Multivariate Polynomial Optimization in Complex Variables Is a (Rectangular) Multiparameter Eigenvalue Problem

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Problem statement

We deal with real-valued multivariate polynomial cost functions $f(\boldsymbol{z}) \in \mathcal{P}^n$:

$$f : \mathbb{C}^n \to \mathbb{R} : \boldsymbol{z} \mapsto f(\boldsymbol{z}).$$

The problem that we consider is the unconstrained minimization problem, i.e.,

$$\min_{\boldsymbol{z}} f(\boldsymbol{z}).$$

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A prototype example is the nonlinear least-squares problem,

$$\min_{\boldsymbol{z}} \|\boldsymbol{\mathcal{F}}(\boldsymbol{z})\|_{\mathsf{F}}^2,$$

with the matrix polynomial $\boldsymbol{\mathcal{F}}(\boldsymbol{z})$,

$$\boldsymbol{\mathcal{F}} : \mathbb{C}^n \to \mathbb{C}^{m_1 \times m_2} : \boldsymbol{z} \mapsto \boldsymbol{\mathcal{F}}(\boldsymbol{z}).$$

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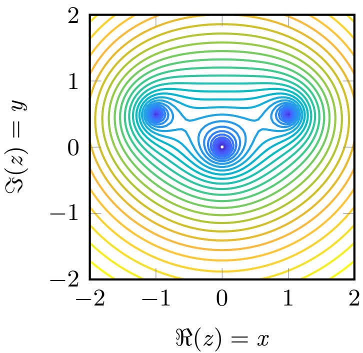
Disclaimer: this is a didactic exposition of a novel alternative approach!

Motivational example

We use the following motivational example to explain our approach:

$$\min_z \left\| z(z-0.5\mathrm{i})^2 - z \right\|_2^2.$$



Contour lines of the real-valued polynomial cost function $f(z)$ of the motivational example.

Complex differentiability

Consider a multivariate complex-valued function $f(\boldsymbol{z})$:

$$f : \mathbb{C}^n \to \mathbb{C} : \boldsymbol{z} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y} \mapsto f(\boldsymbol{z}) = u(\boldsymbol{x}, \boldsymbol{y}) + \mathrm{i}v(\boldsymbol{x}, \boldsymbol{y})$$

- $f(\boldsymbol{z})$ is not necessarily polynomial
- $u(\boldsymbol{x}, \boldsymbol{y})$ and $v(\boldsymbol{x}, \boldsymbol{y})$ are real-valued functions

The complex-valued function is said to be **differentiable** at a point $\boldsymbol{z}_0 \in \mathbb{C}^n$ if

$$\lim_{\Delta\boldsymbol{z}\to\boldsymbol{0}} \frac{f(\boldsymbol{z}_0 + \Delta\boldsymbol{z}) - f(\boldsymbol{z}_0)}{\Delta\boldsymbol{z}}$$

exists. This requirement is formalized in the **Cauchy–Riemann conditions**!

(Sorber et al., 2012; Adali and Schreier, 2014)

Wirtinger derivatives

Wirtinger derivatives are the partial derivatives with respect to z or $\bar{z}$.

cogradient operator

$$\frac{\partial f(z,\bar{z})}{\partial z} = \frac{\partial f(z)}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f(z)}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f(z)}{\partial x} - \mathrm{i}\frac{\partial f(z)}{\partial y}\right)$$

conjugate cogradient operator

$$\frac{\partial f(z,\bar{z})}{\partial \bar{z}} = \frac{\partial f(z)}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f(z)}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f(z)}{\partial x} + \mathrm{i}\frac{\partial f(z)}{\partial y}\right)$$

We define the **complex gradient operator** $\nabla(\cdot)$ as

$$\nabla(\cdot) = \left(\frac{\partial(\cdot)}{\partial z}, \frac{\partial(\cdot)}{\partial \bar{z}}\right).$$

(Poincaré, 1899; Wirtinger, 1927; Brandwood, 1983)

Outline

Outline

First-order necessary conditions

Wirtinger calculus provides an elegant alternative framework to compute the first-order necessary conditions for optimality:

$$\min_{\boldsymbol{z},\bar{\boldsymbol{z}}} f(\boldsymbol{z},\bar{\boldsymbol{z}}) \Rightarrow \begin{cases} p_i(\boldsymbol{z},\bar{\boldsymbol{z}}) = \dfrac{\partial f(\boldsymbol{z},\bar{\boldsymbol{z}})}{\partial z_i} = 0, & \text{for } i = 1,\dots,n, \\ p_i(\boldsymbol{z},\bar{\boldsymbol{z}}) = \dfrac{\partial f(\boldsymbol{z},\bar{\boldsymbol{z}})}{\partial \bar{z}_{i-n}} = 0, & \text{for } i = n+1,\dots,2n. \end{cases}$$

The common roots $(\boldsymbol{z}_0, \bar{\boldsymbol{z}}_0)$ of this square system of $2n$ multivariate polynomial equations in $\boldsymbol{z}$ and $\bar{\boldsymbol{z}}$ correspond to the stationary points:

$$\mathcal{V}_{\mathbb{C}} = \{\boldsymbol{z}_0 \in \mathbb{C}^n : p_i(\boldsymbol{z}_0, \bar{\boldsymbol{z}}_0) = 0, \forall i = 1,\dots,2n\}.$$

Macaulay matrix

$$p_i(\boldsymbol{z}, \bar{\boldsymbol{z}}) = \sum_{\{\boldsymbol{\alpha}\}} p_i^{(\boldsymbol{\alpha})}(\boldsymbol{z})\bar{\boldsymbol{z}}^{\boldsymbol{\alpha}}, \quad \forall i = 1, \ldots, 2n$$

$$\Downarrow$$

$$p_1(\boldsymbol{z}, \bar{\boldsymbol{z}}) = p^{(00)} + p^{(10)}\bar{z}_1 + p^{(01)}\bar{z}_2 + p^{(20)}\bar{z}_1^2 + p^{(11)}\bar{z}_1\bar{z}_2 + p^{(02)}\bar{z}_2^2 = 0$$

$$\bar{z}_1 p_1(\boldsymbol{z}, \bar{\boldsymbol{z}}) = p^{(00)}\bar{z}_1 + p^{(10)}\bar{z}_1^2 + p^{(01)}\bar{z}_1\bar{z}_2 + p^{(20)}\bar{z}_1^3 + p^{(11)}\bar{z}_1^2\bar{z}_2 + p^{(02)}\bar{z}_1\bar{z}_2^2 = 0$$

$$\bar{z}_2 p_1(\boldsymbol{z}, \bar{\boldsymbol{z}}) = p^{(00)}\bar{z}_2 + p^{(10)}\bar{z}_1\bar{z}_2 + p^{(01)}\bar{z}_2^2 + p^{(20)}\bar{z}_1^2\bar{z}_2 + p^{(11)}\bar{z}_1\bar{z}_2^2 + p^{(02)}\bar{z}_2^3 = 0$$

$$\Downarrow$$

$$\boldsymbol{\mathcal{M}}(\boldsymbol{z}) = \begin{array}{r} \\ p_1(\boldsymbol{z},\bar{\boldsymbol{z}}) \\ \bar{z}_1 p_1(\boldsymbol{z},\bar{\boldsymbol{z}}) \\ \bar{z}_2 p_1(\boldsymbol{z},\bar{\boldsymbol{z}}) \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & \bar{z}_1 & \bar{z}_2 & \bar{z}_1^2 & \bar{z}_1\bar{z}_2 & \bar{z}_2^2 & \bar{z}_1^3 & \bar{z}_1^2\bar{z}_2 & \bar{z}_1\bar{z}_2^2 & \bar{z}_2^3 \end{array} \\ \begin{bmatrix} p^{(00)} & p^{(10)} & p^{(01)} & p^{(20)} & p^{(11)} & p^{(02)} & 0 & 0 & 0 & 0 \\ 0 & p^{(00)} & 0 & p^{(10)} & p^{(01)} & 0 & p^{(20)} & p^{(11)} & p^{(02)} & 0 \\ 0 & 0 & p^{(00)} & 0 & p^{(10)} & p^{(01)} & 0 & p^{(20)} & p^{(11)} & p^{(02)} \end{bmatrix} \end{array}$$

(Vermeersch and De Moor, 2021)

Macaulay matrix

For example, in the case $n = 2$,

$$\begin{aligned} p_1(\boldsymbol{z}, \bar{\boldsymbol{z}}) &= 2 + z_2 + 3z_1 z_2 \bar{z}_1 + z_1^2 \bar{z}_2 \\ &= p_1^{(00)}(\boldsymbol{z}) + p_1^{(10)}(\boldsymbol{z})\bar{z}_1 + p_1^{(01)}(\boldsymbol{z})\bar{z}_2 \end{aligned}$$

has a degree in $\bar{\boldsymbol{z}}$ equal to 1 and the corresponding polynomial coefficients $p^{(\boldsymbol{\alpha})}(\boldsymbol{z})$ are

$$p_1^{(00)}(\boldsymbol{z}) = 2 + z_2, \quad p_1^{(10)}(\boldsymbol{z}) = 3z_1 z_2, \quad \text{and} \quad p_1^{(01)}(\boldsymbol{z}) = z_1^2.$$

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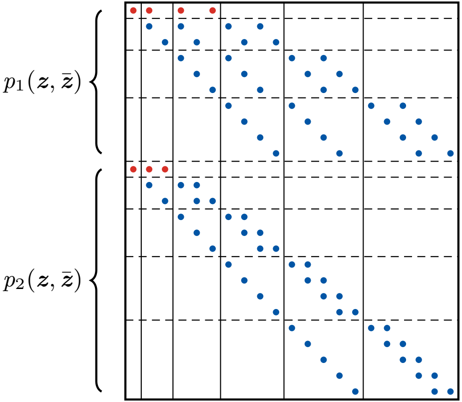
$$p_1^{(00)}(\boldsymbol{z}) = 2 + z_2, \quad p_1^{(10)}(\boldsymbol{z}) = 3z_1 z_2, \quad \text{and} \quad p_1^{(01)}(\boldsymbol{z}) = z_1^2.$$

The corresponding Macaulay matrix (degree $d = 2$) is

$$\boldsymbol{\mathcal{M}}(\boldsymbol{z}) = \begin{bmatrix} 2 + z_2 & 3z_1 z_2 & z_1^2 & 0 & 0 & 0 \\ 0 & 2 + z_2 & 0 & 3z_1 z_2 & z_1^2 & 0 \\ 0 & 0 & 2 + z_2 & 0 & 3z_1 z_2 & z_1^2 \end{bmatrix}.$$

Macaulay matrix



The Macaulay matrix is very sparse and structured. Every element (• or •) is a polynomial $p_i^{(\boldsymbol{\alpha})}(\boldsymbol{z})$ in $\boldsymbol{z}$.

Motivational example

$$\min_z \left\| z(z-0.5j)^2 - z \right\|_2^2$$

$$\Downarrow$$

$$\begin{cases} p_1(z,\bar{z}) = 3z^2\bar{z}^3 + 3\mathrm{i}z^2\bar{z}^2 - 3.75z^2\bar{z} - 2\mathrm{i}z\bar{z}^3 + 2z\bar{z}^2 + 2.5\mathrm{i}z\bar{z} - 1.25\bar{z}^3 \\ \qquad - 1.25\mathrm{i}\bar{z}^2 + 1.5625\bar{z} = 0 \\ p_2(z,\bar{z}) = 3z^3\bar{z}^2 - 3\mathrm{i}z^2\bar{z}^2 - 3.75z\bar{z}^2 + 2\mathrm{i}z^3\bar{z} + 2z^2\bar{z} - 2.5\mathrm{i}z\bar{z} - 1.25z^3 \\ \qquad + 1.25\mathrm{i}z^2 + 1.5625z = 0 \end{cases}$$

$$\Downarrow$$

$$\boldsymbol{\mathcal{M}}(z) = \left[\begin{array}{ccccc} p_1^{(0)}(z) & p_1^{(1)}(z) & p_1^{(2)}(z) & p_1^{(3)}(z) & 0 \\ 0 & p_1^{(0)}(z) & p_1^{(1)}(z) & p_1^{(2)}(z) & p_1^{(3)}(z) \\ \hline p_2^{(0)}(z) & p_2^{(1)}(z) & p_2^{(2)}(z) & 0 & 0 \\ 0 & p_2^{(0)}(z) & p_2^{(1)}(z) & p_2^{(2)}(z) & 0 \\ 0 & 0 & p_2^{(0)}(z) & p_2^{(1)}(z) & p_2^{(2)}(z) \end{array}\right]$$

Multiparameter eigenvalue problem

$$\begin{cases} p_1(\boldsymbol{z}, \bar{\boldsymbol{z}}) = 0 \\ \vdots \\ p_{2n}(\boldsymbol{z}, \bar{\boldsymbol{z}}) = 0 \end{cases}$$

$$\Downarrow$$

$$\boldsymbol{\mathcal{M}}(\boldsymbol{z}) \begin{bmatrix} 1 \\ \bar{z}_1 \\ \vdots \\ \bar{z}_n^d \end{bmatrix} = \mathbf{0}$$

For example, in the case $n = 2$,

$$\begin{aligned} p_1(\boldsymbol{z}, \bar{\boldsymbol{z}}) &= 2 + z_2 + 3z_1 z_2 \bar{z}_1 + z_1^2 \bar{z}_2 \\ &= p_1^{(00)}(\boldsymbol{z}) + p_1^{(10)}(\boldsymbol{z})\bar{z}_1 + p_1^{(01)}(\boldsymbol{z})\bar{z}_2 \end{aligned}$$

leads to

$$\underbrace{\begin{bmatrix} 2+z_2 & 3z_1z_2 & z_1^2 & 0 & 0 & 0 \\ 0 & 2+z_2 & 0 & 3z_1z_2 & z_1^2 & 0 \\ 0 & 0 & 2+z_2 & 0 & 3z_1z_2 & z_1^2 \end{bmatrix}}_{\boldsymbol{\mathcal{M}}(\boldsymbol{z})} \underbrace{\begin{bmatrix} 1 \\ \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_1^2 \\ \bar{z}_1\bar{z}_2 \\ \bar{z}_2^2 \end{bmatrix}}_{\boldsymbol{q}} = \mathbf{0}.$$

Multiparameter eigenvalue problem

The **multiparameter eigenvalue problem (MEP)** $\mathcal{M}(\boldsymbol{\lambda})\boldsymbol{z} = \mathbf{0}$ consists in finding all n-tuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and corresponding vectors $\boldsymbol{z} \in \mathbb{C}^{l \times 1} \setminus \{\mathbf{0}\}$, so that

$$\mathcal{M}(\boldsymbol{\lambda})\boldsymbol{z} = \left(\sum_{\{\boldsymbol{\omega}\}} \boldsymbol{A}_{\boldsymbol{\omega}} \boldsymbol{\lambda}^{\boldsymbol{\omega}} \right) \boldsymbol{z} = \mathbf{0},$$

with $\|\boldsymbol{z}\|_2 = 1$.

- $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ indexes the monomials $\boldsymbol{\lambda}^{\boldsymbol{\omega}}$ and coefficient matrices $\boldsymbol{A}_{\boldsymbol{\omega}}$
- rectangular[†] coefficient matrices $\boldsymbol{A}_{\boldsymbol{\omega}} = \boldsymbol{A}_{(\omega_1, \ldots, \omega_n)} \in \mathbb{R}^{k \times l}$ with $k \geq l + n - 1$
- example: $\left(\boldsymbol{A}_{000} + \boldsymbol{A}_{200}\lambda_1^2 + \boldsymbol{A}_{013}\lambda_2\lambda_3^3\right)\boldsymbol{z} = \mathbf{0}$

[†]We consider only rectangular problems in this presentation, so we no longer mention the term "rectangular" explicitly. — (Vermeersch and De Moor, 2022)

Motivational example

$$\left[\begin{array}{ccccc} p_1^{(0)}(z) & p_1^{(1)}(z) & p_1^{(2)}(z) & p_1^{(3)}(z) & 0 \\ 0 & p_1^{(0)}(z) & p_1^{(1)}(z) & p_1^{(2)}(z) & p_1^{(3)}(z) \\ \hline p_2^{(0)}(z) & p_2^{(1)}(z) & p_2^{(2)}(z) & 0 & 0 \\ 0 & p_2^{(0)}(z) & p_2^{(1)}(z) & p_2^{(2)}(z) & 0 \\ 0 & 0 & p_2^{(0)}(z) & p_2^{(1)}(z) & p_2^{(2)}(z) \end{array}\right] \begin{bmatrix} 1 \\ \bar{z} \\ \bar{z}^2 \\ \bar{z}^3 \\ \bar{z}^4 \end{bmatrix} = \mathbf{0}$$

$$\Downarrow$$

$$\left(\boldsymbol{M}_0 + \boldsymbol{M}_1 z + \boldsymbol{M}_2 z^2 + \boldsymbol{M}_3 z^3\right)\boldsymbol{q} = \mathbf{0}$$

Motivational example

$$\left[\begin{array}{ccccc} p_1^{(0)}(z) & p_1^{(1)}(z) & p_1^{(2)}(z) & p_1^{(3)}(z) & 0 \\ 0 & p_1^{(0)}(z) & p_1^{(1)}(z) & p_1^{(2)}(z) & p_1^{(3)}(z) \\ \hline p_2^{(0)}(z) & p_2^{(1)}(z) & p_2^{(2)}(z) & 0 & 0 \\ 0 & p_2^{(0)}(z) & p_2^{(1)}(z) & p_2^{(2)}(z) & 0 \\ 0 & 0 & p_2^{(0)}(z) & p_2^{(1)}(z) & p_2^{(2)}(z) \end{array}\right] \begin{bmatrix} 1 \\ \bar{z} \\ \bar{z}^2 \\ \bar{z}^3 \\ \bar{z}^4 \end{bmatrix} = \mathbf{0}$$

$$\Downarrow$$

$$\left(\boldsymbol{M}_0 + \boldsymbol{M}_1 z + \boldsymbol{M}_2 z^2 + \boldsymbol{M}_3 z^3\right) \boldsymbol{q} = \mathbf{0}$$

$$\boldsymbol{M}_2 = \left[\begin{array}{ccccc} 0 & -3.75 & 3\mathrm{i} & 3 & 0 \\ 0 & 0 & -3.75 & 3\mathrm{i} & 3 \\ \hline 1.25\mathrm{i} & 2 & -3\mathrm{i} & 0 & 0 \\ 0 & 1.25\mathrm{i} & 2 & -3\mathrm{i} & 0 \\ 0 & 0 & 1.25\mathrm{i} & 2 & -3\mathrm{i} \end{array}\right]$$

$$\begin{aligned} p_1(z, \bar{z}) = {} & 3z^2\bar{z}^3 + 3\mathrm{i}z^2\bar{z}^2 - 3.75z^2\bar{z} \\ & - 2\mathrm{i}z\bar{z}^3 + 2z\bar{z}^2 + 2.5\mathrm{i}z\bar{z} \\ & - 1.25\bar{z}^3 - 1.25\mathrm{i}\bar{z}^2 + 1.5625\bar{z} \end{aligned}$$

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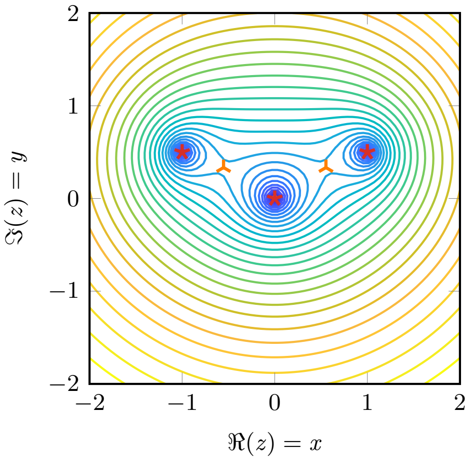
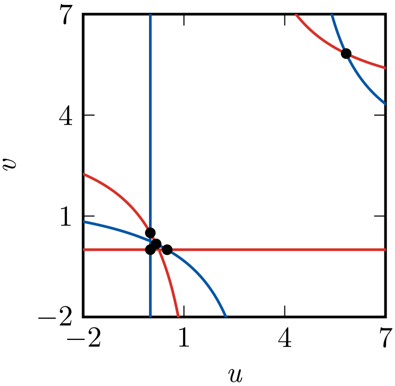
Motivational example



Contour lines of the real-valued polynomial cost function $f(z, \bar{z})$ of the motivational example: the optimization problem has three minimizers (⋆) and two saddle points (⅄).

Outline

Ghost solutions



The solution candidates (•) when the variables $\boldsymbol{z}$ and $\bar{\boldsymbol{z}}$ are considered as independent variables $\boldsymbol{u}$ and $\boldsymbol{v}$.

(Sorber et al., 2014)

We actually find the candidate solution set

$$\mathcal{V}_{\tilde{\mathbb{C}}} = \{(\boldsymbol{u}_0, \boldsymbol{v}_0) \in \mathbb{C}^{2n} : p_i(\boldsymbol{u}_0, \boldsymbol{v}_0) = 0, \forall i = 1, \ldots, 2n\}$$

instead of the desired solution set

$$\mathcal{V}_{\mathbb{C}} = \{\boldsymbol{z}_0 \in \mathbb{C}^n : p_i(\boldsymbol{z}_0, \bar{\boldsymbol{z}}_0) = 0, \forall i = 1, \ldots, 2n\}.$$

These "wrong" solutions are called **ghost solutions**!

Motivational example

Numerical values of the candidate solutions (u_0, v_0) of the motivational example.

u_0	v_0	classification
$1.0000 + 0.5000i$	$1.0000 - 0.5000i$	minimizer
$-1.0000 + 0.5000i$	$-1.0000 - 0.5000i$	minimizer
$0.0000 + 0.0000i$	$0.0000 + 0.0000i$	minimizer
$\pm 0.5528 + 0.3333i$	$\pm 0.5528 - 0.3333i$	saddle point
$\pm 1.0000 + 0.5000i$	$0.0000 + 0.0000i$	ghost solution
$\pm 1.0000 + 0.5000i$	$\mp 1.0000 - 0.5000i$	ghost solution
$0.0000 + 0.0000i$	$\pm 1.0000 - 0.5000i$	ghost solution
$\pm 0.5528 + 0.3333i$	$\mp 0.5528 - 0.3333i$	ghost solution

Outline

Complex matrix decomposition

We consider the problem where we try to fit a rank-1 matrix to a given complex 2×2 matrix $\boldsymbol{A} \in \mathbb{C}^{2\times 2}$:

$$\min_{\boldsymbol{z}} \left\| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} z_1^2 & z_1 z_2 \\ z_1 z_2 & z_2^2 \end{bmatrix} \right\|_{\mathsf{F}}^2,$$

which is an example of a nonlinear least-squares optimization problem.

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which is an example of a nonlinear least-squares optimization problem.

$$\begin{cases} p_1(\boldsymbol{z}, \bar{\boldsymbol{z}}) = -2\bar{a}_{11} z_1 + 2 z_1 \bar{z}_1^2 - (\bar{a}_{12} + \bar{a}_{21}) z_2 + 2 z_2 \bar{z}_1 \bar{z}_2 = 0, \\ p_2(\boldsymbol{z}, \bar{\boldsymbol{z}}) = -2\bar{a}_{22} z_2 + 2 z_2 \bar{z}_2^2 - (\bar{a}_{12} + \bar{a}_{21}) z_1 + 2 z_1 \bar{z}_1 \bar{z}_2 = 0, \\ p_3(\boldsymbol{z}, \bar{\boldsymbol{z}}) = -2 a_{11} \bar{z}_1 + 2 z_1^2 \bar{z}_1 - (a_{12} + a_{21}) \bar{z}_2 + 2 z_1 z_2 \bar{z}_2 = 0, \\ p_4(\boldsymbol{z}, \bar{\boldsymbol{z}}) = -2 a_{22} \bar{z}_2 + 2 z_2^2 \bar{z}_2 - (a_{12} + a_{21}) \bar{z}_1 + 2 z_1 z_2 \bar{z}_1 = 0, \end{cases}$$

Complex matrix decomposition

This can be solved by tackling

$$\left(\boldsymbol{M}_{00} + \boldsymbol{M}_{10} z_1 + \boldsymbol{M}_{01} z_2 + \boldsymbol{M}_{20} z_1^2 + \boldsymbol{M}_{11} z_1 z_2 + \boldsymbol{M}_{02} z_2^2\right) \boldsymbol{q} = \boldsymbol{0},$$

which we can solve, for example, via the block Macaulay matrix or Kronecker operator determinants.

(Vermeersch and De Moor, 2022)

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If we consider the given matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1\mathrm{i} \\ 1\mathrm{i} & -2 \end{bmatrix},$$

then we obtain nine stationary points after solving the MEP. The global minimizer is $(0.8507, 1.3764\mathrm{i})$, which also corresponds to the first triplet obtained via the complex singular value decomposition of $\boldsymbol{A}$.

(Vermeersch and De Moor, 2022)

Outline

Conclusions and Future Work

This reformulation provides **a novel, alternative approach** to multivariate polynomial optimization in complex variables and extends the existing univariate relation.

However, it invokes several **new research challenges**:

- What is the computational complexity of this approach?
- Are there specific conditions that need to be satisfied for the MEP to be solvable?
- Can we exploit the structure of the coefficient matrices of the MEP?
- Does this approach have better numerical properties than solving the system of multivariate polynomial equations directly?

Any questions?

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