System Identification Problems as Multiparameter Eigenvalue Problems

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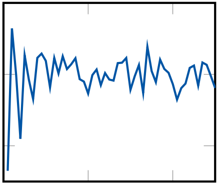
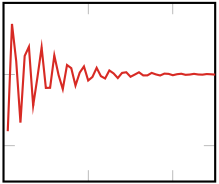
Least-squares realization problem

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}}_{\text{given}} \longrightarrow \underbrace{\begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_N \end{bmatrix}}_{\text{unknown}} \quad \text{such that } \hat{\boldsymbol{y}}_k = \boldsymbol{C}\boldsymbol{A}^k\boldsymbol{x}_0$$





$$\min \|\boldsymbol{y} - \hat{\boldsymbol{y}}\|_2^2$$
$$\text{subject to } \boldsymbol{T}_\alpha \hat{\boldsymbol{y}} = \boldsymbol{0}$$

(De Moor, 2019, 2020)

Multiparameter eigenvalue problem

The **multiparameter eigenvalue problem** $\boldsymbol{\mathcal{M}}(\lambda_1, \ldots, \lambda_n)\, \boldsymbol{z} = \mathbf{0}$ consists in finding all n-tuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and corresponding vectors $\boldsymbol{z} \in \mathbb{C}^{l \times 1} \setminus \{\mathbf{0}\}$, so that

$$\boldsymbol{\mathcal{M}}(\lambda_1, \ldots, \lambda_n)\, \boldsymbol{z} = \left(\sum_{\{\boldsymbol{\omega}\}} \boldsymbol{A}_{\boldsymbol{\omega}} \boldsymbol{\lambda}^{\boldsymbol{\omega}} \right) \boldsymbol{z} = \mathbf{0},$$

with $\|\boldsymbol{z}\|_2 = 1$.

- coefficient matrices $\boldsymbol{A}_{\boldsymbol{\omega}} = \boldsymbol{A}_{(\omega_1, \ldots, \omega_n)} \in \mathbb{R}^{k \times l}$ with $k \geq l + n - 1$
- eigentuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and eigenvectors $\boldsymbol{z} \neq \mathbf{0}$
- example: $\left(\boldsymbol{A}_{000} + \boldsymbol{A}_{100}\lambda_1 + \boldsymbol{A}_{032}\lambda_2^3\lambda_3^2\right) \boldsymbol{z} = \mathbf{0}$

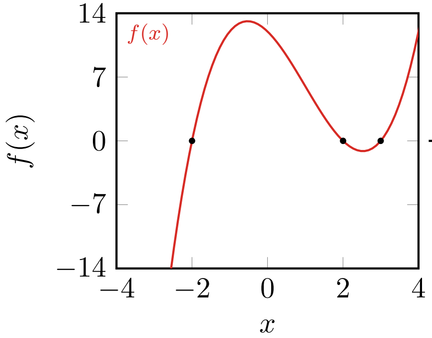
(Vermeersch and De Moor, 2019, 2022b)

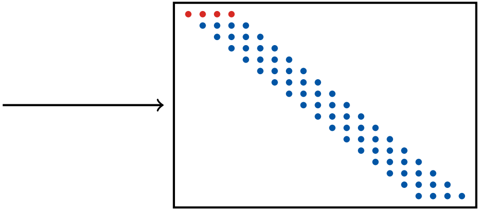
Outline

Outline

Case I: univariate polynomial equation

$$f(x) = 12 + (-4)x + -3x^2 + 1x^3 = 0$$





Toeplitz matrix

(De Cock and De Moor, 2021)

Case I: Toeplitz matrix and scalar FsSRs

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0$$

$$\begin{array}{r} f(x) \\ x f(x) \\ x^2 f(x) \\ x^3 f(x) \end{array} \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{bmatrix} = \mathbf{0}$$

use **scalar forward single-shift recursions (scalar FsSRs)** to generate the Toeplitz matrix $\boldsymbol{T}$ from $f(x)$

The solution vectors span the null space:

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 3 \\ 4 & 4 & 9 \\ -8 & 8 & 27 \\ 16 & 16 & 81 \\ -32 & 32 & 243 \\ 64 & 64 & 729 \end{bmatrix}$$

$$\boldsymbol{T}\boldsymbol{V} = \mathbf{0}$$

Case I: scalar single-shift-invariance

one row

one variable

$$\underbrace{\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{bmatrix}}_{\boldsymbol{S}_1 \, \boldsymbol{v}|_{(j)}} \xrightarrow{\;x\;} \underbrace{\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{bmatrix}}_{\boldsymbol{S}_x \, \boldsymbol{v}|_{(j)}}$$

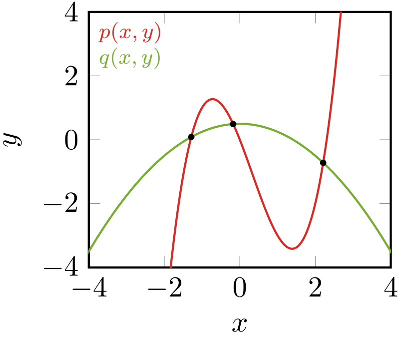
one column/solution

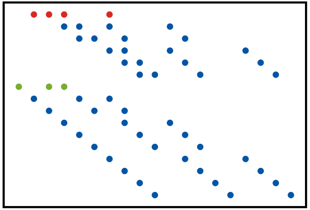
We can shift the rows of $\boldsymbol{V}$ with the three roots of $f(x)$:

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 3 \\ 4 & 4 & 9 \end{bmatrix} \underbrace{\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{\text{roots}} = \begin{bmatrix} -2 & 2 & 3 \\ 4 & 4 & 9 \\ -8 & 8 & 27 \end{bmatrix}$$

Case II: system of multivariate polynomial equations

$$\begin{cases} p(x,y) = 1y + 3x + 1x^2 + (-1)x^3 = 0 \\ q(x,y) = 2 + (-4)y + (-1)x^2 = 0 \end{cases}$$





Macaulay matrix

Case II: Macaulay matrix and scalar FmSRs

$$\begin{cases} p(x,y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{30}x^3 = 0 \\ q(x,y) = b_{00} + b_{01}y + b_{20}x^2 = 0 \end{cases}$$

$$\begin{array}{r} p(x,y) \\ q(x,y) \\ xq(x,y) \\ yq(x,y) \end{array} \begin{bmatrix} 0 & a_{10} & a_{01} & a_{20} & 0 & 0 & a_{30} & 0 & 0 & 0 \\ b_{00} & 0 & b_{01} & b_{20} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{00} & 0 & 0 & b_{01} & 0 & b_{20} & 0 & 0 & 0 \\ 0 & 0 & b_{00} & 0 & 0 & b_{01} & 0 & b_{20} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix} = \mathbf{0}$$

use **scalar forward multi-shift recursions (scalar FmSRs)** to generate the Macaulay matrix $\boldsymbol{M}$ from $p(x,y)$ and $q(x,y)$

Case II: scalar multi-shift-invariance

one row

multiple variables

$$\underbrace{\begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix}}_{\boldsymbol{S}_1\, \boldsymbol{v}|_{(j)}} \xrightarrow{\quad x \quad} \underbrace{\begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix}}_{\boldsymbol{S}_x\, \boldsymbol{v}|_{(j)}}$$

one column/solution

Case II: scalar multi-shift-invariance

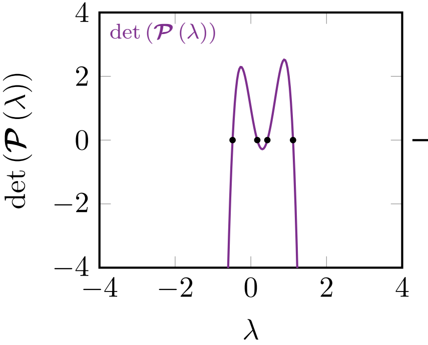
one row multiple variables

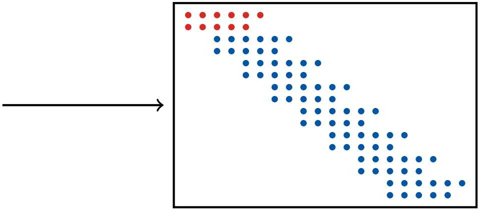
$$\underbrace{\begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix}}_{\boldsymbol{S}_1 \boldsymbol{v}|_{(j)}} \xrightarrow{\quad y \quad} \underbrace{\begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix}}_{\boldsymbol{S}_y \boldsymbol{v}|_{(j)}}$$

one column/solution

Case III: one-parameter eigenvalue problem

$$\boldsymbol{\mathcal{P}}(\lambda)\,\boldsymbol{z} = \left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}\lambda + \begin{bmatrix} 1 & -5 \\ -5 & 0 \end{bmatrix}\lambda^2\right)\boldsymbol{z} = \mathbf{0}$$





block Toeplitz matrix

Case III: block Toeplitz matrix and block FsSRs

$$\boldsymbol{\mathcal{P}}(\lambda)\,\boldsymbol{z} = \left(\boldsymbol{A}_0 + \boldsymbol{A}_1\lambda + \boldsymbol{A}_2\lambda^2\right)\boldsymbol{z} = \boldsymbol{0}$$

$$\begin{array}{r} \boldsymbol{\mathcal{P}}(\lambda) \\ \lambda\boldsymbol{\mathcal{P}}(\lambda) \\ \lambda^2\boldsymbol{\mathcal{P}}(\lambda) \\ \lambda^3\boldsymbol{\mathcal{P}}(\lambda) \end{array}
\begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & 0 & 0 & 0 \\ 0 & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & 0 & 0 \\ 0 & 0 & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & 0 \\ 0 & 0 & 0 & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 \end{bmatrix}
\begin{bmatrix} \boldsymbol{z} \\ \lambda\boldsymbol{z} \\ \lambda^2\boldsymbol{z} \\ \lambda^3\boldsymbol{z} \\ \lambda^4\boldsymbol{z} \\ \lambda^5\boldsymbol{z} \end{bmatrix} = \boldsymbol{0}$$

use **block forward single-shift recursions (block FsSRs)** to generate the block Toeplitz matrix $\boldsymbol{T}$ from $\boldsymbol{\mathcal{P}}(\lambda)\,\boldsymbol{z}$

Case III: block single-shift-invariance

multiple rows

one variable

$$\underbrace{\begin{bmatrix} \boldsymbol{z} \\ \lambda \boldsymbol{z} \\ \lambda^2 \boldsymbol{z} \\ \lambda^3 \boldsymbol{z} \\ \lambda^4 \boldsymbol{z} \\ \lambda^5 \boldsymbol{z} \end{bmatrix}}_{\boldsymbol{S}_1 \boldsymbol{v}|_{(j)}} \xrightarrow{\quad\lambda\quad} \underbrace{\begin{bmatrix} \boldsymbol{z} \\ \lambda \boldsymbol{z} \\ \lambda^2 \boldsymbol{z} \\ \lambda^3 \boldsymbol{z} \\ \lambda^4 \boldsymbol{z} \\ \lambda^5 \boldsymbol{z} \end{bmatrix}}_{\boldsymbol{S}_\lambda \boldsymbol{v}|_{(j)}}$$

one column/solution

Case III: block single-shift-invariance

multiple rows

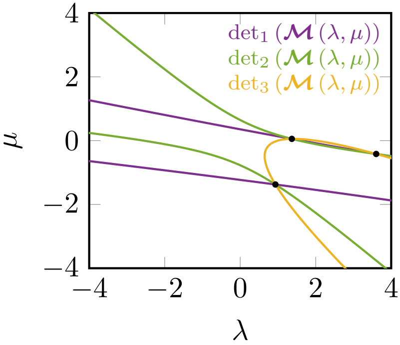
one variable

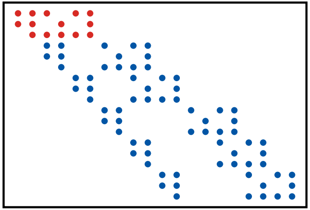
$$\begin{bmatrix} z_1 \\ \vdots \\ z_l \end{bmatrix} \quad \underbrace{\begin{bmatrix} \boldsymbol{z} \\ \lambda \boldsymbol{z} \\ \lambda^2 \boldsymbol{z} \\ \lambda^3 \boldsymbol{z} \\ \lambda^4 \boldsymbol{z} \\ \lambda^5 \boldsymbol{z} \end{bmatrix}}_{\boldsymbol{S}_1 \, \boldsymbol{v}|_{(j)}} \xrightarrow{\quad \lambda \quad} \underbrace{\begin{bmatrix} \boldsymbol{z} \\ \lambda \boldsymbol{z} \\ \lambda^2 \boldsymbol{z} \\ \lambda^3 \boldsymbol{z} \\ \lambda^4 \boldsymbol{z} \\ \lambda^5 \boldsymbol{z} \end{bmatrix}}_{\boldsymbol{S}_\lambda \, \boldsymbol{v}|_{(j)}}$$

one column/solution

Case IV: multiparameter eigenvalue problem

$$\mathcal{M}(\lambda,\mu)\,\boldsymbol{z} = \left(\begin{bmatrix} 2 & 6 \\ 4 & 5 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \lambda + \begin{bmatrix} 4 & 2 \\ 0 & 8 \\ 1 & 1 \end{bmatrix} \mu \right) \boldsymbol{z} = \boldsymbol{0}$$





block Macaulay matrix

Case IV: block Macaulay matrix and block FmSRs

$$\boldsymbol{\mathcal{M}}\left(\lambda,\mu\right)z = \left(\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\lambda + \boldsymbol{A}_{01}\mu\right)\boldsymbol{z} = \mathbf{0}$$

$$\begin{array}{r} \boldsymbol{\mathcal{M}}\left(\lambda\right) \\ \lambda\boldsymbol{\mathcal{M}}\left(\lambda\right) \\ \mu\boldsymbol{\mathcal{M}}\left(\lambda\right) \\ \lambda^2\boldsymbol{\mathcal{M}}\left(\lambda\right) \end{array}
\begin{bmatrix}
\boldsymbol{A}_{00} & \boldsymbol{A}_{10} & \boldsymbol{A}_{01} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \boldsymbol{A}_{00} & 0 & \boldsymbol{A}_{10} & \boldsymbol{A}_{01} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \boldsymbol{A}_{00} & 0 & \boldsymbol{A}_{10} & \boldsymbol{A}_{01} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \boldsymbol{A}_{00} & 0 & 0 & \boldsymbol{A}_{10} & \boldsymbol{A}_{01} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\boldsymbol{z} \\ \lambda\boldsymbol{z} \\ \mu\boldsymbol{z} \\ \lambda^2\boldsymbol{z} \\ \lambda\mu\boldsymbol{z} \\ \lambda^3\boldsymbol{z} \\ \lambda^2\mu\boldsymbol{z} \\ \lambda\mu^2\boldsymbol{z} \\ \mu^3\boldsymbol{z}
\end{bmatrix} = \mathbf{0}$$

use **block forward multi-shift recursions (block FmSRs)** to generate the block Macaulay matrix $\boldsymbol{M}$ from $\boldsymbol{\mathcal{M}}\left(\lambda\right)\boldsymbol{z}$

Case IV: block multi-shift-invariance

multiple rows multiple variables

$$\underbrace{\begin{bmatrix} \boldsymbol{z} \\ \lambda \boldsymbol{z} \\ \mu \boldsymbol{z} \\ \lambda^2 \boldsymbol{z} \\ \lambda\mu \boldsymbol{z} \\ \lambda^3 \boldsymbol{z} \\ \lambda^2\mu \boldsymbol{z} \\ \lambda\mu^2 \boldsymbol{z} \\ \mu^3 \boldsymbol{z} \end{bmatrix}}_{\boldsymbol{S}_1\, \boldsymbol{v}|_{(j)}} \xrightarrow{\quad\lambda\quad} \underbrace{\begin{bmatrix} \boldsymbol{z} \\ \lambda \boldsymbol{z} \\ \mu \boldsymbol{z} \\ \lambda^2 \boldsymbol{z} \\ \lambda\mu \boldsymbol{z} \\ \lambda^3 \boldsymbol{z} \\ \lambda^2\mu \boldsymbol{z} \\ \lambda\mu^2 \boldsymbol{z} \\ \mu^3 \boldsymbol{z} \end{bmatrix}}_{\boldsymbol{S}_\lambda\, \boldsymbol{v}|_{(j)}}$$

one column/solutions

Unifying block Macaulay matrix framework

	one variable	**multiple variables**
one row	**Case I** scalar single-shift-invariance Toeplitz matrix univariate polynomials	**Case II** scalar multi-shift-invariance Macaulay matrix multivariate polynomials
multiple rows	**Case III** block single-shift-invariance block Toeplitz matrix SEPs, GEPs, PEPs	**Case IV** block multi-shift-invariance block Macaulay matrix MEPs

Unifying block Macaulay matrix framework

	one variable	multiple variables
one row	**Case I** scalar single-shift-invariance Toeplitz matrix univariate polynomials	**Case II** scalar multi-shift-invariance Macaulay matrix multivariate polynomials
multiple rows	**Case III** block single-shift-invariance block Toeplitz matrix SEPs, GEPs, PEPs	**Case IV** block multi-shift-invariance block Macaulay matrix MEPs

Multidimensional realization problem

Assume only simple and affine solutions

- Solutions generate vectors in the null space of block Macaulay matrix $\boldsymbol{M}$

$$\boldsymbol{MV} = \boldsymbol{0}$$

- Nullity corresponds to the number of solutions m_a
- Null space has a **block multi-shift-invariant** structure
- Similar expositions exist in the other three cases

block multivariate Vandermonde basis matrix

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{z}|_{(1)} & \cdots & \boldsymbol{z}|_{(m_a)} \\ \hline (\lambda\boldsymbol{z})|_{(1)} & \cdots & (\lambda\boldsymbol{z})|_{(m_a)} \\ (\mu\boldsymbol{z})|_{(1)} & \cdots & (\mu\boldsymbol{z})|_{(m_a)} \\ \hline \left(\lambda^2\boldsymbol{z}\right)\Big|_{(1)} & \cdots & \left(\lambda^2\boldsymbol{z}\right)\Big|_{(m_a)} \\ (\lambda\mu\boldsymbol{z})|_{(1)} & \cdots & (\lambda\mu\boldsymbol{z})|_{(m_a)} \\ \left(\mu^2\boldsymbol{z}\right)\Big|_{(1)} & \cdots & \left(\mu^2\boldsymbol{z}\right)\Big|_{(m_a)} \\ \hline \left(\lambda^3\boldsymbol{z}\right)\Big|_{(1)} & \cdots & \left(\lambda^3\boldsymbol{z}\right)\Big|_{(m_a)} \\ \vdots & & \vdots \end{bmatrix}$$

Multidimensional realization theory

$$\left[\begin{array}{c} \boldsymbol{z} \\ \hline \lambda\boldsymbol{z} \\ \mu\boldsymbol{z} \\ \hline \lambda^2\boldsymbol{z} \\ \lambda\mu\boldsymbol{z} \\ \mu^2\boldsymbol{z} \\ \hline \lambda^3\boldsymbol{z} \\ \vdots \end{array}\right] \xrightarrow{\quad\lambda\quad} \left[\begin{array}{c} \boldsymbol{z} \\ \hline \lambda\boldsymbol{z} \\ \mu\boldsymbol{z} \\ \hline \lambda^2\boldsymbol{z} \\ \lambda\mu\boldsymbol{z} \\ \mu^2\boldsymbol{z} \\ \hline \lambda^3\boldsymbol{z} \\ \vdots \end{array}\right]$$

$$\boldsymbol{S}_1\boldsymbol{v}|_{(j)}\lambda = \boldsymbol{S}_\lambda\boldsymbol{v}|_{(j)}$$

Multidimensional realization theory

$$\left[\begin{array}{ccc} \boldsymbol{z}\big|_{(1)} & \cdots & \boldsymbol{z}\big|_{(m_a)} \\ \hline (\lambda\boldsymbol{z})\big|_{(1)} & \cdots & (\lambda\boldsymbol{z})\big|_{(m_a)} \\ (\mu\boldsymbol{z})\big|_{(1)} & \cdots & (\mu\boldsymbol{z})\big|_{(m_a)} \\ \hline \left(\lambda^2\boldsymbol{z}\right)\big|_{(1)} & \cdots & \left(\lambda^2\boldsymbol{z}\right)\big|_{(m_a)} \\ (\lambda\mu\boldsymbol{z})\big|_{(1)} & \cdots & (\lambda\mu\boldsymbol{z})\big|_{(m_a)} \\ \left(\mu^2\boldsymbol{z}\right)\big|_{(1)} & \cdots & \left(\mu^2\boldsymbol{z}\right)\big|_{(m_a)} \\ \hline \left(\lambda^3\boldsymbol{z}\right)\big|_{(1)} & \cdots & \left(\lambda^3\boldsymbol{z}\right)\big|_{(m_a)} \\ \vdots & & \vdots \end{array}\right] \xrightarrow{\quad\lambda\quad} \left[\begin{array}{ccc} \boldsymbol{z}\big|_{(1)} & \cdots & \boldsymbol{z}\big|_{(m_a)} \\ \hline (\lambda\boldsymbol{z})\big|_{(1)} & \cdots & (\lambda\boldsymbol{z})\big|_{(m_a)} \\ (\mu\boldsymbol{z})\big|_{(1)} & \cdots & (\mu\boldsymbol{z})\big|_{(m_a)} \\ \hline \left(\lambda^2\boldsymbol{z}\right)\big|_{(1)} & \cdots & \left(\lambda^2\boldsymbol{z}\right)\big|_{(m_a)} \\ (\lambda\mu\boldsymbol{z})\big|_{(1)} & \cdots & (\lambda\mu\boldsymbol{z})\big|_{(m_a)} \\ \left(\mu^2\boldsymbol{z}\right)\big|_{(1)} & \cdots & \left(\mu^2\boldsymbol{z}\right)\big|_{(m_a)} \\ \hline \left(\lambda^3\boldsymbol{z}\right)\big|_{(1)} & \cdots & \left(\lambda^3\boldsymbol{z}\right)\big|_{(m_a)} \\ \vdots & & \vdots \end{array}\right]$$

$$\boldsymbol{S}_1\boldsymbol{V}\underbrace{\begin{bmatrix} \lambda\big|_{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda\big|_{(m_a)} \end{bmatrix}}_{\boldsymbol{D}_\lambda} = \boldsymbol{S}_\lambda\boldsymbol{V}$$

Multidimensional realization theory

numerical basis for the null space

$$\boldsymbol{S}_1 \boldsymbol{V} \boldsymbol{D}_\lambda = \boldsymbol{S}_\lambda \boldsymbol{V}$$

for example, calculated via the SVD

- Solutions are not known in advance
- Consider a **numerical basis for the null space** $\boldsymbol{Z}$

$$\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{T}$$

non-singular matrix $\boldsymbol{T}$

- This results in

$$(\boldsymbol{S}_1 \boldsymbol{Z})\, \boldsymbol{T} \boldsymbol{D}_\lambda = (\boldsymbol{S}_\lambda \boldsymbol{Z})\, \boldsymbol{T}$$

Multidimensional realization theory

other shift functions

- It is **possible to shift with any polynomial** in the eigenvalues – for example with $g(\lambda, \mu) = 3\lambda + 2\mu^3$

$$(\boldsymbol{S}_1 \boldsymbol{Z}) \boldsymbol{T} \underbrace{\begin{bmatrix} g(\lambda,\mu)|_{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g(\lambda,\mu)|_{(m_a)} \end{bmatrix}}_{\boldsymbol{D}_g} = (\boldsymbol{S}_g \boldsymbol{Z}) \boldsymbol{T}$$

- This leads to the same eigenvectors $\boldsymbol{T}$

(Vermeersch and De Moor, 2023)

Multidimensional realization theory

standard eigenvalue problem

Realization theory for any shift polynomial $g(\lambda, \mu)$:

$$(\boldsymbol{S}_1 \boldsymbol{Z}) \boldsymbol{T} \boldsymbol{D}_g = (\boldsymbol{S}_g \boldsymbol{Z}) \boldsymbol{T},$$

where $\boldsymbol{S}_1$ and $\boldsymbol{S}_g$ select (block) rows from $\boldsymbol{Z}$

- Generalized eigenvalue problem, with $\boldsymbol{T}$ the matrix of eigenvectors
- We can rewrite this as a **standard eigenvalue problem**

$$\boldsymbol{T} \boldsymbol{D}_g \boldsymbol{T}^{-1} = (\boldsymbol{S}_1 \boldsymbol{Z})^{\dagger} (\boldsymbol{S}_g \boldsymbol{Z})$$

Multiplicity and solutions at infinity

- **Multiple solutions** lead to a confluent block multivariate Vandermonde basis matrix and the Jordan normal form, but we can avoid this via multiple Schur decompositions
- **Solutions at infinity** can be deflated from the numerical basis matrix via a column compression

Outline

Prediction error methods

$$\min_{\boldsymbol{\theta}} \sum_{k=0}^{N} \left(\hat{y}_k\left(\boldsymbol{\theta}, k-1\right) - y_k\right)$$

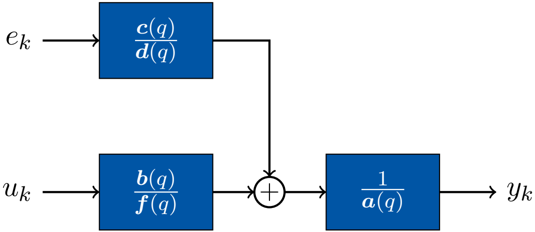
$$\text{subject to } \boldsymbol{a}\left(q\right) y_k = \frac{\boldsymbol{b}\left(q\right)}{\boldsymbol{f}\left(q\right)} u_k + \frac{\boldsymbol{c}\left(q\right)}{\boldsymbol{d}\left(q\right)} e_k$$

$\downarrow$

multivariate polynomial optimization problem

$\downarrow$

system of multivariate polynomial equations



(Batselier et al., 2012)

Vibration analysis

$$\boldsymbol{M}\frac{d^2q(t)}{dt^2} + \boldsymbol{C}\frac{dq(t)}{dt} + \boldsymbol{K}q(t) = f(t)$$

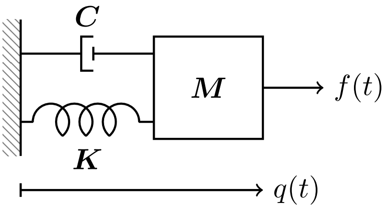
$$\downarrow$$

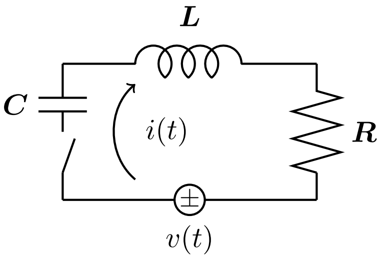
$$\left(\boldsymbol{K} + \boldsymbol{C}\lambda + \boldsymbol{M}\lambda^2\right)\boldsymbol{z} = \boldsymbol{0}$$

$$\boldsymbol{L}\frac{d^2i(t)}{dt^2} + \boldsymbol{R}\frac{di(t)}{dt} + \frac{1}{\boldsymbol{C}}i(t) = \frac{dv(t)}{dt}$$

$$\downarrow$$

$$\left(\boldsymbol{C}^{-1} + \boldsymbol{R}\lambda + \boldsymbol{L}\lambda^2\right)\boldsymbol{z} = \boldsymbol{0}$$





(Tisseur and Meerbergen, 2001)

Partial differential equations

Three-dimensional Helmholtz equation in parabolic cilinder coordinates (μ, ν, z):

$$\nabla^2 u = -\omega u$$

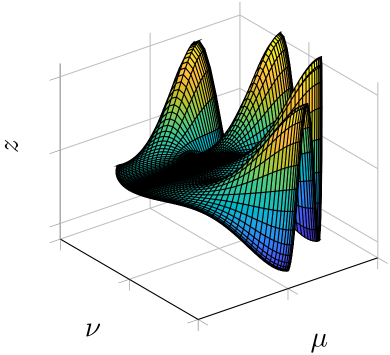
$$\begin{cases} \ddot{M}(\mu) - \left(\alpha + \beta\mu^2\right) M(\mu) = 0 \\ \ddot{N}(\nu) + \left(\alpha - \beta\nu^2\right) N(\nu) = 0 \\ \ddot{Z}(z) + (\omega + \beta)\, Z(z) = 0 \end{cases}$$

$$\downarrow$$

$$\begin{cases} (\boldsymbol{A}_1 + \boldsymbol{B}_1\alpha + \boldsymbol{C}_1\beta + \boldsymbol{D}_1\omega)\, \boldsymbol{x} = \boldsymbol{0} \\ (\boldsymbol{A}_2 + \boldsymbol{B}_2\alpha + \boldsymbol{C}_2\beta + \boldsymbol{D}_2\omega)\, \boldsymbol{y} = \boldsymbol{0} \\ (\boldsymbol{A}_3 + \boldsymbol{B}_3\alpha + \boldsymbol{C}_3\beta + \boldsymbol{D}_3\omega)\, \boldsymbol{z} = \boldsymbol{0} \end{cases}$$

$$\downarrow$$

$$\left(\begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \\ \boldsymbol{\Delta}_3 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Delta}_0 \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \alpha - \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\Delta}_0 \\ \boldsymbol{0} \end{bmatrix} \beta - \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{\Delta}_0 \end{bmatrix} \omega \right) \boldsymbol{z} = \boldsymbol{0}$$

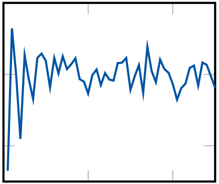


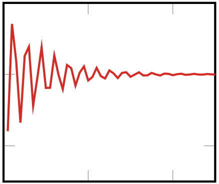
One of the modes of the solutions

(Plestenjak et al., 2015; Vermeersch and De Moor, 2022b)

Least-squares realization problem

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}}_{\text{given}} \longrightarrow \underbrace{\begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_N \end{bmatrix}}_{\text{unknown}} \qquad \text{such that } \hat{\boldsymbol{y}}_k = \boldsymbol{C}\boldsymbol{A}^k\boldsymbol{x}_0$$



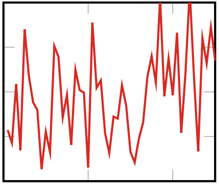


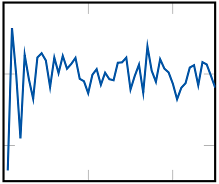
$$\min \|\boldsymbol{y} - \hat{\boldsymbol{y}}\|_2^2$$
$$\text{subject to } \boldsymbol{T}_\alpha \hat{\boldsymbol{y}} = \boldsymbol{0}$$

(De Moor, 2019, 2020)

ARMA model identification problem

$$\underbrace{\begin{bmatrix} e_1 \\ \vdots \\ e_M \end{bmatrix}}_{\text{unknown, latent input}} \longrightarrow \boxed{\frac{\boldsymbol{c}(q)}{\boldsymbol{a}(q)}} \longrightarrow \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}}_{\text{known output}}$$





$$\text{such that } \sum_{i=0}^{n_a} \alpha_i y_{k-i} = \sum_{i=0}^{n_c} \gamma_i e_{k-i}$$

$$\min \|\boldsymbol{e}\|_2^2$$

$$\text{subject to } \boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$$

(Vermeersch and De Moor, 2019)

Multiparameter eigenvalue problem at the core

for example, the ARMA$(1, 1)$ model

$$\boldsymbol{y} \in \mathbb{R}^N$$

$$\downarrow$$

$$\min \|\boldsymbol{e}\|_2^2$$

$$\text{subject to } \boldsymbol{T}_\alpha \boldsymbol{y} = \boldsymbol{T}_\gamma \boldsymbol{e}$$

$$\downarrow$$

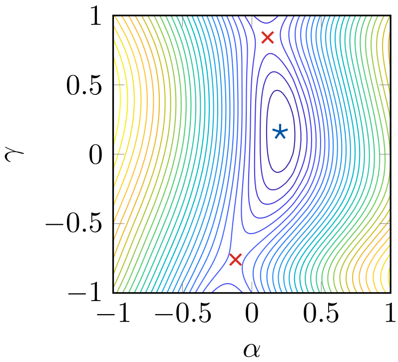
$$\left(\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\alpha + \boldsymbol{A}_{01}\gamma + \boldsymbol{A}_{02}\gamma^2\right)\boldsymbol{z} = \boldsymbol{0}$$

$$\downarrow$$

block Macaulay matrix and shift-invariance

$$\downarrow$$

parameters α and γ



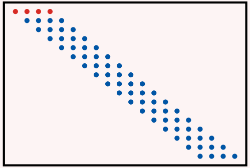
Contour plot of the cost function with one minimum ($\star$) and two saddle points ($\times$)

(Vermeersch and De Moor, 2019, 2022b)

Outline

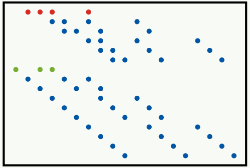
Conclusion

Case I: scalar single-shift-invariance



Toeplitz matrix

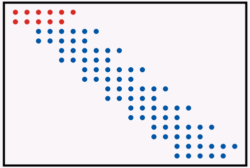
Case II: scalar multi-shift-invariance



Macaulay matrix

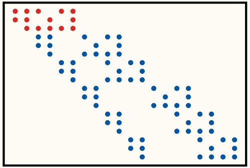
We can solve various system identification problems via the shift-invariant null space of a structured matrix

block Toeplitz matrix



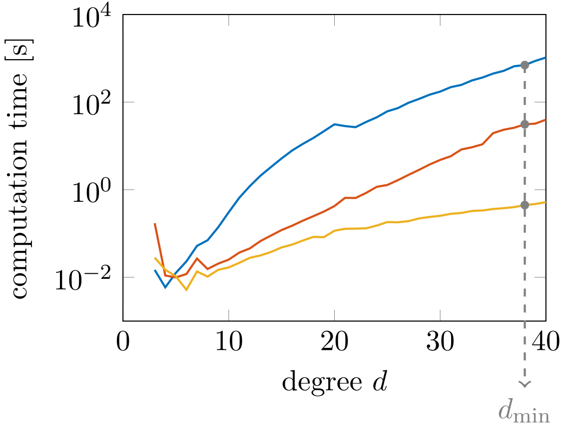
Case III: block single-shift-invariance

block Macaulay matrix



Case IV: block multi-shift-invariance

Future work



Comparison of the standard (—), recursive (—), and sparse (—) approach

(Vermeersch and De Moor, 2022a, 2023)

Any questions?

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