left(A_1 + A_\alpha \alpha + A_\gamma \gamma + A_{\alpha^2} \alpha^2 + A_{\alpha\gamma} \alpha\gamma + A_{\gamma^2} \gamma^2\right) z = 0$$

Block Macaulay matrix

$$\left(A_1 + A_\alpha \alpha + A_\gamma \gamma + A_{\alpha^2}\alpha^2 + A_{\alpha\gamma}\alpha\gamma + A_{\gamma^2}\gamma^2\right) z = 0$$

↳ **"shift"** with the eigenvalues α and γ

$$\alpha\left(A_1 + A_\alpha \alpha + A_\gamma \gamma + A_{\alpha^2}\alpha^2 + A_{\alpha\gamma}\alpha\gamma + A_{\gamma^2}\gamma^2\right) z = 0$$

$$\gamma\left(A_1 + A_\alpha \alpha + A_\gamma \gamma + A_{\alpha^2}\alpha^2 + A_{\alpha\gamma}\alpha\gamma + A_{\gamma^2}\gamma^2\right) z = 0$$

$$\alpha^2\left(A_1 + A_\alpha \alpha + A_\gamma \gamma + A_{\alpha^2}\alpha^2 + A_{\alpha\gamma}\alpha\gamma + A_{\gamma^2}\gamma^2\right) z = 0$$

$$\vdots$$

Block Macaulay matrix

stack the MEP and its shifts as a **block Macaulay matrix**

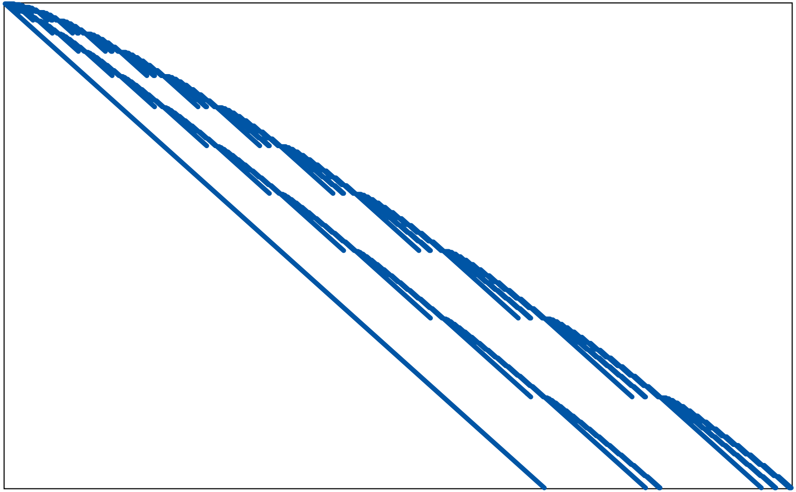
$$M(d) = \begin{array}{c} \begin{matrix} z & z\alpha & z\gamma & z\alpha^2 & z\alpha\gamma & z\gamma^2 & z\alpha^3 & \end{matrix} \\ \begin{bmatrix} A_1 & A_\alpha & A_\gamma & A_{\alpha^2} & A_{\alpha\gamma} & A_{\gamma^2} & 0 & \cdots \\ 0 & A_1 & 0 & A_\alpha & A_{\alpha^2} & 0 & A_{\alpha^2} & \cdots \\ 0 & 0 & A_1 & 0 & A_\alpha & A_{\alpha^2} & 0 & \cdots \\ 0 & 0 & 0 & A_1 & 0 & 0 & A_\alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

Block Macaulay matrix

a **quasi-Toeplitz structure** emerges

$$
\begin{array}{c} \\ M(d) = \end{array}
\begin{array}{c}
\begin{array}{cccccccc} z & z\alpha & z\gamma & z\alpha^2 & z\alpha\gamma & z\gamma^2 & z\alpha^3 & \end{array} \\
\begin{bmatrix}
A_1 & A_\alpha & A_\gamma & A_{\alpha^2} & A_{\alpha\gamma} & A_{\gamma^2} & 0 & \cdots \\
0 & A_1 & 0 & A_\alpha & A_\gamma & 0 & A_{\alpha^2} & \cdots \\
0 & 0 & A_1 & 0 & A_\alpha & A_\gamma & 0 & \cdots \\
0 & 0 & 0 & A_1 & 0 & 0 & A_\alpha & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{array}
$$

Block Macaulay matrix



Block Macaulay matrix

rewrite the MEP and its shifts using the block Macaulay matrix

$$\begin{array}{cccccccc} z & z\alpha & z\gamma & z\alpha^2 & z\alpha\gamma & z\gamma^2 & z\alpha^3 & \end{array}$$

$$\begin{bmatrix} A_1 & A_\alpha & A_\gamma & A_{\alpha^2} & A_{\alpha\gamma} & A_{\gamma^2} & 0 & \cdots \\ 0 & A_1 & 0 & A_\alpha & A_\gamma & 0 & A_{\alpha^2} & \cdots \\ 0 & 0 & A_1 & 0 & A_\alpha & A_\gamma & 0 & \cdots \\ 0 & 0 & 0 & A_1 & 0 & 0 & A_\alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} z \\ \hline z\alpha \\ z\gamma \\ \hline z\alpha^2 \\ z\alpha\gamma \\ z\gamma^2 \\ \hline z\alpha^3 \\ \vdots \end{bmatrix} = 0$$

Block multi-shift-invariant null space

- Solutions generate vectors in the null space of M
  $$MK = 0$$
- Nullity corresponds to the number of solutions m_b
  $$m_b = \prod_{i=1}^{n} d_i$$
- Null space has a **block multi-shift-invariant** structure

block multivariate Vandermonde basis K

$$K = \begin{bmatrix} z & \cdots & z \\ \hline z\alpha & \cdots & z\alpha \\ z\gamma & \cdots & z\gamma \\ \hline z\alpha^2 & \cdots & z\alpha^2 \\ z\alpha\gamma & \cdots & z\alpha\gamma \\ z\gamma^2 & \cdots & z\gamma^2 \\ \hline z\alpha^3 & \cdots & z\alpha^3 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Multidimensional realization theory

for one solution

$$\begin{bmatrix} z \\ \hline z\alpha \\ z\gamma \\ \hline z\alpha^2 \\ z\alpha\gamma \\ z\gamma^2 \\ \hline z\alpha^3 \\ \vdots \end{bmatrix} \xrightarrow{\alpha} \begin{bmatrix} z \\ \hline z\alpha \\ z\gamma \\ \hline z\alpha^2 \\ z\alpha\gamma \\ z\gamma^2 \\ \hline z\alpha^3 \\ \vdots \end{bmatrix}$$

$$S_1 k\alpha = S_\alpha k$$

Multidimensional realization theory

for all solutions

$$\begin{bmatrix} z & \cdots & z \\ \hline z\alpha & \cdots & z\alpha \\ z\gamma & \cdots & z\gamma \\ \hline z\alpha^2 & \cdots & z\alpha^2 \\ z\alpha\gamma & \cdots & z\alpha\gamma \\ z\gamma^2 & \cdots & z\gamma^2 \\ \hline z\alpha^3 & \cdots & z\alpha^3 \\ \vdots & \vdots & \vdots \end{bmatrix} \xrightarrow{\alpha} \begin{bmatrix} z & \cdots & z \\ \hline z\alpha & \cdots & z\alpha \\ z\gamma & \cdots & z\gamma \\ \hline z\alpha^2 & \cdots & z\alpha^2 \\ z\alpha\gamma & \cdots & z\alpha\gamma \\ z\gamma^2 & \cdots & z\gamma^2 \\ \hline z\alpha^3 & \cdots & z\alpha^3 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$S_1 K D_\alpha = S_\alpha K$$

Multidimensional realization theory

consider m_a affine and simple solutions

Realization theory for an unknown α:

$$S_1 K D_\alpha = S_\alpha K,$$

where S_1 and S_α select blocks from K

Multidimensional realization theory

numerical basis for the null space

$$S_1 K D_\alpha = S_\alpha K$$

Multidimensional realization theory

numerical basis for the null space

$$S_1 K D_\alpha = S_\alpha K$$

- The solutions are not known in advance
- Consider a **numerical basis for the null space** Z

$$K = ZT$$

- This results in

$$(S_1 Z) T D_\alpha = (S_\alpha Z) T$$

Multidimensional realization theory

numerical basis for the null space

$$S_1 K D_\alpha = S_\alpha K$$

for example, calculated via the SVD

- The solutions are not known in advance
- Consider a **numerical basis for the null space** Z

$$K = ZT$$

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Multidimensional realization theory

numerical basis for the null space

$$S_1 K D_\alpha = S_\alpha K$$

for example, calculated via the SVD

- The solutions are not known in advance
- Consider a **numerical basis for the null space** Z

$$K = ZT$$

non-singular matrix T

- This results in

$$(S_1 Z)\, T D_\alpha = (S_\alpha Z)\, T$$

Multidimensional realization theory

standard eigenvalue problem

Realization theory for an unknown α:

$$(S_1 Z)\, T D_\alpha = (S_\alpha Z)\, T,$$

where S_1 and S_α select blocks from Z

- Generalized eigenvalue problem, with T the matrix of eigenvectors
- We can rewrite this as a **standard eigenvalue problem**

$$T D_\alpha T^{-1} = (S_1 Z)^\dagger (S_\alpha Z)$$

Multidimensional realization theory

Other shift functions

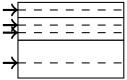
- It is **possible to shift with any polynomial** in the eigenvalues – for example with γ

$$(S_1 Z)\, T D_\gamma = (S_\gamma Z)\, T$$

- This leads to the same eigenvectors T

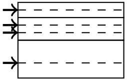
Solutions at infinity

$d = 3$

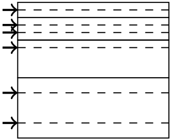


Solutions at infinity

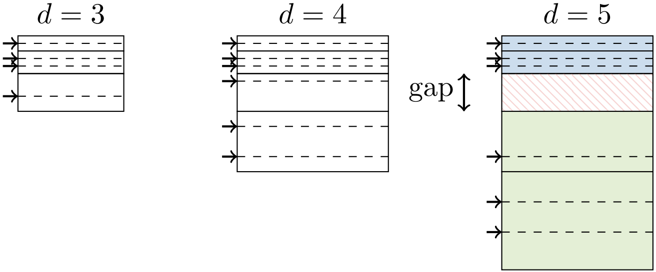
$d = 3$



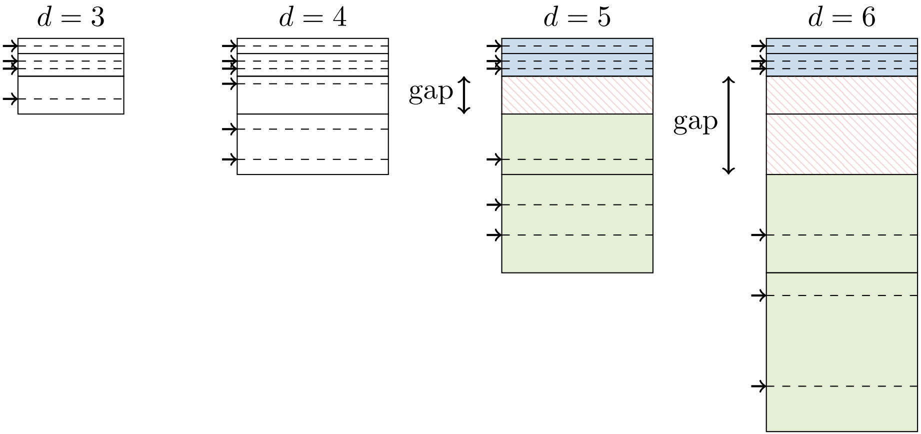
$d = 4$



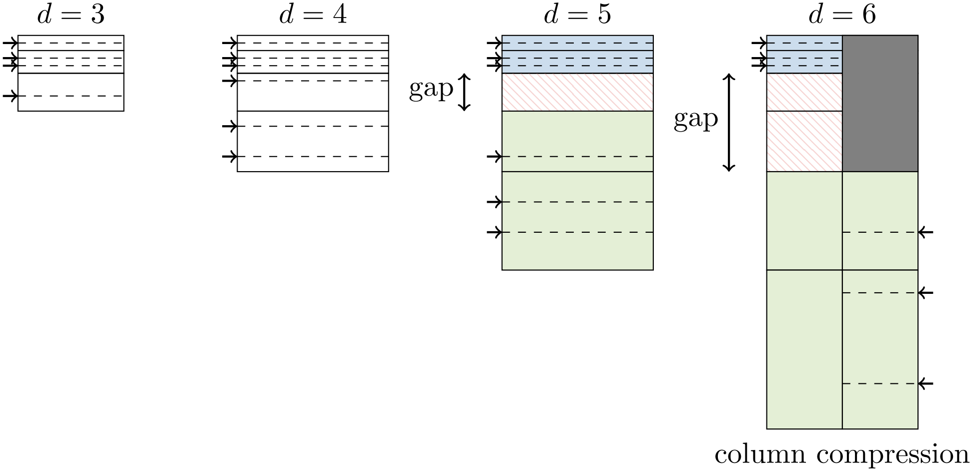
Solutions at infinity



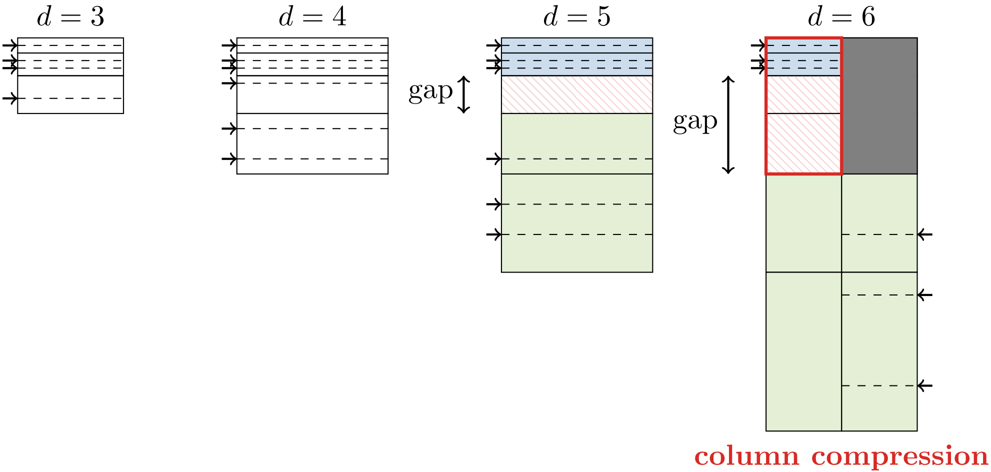
Solutions at infinity



Solutions at infinity



Solutions at infinity



Multiplicity higher than one

simple roots

$$\begin{bmatrix} | & | & \\ k^{(1)} & k^{(2)} & \cdots \\ | & | & \end{bmatrix}$$

multiple roots

$$\begin{bmatrix} | & | & \\ k^{(1)} & \partial k^{(1)} & \cdots \\ | & | & \end{bmatrix}$$

System theoretic interpretations

- Corresponds to a **multidimensional observability matrix**

$$H = \begin{bmatrix} \Gamma_R & \Gamma_S \end{bmatrix}$$

- Generated by a multidimensional descriptor system
- Contains the block multi-shift-invariance

$$S_\alpha H A_\gamma = S_{\alpha\gamma} H,$$

$$A_\gamma = \left(S_\alpha H\right)^\dagger \left(S_{\alpha\gamma} H\right)$$

block column echelon basis H

$$H = \left[\begin{array}{cc} C_R & 0 \\ \hline C_R A_\alpha & 0 \\ C_R A_\gamma & 0 \\ \hline C_R A_\alpha^2 & 0 \\ C_R A_\alpha A_\gamma & 0 \\ C_R A_\alpha^2 & 0 \\ \hline \vdots & \vdots \\ \hline C_R A_\alpha^n & 0 \\ \vdots & \vdots \\ \hline \times & C_S E_\alpha^{m-1} \\ \vdots & \vdots \end{array}\right]$$

Outline

Conclusion

- Identification of ARMA models is an MEP
- MEP can be solved via the block Macaulay matrix
- Null space of block Macaulay matrix yields an SEP

Future work

- Investigate rigorous properties of block Macaulay matrix
- Extend to other model classes
- Restrict the calculations to the optimizing solution only
- Solve large real-life problems