Using the (Block) Macaulay Matrix in the Chebyshev Polynomial Basis*

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Poster for the 31st ERNSI Workshop in System Identification

September 25, 2023

About the Macaulay matrix approach

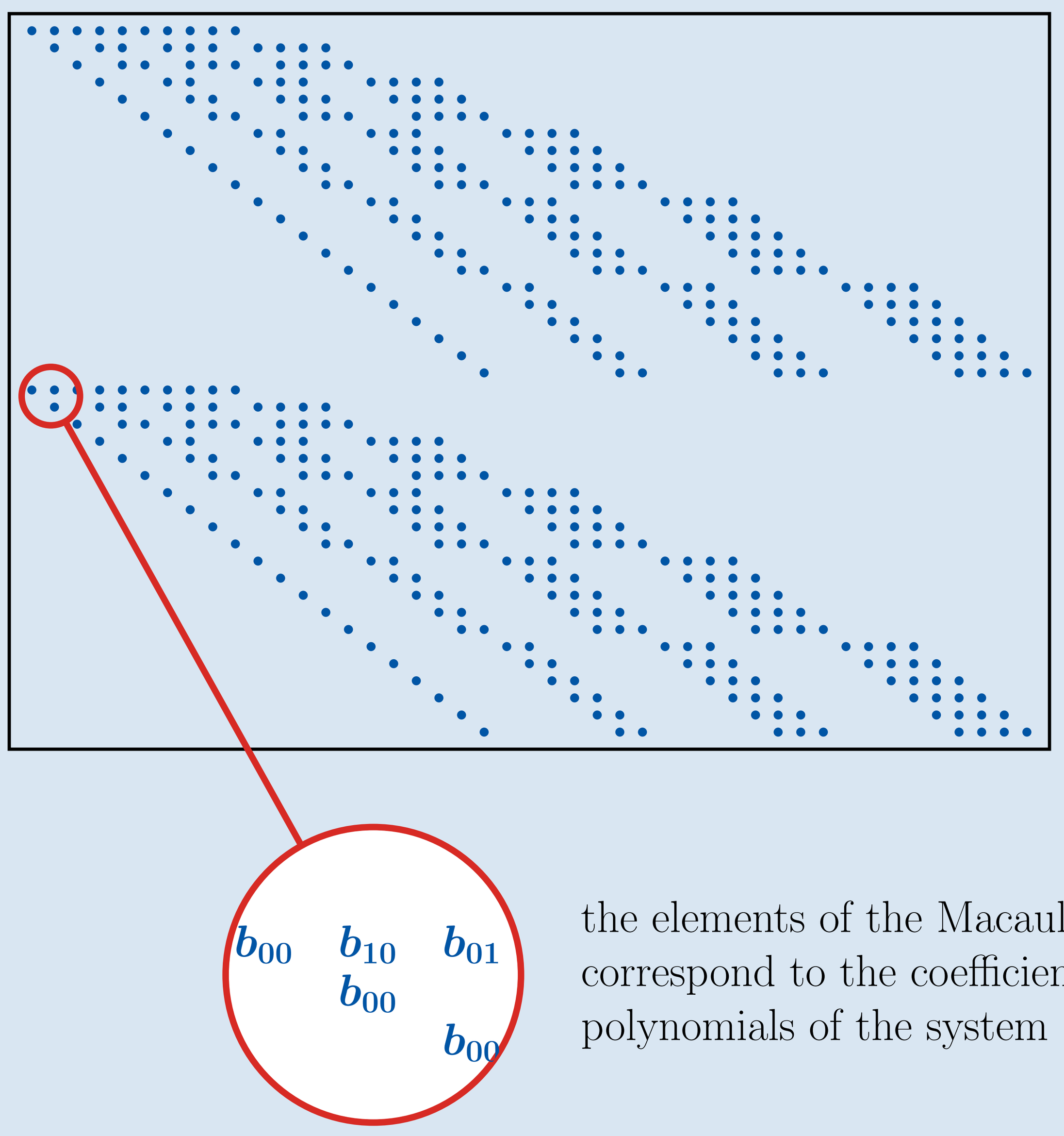
Consider a **system of multivariate polynomial equations**, for example, with $s = 2$ equations in $n = 2$ variables,

$$\begin{cases} p_1(\boldsymbol{x}) = a_{00} + a_{10}x_1 + a_{01}x_2 + \cdots + a_{0d_1}x_2^{d_1} = 0, \\ p_2(\boldsymbol{x}) = b_{00} + b_{10}x_1 + b_{01}x_2 + \cdots + b_{0d_2}x_2^{d_2} = 0, \end{cases}$$

which is given in the **standard monomial basis**. The basis polynomials $\varphi_{\boldsymbol{\alpha}}(\boldsymbol{x})$ are powers of the variables: $\varphi_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

↓

The Macaulay matrix is constructed from these polynomials:



$$\begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \\ x_1^3 \\ \vdots \\ x_2^d \end{bmatrix} = \boldsymbol{0}, \quad \text{(vector } \boldsymbol{v}\text{)}$$

the elements of the Macaulay matrix correspond to the coefficients of the polynomials of the system

Every solution $\boldsymbol{x}|_{(j)}$ of the system corresponds to one vector, $\boldsymbol{v}|_{(j)}$, in the basis matrix $\boldsymbol{V}$ of the right null space of that Macaulay matrix.

↓

In the **(backward) shift-invariant structure** of the right null space lies the key to finding the unknown solutions of the system:

$$\begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} x_1 = \begin{bmatrix} x_1 \\ x_1^2 \\ x_1x_2 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{S}_1\boldsymbol{V}\boldsymbol{D}_g = \boldsymbol{S}_g\boldsymbol{V}$$

The matrix $\boldsymbol{V}$ is not known in advance, because it consists out of $\boldsymbol{x}|_{(j)}$.

↓ numerical basis matrix $\boldsymbol{Z}$ with $\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{T}$

Eigenvalue problems, for $g(\boldsymbol{x}) = x_i$, give the solutions of the system:

$$(\boldsymbol{S}_1\boldsymbol{Z})\,\boldsymbol{T}\boldsymbol{D}_g = (\boldsymbol{S}_g\boldsymbol{Z})\,\boldsymbol{T}.$$

Of course, a lot of details are not shown in this short summary [3]!

Rectangular multiparameter eigenvalue problems

It is also possible to build the **block Macaulay matrix** from the coefficient matrices of a **rectangular multiparameter eigenvalue problem**.

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{z} = (\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\lambda_1 + \boldsymbol{A}_{01}\lambda_2)\,\boldsymbol{z} = \boldsymbol{0}$$

↓

$$\begin{bmatrix} \boldsymbol{A}_{00} & \boldsymbol{A}_{10} & \boldsymbol{A}_{01} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots \\ \boldsymbol{0} & \boldsymbol{A}_{00} & \boldsymbol{0} & \boldsymbol{A}_{10} & \boldsymbol{A}_{01} & \boldsymbol{0} & \cdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_{00} & \boldsymbol{0} & \boldsymbol{A}_{10} & \boldsymbol{A}_{01} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \lambda_1\boldsymbol{z} \\ \lambda_2\boldsymbol{z} \\ \lambda_1^2\boldsymbol{z} \\ \vdots \end{bmatrix} = \boldsymbol{0}$$

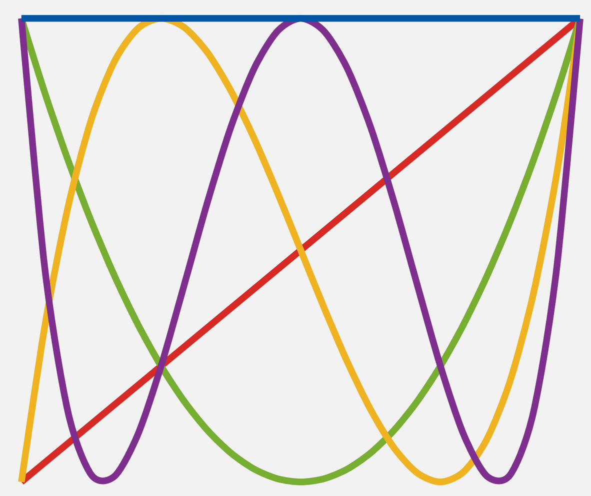
The above-mentioned approach can be extended to solve this problem [4].

Systems in the Chebyshev polynomial basis

A system of multivariate polynomial equations can also be expanded in a different polynomial basis. For example,

$$\begin{cases} p_1(\boldsymbol{x}) = \tilde{a}_{00}t_{00}(\boldsymbol{x}) + \tilde{a}_{10}t_{10}(\boldsymbol{x}) + \tilde{a}_{01}t_{01}(\boldsymbol{x}) + \cdots + \tilde{a}_{0d_1}t_{0d_1}(\boldsymbol{x}) = 0, \\ p_2(\boldsymbol{x}) = \tilde{b}_{00}t_{00}(\boldsymbol{x}) + \tilde{b}_{10}t_{10}(\boldsymbol{x}) + \tilde{b}_{01}t_{01}(\boldsymbol{x}) + \cdots + \tilde{b}_{0d_2}t_{0d_2}(\boldsymbol{x}) = 0, \end{cases}$$

is given in the **Chebyshev polynomial basis**. The basis polynomials $\varphi_{\boldsymbol{\beta}}(\boldsymbol{x}) = t_{\boldsymbol{\beta}}(\boldsymbol{x})$ are multivariate products of Chebyshev polynomials.

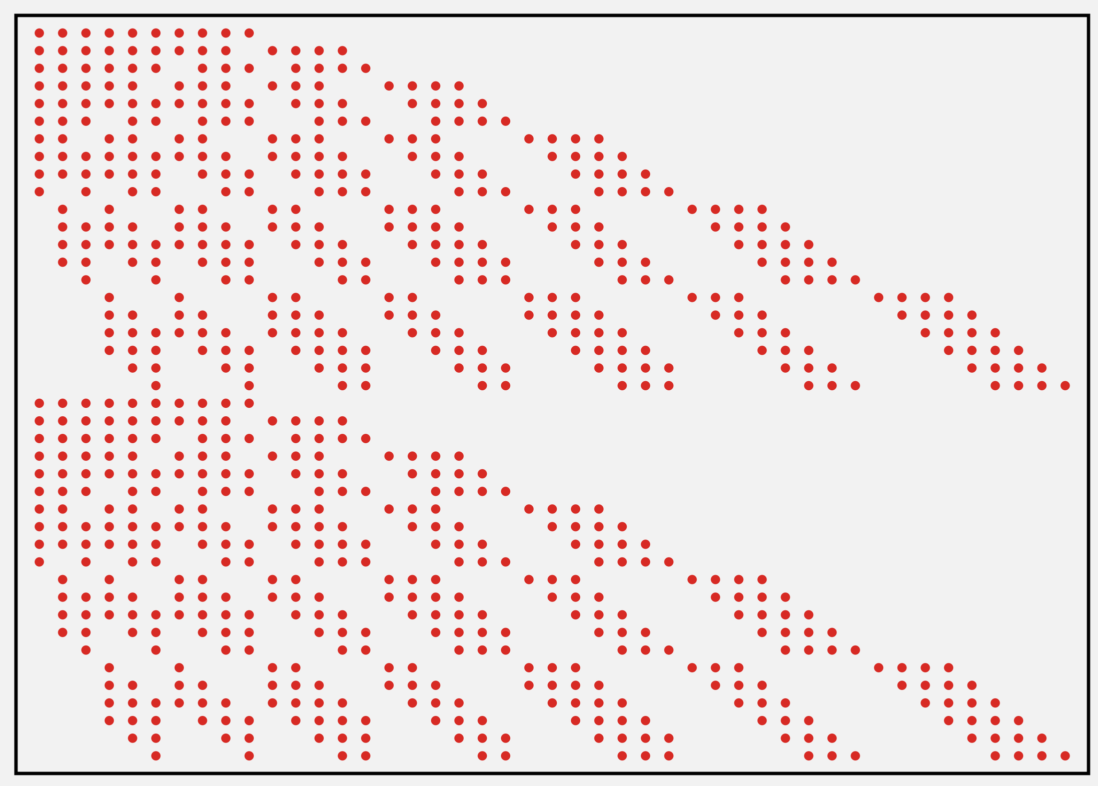


- $t_0(x) = 1$
- $t_1(x) = x$
- $t_2(x) = 2x^2 - 1$
- $t_3(x) = 4x^3 - 3x$
- $t_4(x) = 8x^4 - 8x^2 + 1$

Good numerical properties [2]!

Computational advantages

The change of basis polynomials results in a different Macaulay matrix:



↓

"Can the structure result in computational advantages?"

- Sparsity of the Macaulay matrix reduces, but the link between the FFT and Chebyshev polynomials may be useful.
- Relation between this Macaulay matrix and a Cauchy matrix for bivariate systems still exists, resulting in a faster approach to compute $\boldsymbol{Z}$ [1].

Numerical advantages

The (backward) shift-invariant structure of the right null space changes:

$$\begin{bmatrix} t_{00}(\boldsymbol{x}) \\ t_{10}(\boldsymbol{x}) \\ t_{01}(\boldsymbol{x}) \end{bmatrix} t_{10}(\boldsymbol{x}) = \frac{1}{2}\begin{bmatrix} t_{10}(\boldsymbol{x}) \\ t_{20}(\boldsymbol{x}) \\ t_{11}(\boldsymbol{x}) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} t_{10}(\boldsymbol{x}) \\ t_{00}(\boldsymbol{x}) \\ t_{11}(\boldsymbol{x}) \end{bmatrix}$$

Eigenvalue problems, for $g(\boldsymbol{x}) = x_i$, yield again the solutions of the system.

↓

"Are there numerical advantages to using the adapted Macaulay matrix approach?"

- Preliminary results suggest that the Chebyshev polynomial basis behaves numerical better for solutions in the real hyperplane.
- Basis transformation may be very ill-conditioned!

Adapted block Macaulay matrix

It is also possible to express and solve rectangular multiparameter eigenvalue problems in the Chebyshev polynomial basis, for example,

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \left(\tilde{\boldsymbol{A}}_{00}t_{00}(\boldsymbol{\lambda}) + \tilde{\boldsymbol{A}}_{10}t_{10}(\boldsymbol{\lambda}) + \tilde{\boldsymbol{A}}_{12}t_{12}(\boldsymbol{\lambda})\right)\boldsymbol{z} = \boldsymbol{0}.$$

* This poster considers results from the master thesis research of Quinten Peeters.

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‡ The work of Christof Vermeersch was supported in part by FWO Strategic Basic Research fellowship under grant SB/1SA1319N.

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