Equivalence of State Representations for Hidden Markov Models

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***Abstract*— In this paper we consider the following problem for (quasi) hidden Markov models: given a minimal (quasi) hidden Markov model, what can be said about the set of all equivalent (quasi) hidden Markov models of the same order. A distinction is made between Mealy and Moore type of hidden Markov models. A complete solution is presented for the quasi HMM case. For quasi Mealy models, there exists already a description of the set of equivalent models. In this paper, we prove that for minimal quasi Moore models, the set of equivalent models consists of only one element (up to a permutation of the states). Finally, we present some initial results for the positive HMM case and show a motivating simulation example.**

I. INTRODUCTION

Hidden Markov models (HMMs) were introduced in the literature in the late 1950s [2]. Twenty years later, HMMs started to be used in engineering applications, such as speech processing, image processing and bioinformatics. Despite the success in applications, many theoretical questions remain unanswered until now. An example of an open theoretical problem is the *realization problem*: given string probabilities of all possible finite length strings, find all hidden Markov models that realize these string probabilities. The realization problem can be split up into three subproblems. The first subproblem is the realizability problem: derive conditions for string probabilities to be realizable by a hidden Markov model. In [9] almost necessary and sufficient conditions for the realizability of string probabilities are derived. The second subproblem is the realization problem itself: given realizable string probabilities, find a corresponding hidden Markov model. Partial solutions for this problem are given in [1], [8], [9]. The third subproblem concerns the question of finding all possible realizations which are equivalent to a given realization. For Gauss-Markov systems, where both the states and observations take values from continuous sets, this problem was already solved early [3]. However for hidden Markov models, to the best of our knowledge, not much is known about the equivalence problem.

The positive realization problem for hidden Markov models can be written as a nonnegative matrix factorization problem of a certain matrix containing the string probabilities. The equivalence problem for positive hidden Markov models is therefore related to the problem of finding *all* minimal nonnegative factorizations of a given nonnegative matrix [7]. This is a far from trivial problem, as there does not even exist a procedure to find the minimal inner dimension for which a positive factorization exists, nor a procedure to calculate an exact nonnegative factorization. At this moment only approximate nonnegative factorizations procedures exist [4], [6]. In this paper, we consider the equivalence problem for quasi hidden Markov models and give some initial results for the positive hidden Markov case.

In Section II we introduce Moore and Mealy type of hidden Markov models and their quasi forms. In Section III we first recall the description of the complete set of equivalent quasi Mealy models, secondly we give some initial results for the positive Mealy case and finally we perform a simulation example. In Section IV subsequently, we show that, under certain conditions, the class of equivalent (quasi) Moore hidden Markov models consists of only one element (up to a permutation of the states). In Section V we summarize the results and in Section VI finally, we draw some conclusions.

The following notation is used throughout the paper. If X is a matrix, then $X_{i:k,j:l}$ denotes the submatrix of X formed by the i-th to the k-th row and by the j-th to the l-th column of X. With X_{ij} we mean the i,j-th element of X, and with $X_{:,j}$ and $X_{i,:}$, we mean the j-th column and i-th row respectively.

II. MOORE AND MEALY TYPE OF HMM

Hidden Markov models (HMM) are used to model finite-valued output processes y defined on the time axis $\mathbb{N}$. Hidden Markov models assume the existence of an underlying finite-valued Markov process x, called the state process, on which the output process depends in a probabilistic manner. In this section, we introduce two different types of hidden Markov models: *Moore hidden Markov models* and *Mealy hidden Markov models*. We also introduce the so-called *quasi forms* of these two types of models. Finally, we discuss conversions between Moore and Mealy models.

A. Mealy type of HMM

A Mealy type of hidden Markov model assumes that the event of going to a certain state at time instant $t+1$ given the state at time instant t is dependent on the output symbol produced at time instant t. Consequently, a Mealy hidden Markov model is denoted by the quadruple $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ where:

- $\mathbb{X}$ with $|\mathbb{X}| < \infty$ is the state alphabet and $\mathbb{Y}$ with $|\mathbb{Y}| < \infty$ the output alphabet. Whithout loss of generality, we identify $\mathbb{X} = \{1, 2, \dots, |\mathbb{X}|\}$.
- Π is a mapping from $\mathbb{Y}$ to $\mathbb{R}_+^{|\mathbb{X}|\times|\mathbb{X}|}$, where $\Pi_{\mathbb{X}} = \sum_{\mathsf{y}\in\mathbb{Y}} \Pi(\mathsf{y})$ is a stochastic matrix, i.e. $\Pi_{\mathbb{X}} e = e$, where $e := \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^\top$. The element $\Pi(\mathsf{y})_{i,j}$ is equal to $P(x(t+1) = j, y(t) = \mathsf{y} | x(t) = i)$, the probability

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of going from state i to state j while observing output symbol y. The matrix $\Pi_{\mathbb{X}}$ is called the transition matrix;
- $\pi(1)$ is a vector in $\mathbb{R}_+^{|\mathbb{X}|}$ for which $\pi(1)e = 1$. The element $\pi_i(1)$ is $P(x(1) = i)$, the probability that the initial state is i.

The number of states $|\mathbb{X}|$ is called the *order* of the hidden Markov model. The model is called *stationary* if the state distribution is the same at every time instant, i.e. if the initial state distribution vector is the left eigenvector of the transition matrix corresponding to the eigenvalue 1: $\pi(1)\Pi_{\mathbb{X}} = \pi(1)$.

Denote by $\mathbb{Y}^*$ the set of all finite strings with symbols from the set $\mathbb{Y}$ (including the empty string) and by $\mathrm{y} = \mathrm{y}_1\mathrm{y}_2\ldots\mathrm{y}_{|\mathrm{y}|}$ an output sequence from $\mathbb{Y}^*$, where $|\mathrm{y}|$ denotes the length of y. Let $\mathcal{P} : \mathbb{Y}^* \mapsto [0, 1]$ be *string probabilities*, defined as $\mathcal{P}(\mathrm{y}) := P(y(1) = \mathrm{y}_1, y(2) = \mathrm{y}_2, \ldots, \mathrm{y}(|\mathrm{y}|) = \mathrm{y}_{|\mathrm{y}|})$. Of course, the string probabilities satisfy $\mathcal{P}(\phi) = 1$, where ϕ denotes the empty string, and $\sum_{\mathsf{y}\in\mathbb{Y}} \mathcal{P}(\mathrm{y}\mathsf{y}) = \mathcal{P}(\mathrm{y})$[1]. One can easily see that the string probabilities for all $\mathrm{y} = \mathrm{y}_1\mathrm{y}_2\ldots\mathrm{y}_{|\mathrm{y}|} \in \mathbb{Y}^*$ can be calculated as

$$\mathcal{P}(\mathrm{y}) = \pi(1)\Pi(\mathrm{y})e,$$

where $\Pi(\mathrm{y}) = \Pi(\mathrm{y}_1)\Pi(\mathrm{y}_2)\ldots\Pi(\mathrm{y}_{|\mathrm{y}|})$.

Two Mealy hidden Markov models $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ and $(\mathbb{X}', \mathbb{Y}, \Pi', \pi'(1))$ with string probabilities $\mathcal{P}$ and $\mathcal{P}'$ respectively are said to be *equivalent* if and only if they satisfy $\mathcal{P} = \mathcal{P}'$.

A Mealy hidden Markov model is $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ is called *minimal* if and only if for any other equivalent model $(\mathbb{X}', \mathbb{Y}, \Pi', \pi'(1))$ it holds that $|\mathbb{X}| \le |\mathbb{X}'|$.

In the *Mealy realization problem*, we are given all possible output string probabilities $\mathcal{P}$ and the problem is to find a Mealy HMM $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ that realizes $\mathcal{P}$, which means that for all $\mathrm{y} = \mathrm{y}_1\mathrm{y}_2\ldots\mathrm{y}_{|\mathrm{y}|} \in \mathbb{Y}^*$, it holds that $\mathcal{P}(\mathrm{y}) = \pi(1)\Pi(\mathrm{y}_1))\Pi(\mathrm{y}_2)\ldots\Pi(\mathrm{y}_{|\mathrm{y}|})e$.

The realization problem is very hard in practice because of the positivity contraints on $\pi(1)$ and Π. For that reason, one typically (first) solves the quasi realization problem, which is exactly the same problem as the realization problem but without the positivity contraints. However, the quasi model which is found from the quasi realization procedure retains some of the interesting properties of a positive model [8].

A *quasi hidden Markov model* of Mealy type is defined by the pentuple $(\mathbb{X}_q, \mathbb{Y}, \Pi_q, \pi_q(1), e_q)$, where Π_q, $\pi_q(1)$ and e_q are the analogues of Π, $\pi(1)$ and e, but the image of the mapping Π_q are matrices over $\mathbb{R}$ instead of $\mathbb{R}_+$, moreover the vector $\pi_q(1)$ is a row vector over $\mathbb{R}$, and the column vector e_q is a vector over $\mathbb{R}$ instead of a fixed vector e. For the initial quasi state distribution, it holds that $\pi_q(1)e_q = 1$ and for the quasi state transition matrix $\Pi_{\mathbb{X},q} = \sum_{\mathsf{y}\in\mathbb{Y}} \Pi_q(\mathsf{y})$ it holds that $\Pi_{\mathbb{X},q}e_q = e_q$. The number of states $|\mathbb{X}_q|$ is called the *(quasi) order* of the hidden Markov model.

Equivalence and minimality of quasi Mealy hidden Markov models are defined in an analogous way as for positive Mealy hidden Markov models.

[1]With $\mathrm{y}\mathsf{y}$, we mean the concatenation of the string y with the symbol y, concatenation of two strings is defined analogously.

A Mealy hidden Markov model $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ with string probabilities $\mathcal{P}$ and a quasi Mealy hidden Markov model $(\mathbb{X}_q, \mathbb{Y}, \Pi_q, \pi_q(1), e_q)$ with string probabilities $\mathcal{P}_q$ are said to be *equivalent* if and only if they satisfy $\mathcal{P} = \mathcal{P}_q$. The minimal order of a quasi HMM is lower than or equal to the minimal order of an equivalent positive HMM.

B. Moore type of HMM

In a Moore type of hidden Markov model, the generation of the next state and the generation of the output are independent. For a Moore HMM there exists a matrix $\Pi_{\mathbb{X}}$ and a mapping β from $\mathbb{Y}$ to $\mathbb{R}_+^{|\mathbb{X}|}$ such that for each $\mathsf{y} \in \mathbb{Y}$ it holds that

$$\Pi(\mathsf{y}) = \mathrm{diag}(\beta(\mathsf{y}))\Pi_{\mathbb{X}},$$

where $\mathrm{diag}(\cdot)$ is the diagonal matrix with the elements of the vector $\cdot$ on its diagonal. The element $(\Pi_{\mathbb{X}})_{ij}$ is then equal to $P(x(t+1) = j|x(t) = i)$, the probability of going from state i to state j. The element $\beta(\mathsf{y})_i$ is equal to $P(y(t) = \mathsf{y}|x(t) = i)$, the probability of observing the symbol y given that the present state is equal to i. Suppose we have an ordering $(\mathsf{y}_k, k = 1, 2, \ldots, |\mathbb{Y}|)$ of the output symbols of the set $\mathbb{Y}$, then the matrix B is defined as $B := \left[\begin{array}{ccc} \beta(\mathsf{y}_1) & \ldots & \beta(\mathsf{y}_{|\mathbb{Y}|}) \end{array}\right]$. One can easily see that $\Pi_{\mathbb{X}}e = e$ and that $Be = e$. A Moore HMM is fully described by $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, \beta, \pi(1))$ or $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, B, \pi(1))$.

Two Moore hidden Markov models $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, \beta, \pi(1))$ and $(\mathbb{X}', \mathbb{Y}, \Pi'_{\mathbb{X}}, \beta', \pi'(1))$ with string probabilities $\mathcal{P}$ and $\mathcal{P}'$ respectively are said to be *equivalent* if and only if they satisfy $\mathcal{P} = \mathcal{P}'$.

A Moore model $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, \beta, \pi(1))$ is called *minimal* if and only if for any other equivalent Moore model $(\mathbb{X}', \mathbb{Y}, \Pi'_{\mathbb{X}}, \beta', \pi'(1))$, it holds that $|\mathbb{X}| \le |\mathbb{X}'|$.

We define a *quasi hidden Markov model* of Moore type as $(\mathbb{X}_q, \mathbb{Y}, \Pi_{\mathbb{X},q}, \beta_q, \pi_q(1), e_q)$ or as $(\mathbb{X}_q, \mathbb{Y}, \Pi_{\mathbb{X},q}, B_q, \pi_q(1), e_q)$, where $\Pi_{\mathbb{X},q}$, β_q, B_q, $\pi_q(1)$ and e_q are the analogues of $\Pi_{\mathbb{X}}$, β, B, $\pi(1)$ and e, but the vectors $\beta(\mathsf{y})$, the matrix $\Pi_{\mathbb{X}}$ and the row vector $\pi_q(1)$ are over $\mathbb{R}$ instead of $\mathbb{R}_+$, and the column vector e_q is a vector over $\mathbb{R}$ instead of a fixed vector e. For the initial quasi state distribution, it holds that $\pi_q(1)e_q = 1$, for the quasi state transition matrix it holds that $\Pi_{\mathbb{X},q}e_q = e_q$ and for the output matrix B_q it holds that $B_qe = e$.

Equivalence and minimality of quasi Moore hidden Markov models is defined in an analogous way as minimality for Moore hidden Markov models.

A Moore hidden Markov model $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ with string probabilities $\mathcal{P}$ and a quasi Moore hidden Markov model $(\mathbb{X}_q, \mathbb{Y}, \Pi_q, \pi_q(1), e_q)$ with string probabilities $\mathcal{P}_q$ are said to be *equivalent* if and only if they satisfy $\mathcal{P} = \mathcal{P}_q$.

Equivalence between (quasi) Moore and (quasi) Mealy models is defined analogously.

C. Conversions between Moore and Mealy

It can be shown that the expressive power of Moore HMMs and Mealy HMMs is the same [9], which means that a finite valued process is realizable with a Moore hidden Markov model if and only if it is realizable with a Mealy hidden Markov model. However, one easily sees that the minimal order needed to realize a certain finite process with a Mealy model is smaller than or equal to the minimal order of a Moore model.

Converting a minimal Moore model $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, \beta, \pi(1))$ into a Mealy model $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ is always possible, using

$$\Pi(\mathsf{y}) = \mathrm{diag}(\beta(\mathsf{y}))\Pi_{\mathbb{X}}.$$

However, the obtained Mealy model can be nonminimal, even if the Moore model was minimal.

Converting a minimal Mealy model in a Moore model, can be done always by connecting a state of the Moore model to every state transition of the Mealy model and then calculating the state transition probabilities and the output probabilities. Typically, this approach will lead to a nonminimal Moore model.

In section III, we will show that a (quasi) Mealy model with two outputs can always be converted into a quasi Moore model with the same number of states.

III. EQUIVALENCE FOR MEALY HMMS

In this section we investigate the set of equivalent Mealy hidden Markov models: suppose we are given a minimal HMM with certain string probabilities, how can we find (all) equivalent Mealy HMMs? Of course, given a certain quasi or positive Mealy model, one can always obtain an equivalent model by permuting the states. However, there are many more equivalent models than only the ones obtained by permuting states. In addition, for the quasi case, the class of equivalent quasi Mealy hidden Markov models can be characterized in an appropriate way. In the positive hidden Markov case the problem is more complicated and to the best of our knowledge there exists no characterization of the set of equivalent models. We prove some initial results and give a simulation example.

For quasi Mealy HMMs the set of all equivalent realizations is characterized by the following theorem [9].

Theorem 1: Two minimal quasi Mealy models $(\mathbb{X}_q, \mathbb{Y}, \Pi_q, \pi_q(1), e_q)$ and $(\mathbb{X}'_q, \mathbb{Y}, \Pi'_q, \pi'_q(1), e'_q)$ are equivalent (i.e. they have the same string probabilities), if and only if there exists a nonsingular matrix T, such that

$$\begin{aligned} \pi_q(1) &= \pi'_q(1)T^{-1} \\ \Pi_q(\mathsf{y}) &= T\Pi'_q(\mathsf{y})T^{-1} \quad \forall \mathsf{y} \in \mathbb{Y} \\ e_q &= Te'_q \end{aligned}$$

This theorem allows us to prove that a (quasi) Mealy model with two outputs, y_1 and y_2, can always be converted into an equivalent quasi Moore model with the same number of states under the condition that the row space of $\Pi_q(\mathsf{y}_1)$ equals the row space of $\Pi_q(\mathsf{y}_2)$, which is fulfilled for most HMMs. For that conversion, we need to find a nonsingular matrix T such that there exists a matrix $\Pi_{\mathbb{X},q}$ and two vectors $\beta_q(\mathsf{y}_1)$ and $\beta_q(\mathsf{y}_2)$ for which

$$\begin{aligned} T\Pi_q(\mathsf{y}_1)T^{-1} &= \mathrm{diag}(\beta_q(\mathsf{y}_1))\Pi_{\mathbb{X},q} \\ T\Pi_q(\mathsf{y}_2)T^{-1} &= \mathrm{diag}(\beta_q(\mathsf{y}_2))\Pi_{\mathbb{X},q}, \end{aligned}$$

or that there exists a diagonal matrix D such that

$$T\Pi_q(\mathsf{y}_1)T^{-1} = DT\Pi_q(\mathsf{y}_2)T^{-1}.$$

One can easily see that this matrix D always exists under the condition that $\mathrm{row}(\Pi_q(\mathsf{y}_1)) = \mathrm{row}(\Pi_q(\mathsf{y}_2))$.

For Mealy hidden Markov models with more than two outputs the above result does not hold in general anymore, so it is possible that a minimal quasi Moore model equivalent to a minimal quasi Mealy model, has more states than the Mealy model.

For positive Mealy HMMs we prove the following theorem

Theorem 2: For a positive Mealy hidden Markov model $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ the following holds: If there exists a nonsingular matrix T such that

- $Te = e$,
- $\pi(1)T^{-1}$ and $T\Pi(\mathsf{y})T^{-1} \quad \forall \mathsf{y} \in \mathbb{Y}$ are nonnegative

then $(\mathbb{X}, \mathbb{Y}, T\Pi T^{-1}, \pi(1)T^{-1})$ is a *positive* hidden Markov model which is equivalent to the given HMM $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$.

Proof: We first prove that $(\mathbb{X}, \mathbb{Y}, T\Pi T^{-1}, \pi(1)T^{-1})$ is a positive hidden Markov model. As the system matrices are nonnegative by construction, we only need to prove the consistency properties.

The fact that the entries of $\pi(1)T^{-1}$ sum to one, follows from

$$\pi(1)T^{-1}e \;=\; \pi(1)e \;=\; 1.$$

The fact that $\sum_{\mathsf{y}\in\mathbb{Y}} T\Pi(\mathsf{y})T^{-1}$ is a stochastic matrix, follows from

$$\begin{aligned} \sum_{\mathsf{y}\in\mathbb{Y}} T\Pi(\mathsf{y})T^{-1}e &= T\left(\sum_{\mathsf{y}\in\mathbb{Y}} \Pi(\mathsf{y})\right)e \\ &= Te \;=\; e. \end{aligned}$$

Now we prove that $(\mathbb{X}, \mathbb{Y}, T\Pi T^{-1}, \pi(1)T^{-1})$ is equivalent to $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$. This follows from the fact that the string probabilities of $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ are equal to the string probabilities of $(\mathbb{X}, \mathbb{Y}, T\Pi T^{-1}, \pi(1)T^{-1})$ for all $\mathrm{y} = \mathrm{y}_1\mathrm{y}_2\ldots\mathrm{y}_{|\mathrm{y}|} \in \mathbb{Y}^*$, i.e.

$$\begin{aligned} &\pi(1)\Pi(\mathrm{y}_1)\ldots\Pi(\mathrm{y}_{|\mathrm{y}|})e = \\ &\qquad \pi(1)T^{-1}T\Pi(\mathrm{y}_1)T^{-1}\ldots T\Pi(\mathrm{y}_{|\mathrm{y}|})T^{-1}e, \end{aligned}$$

which follows from the fact that $T^{-1}e = e$. ∎

In the same way as in the proof of the theorem, it can be proven that if the given model is stationary (i.e. $\pi(1)\sum_{\mathsf{y}\in\mathbb{Y}}\Pi(\mathsf{y}) = \pi(1)$), that the equivalent model is also stationary.

Notice that the converse of Theorem 2 is not true in general. There exist equivalent positive Mealy models which are not connected trough a similarity transform T. So the theorem does not give a way to find the *complete set* of hidden Markov models equivalent to a given hidden Markov model (in contradiction to the case with quasi Mealy models).

However, in case a positive hidden Markov model is minimal in the set of quasi HMMs then the converse of

Theorem 2 does hold (by combining Theorem 1 and Theorem 2). In that case, one can say that if two positive HMMs are equivalent that there must exist a transformation T fulfilling the conditions of Theorem 2. Consequently, in that case the theorem gives a description of the complete set of equivalent positive hidden Markov models.

We now show a simulation example where we try to get an idea of the set of all equivalent positive models equivalent to $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$, with $|\mathbb{X}| = 3$, $\mathbb{Y} = \{0, 1\}$ and

$$
\begin{aligned}
\Pi(0) &= \begin{bmatrix} 0.05 & 0.4 & 0.1 \\ 0.2 & 0.15 & 0.2 \\ 0.35 & 0.15 & 0.2 \end{bmatrix}, \\
\Pi(1) &= \begin{bmatrix} 0.15 & 0.1 & 0.2 \\ 0.1 & 0.05 & 0.3 \\ 0.05 & 0.15 & 0.1 \end{bmatrix}, \\
\pi(1) &= \begin{bmatrix} 0.3060 & 0.3284 & 0.3657 \end{bmatrix}.
\end{aligned}
$$

One can easily see that this model is minimal as a quasi realization (by calculating the rank of the associated Hankel matrix (see [5])) and that the model is in addition stationary: i.e. $\pi(1) \sum_{\mathsf{y} \in \mathbb{Y}} \Pi(\mathsf{y}) = \pi(1)$.

An equivalent HMM is found by taking a nonsingular matrix T of the form

$$
T = \begin{bmatrix} T_{11} & T_{12} & 1 - (T_{11} + T_{12}) \\ T_{21} & T_{22} & 1 - (T_{21} + T_{22}) \\ T_{31} & T_{32} & 1 - (T_{31} + T_{32}) \end{bmatrix},
$$

and checking whether the second condition of Theorem 2 holds. We now perform a matlab experiment that investigates matrices T on a grid, i.e. $T_{11} = -1, -0.9, -0.8, \ldots, 1$ (with steps of 0.1), and the same for T_{12}, T_{21}, T_{22}, T_{31} and T_{32}. In that way, we find a whole class of realizations which are equivalent to the given HMM. In Figure 1 we show the stationary state distribution $\pi(1)$ of each of these equivalent realizations.

Of course all $\pi(1)$s are lying on the plane $\pi_1(1) + \pi_2(1) + \pi_3(1) = 1$. In the figure, it seems that not all equilibrium state distributions are possible and that the possible equilibrium state distributions make some complex form. The question whether these observations are coincidence and due to the grid size will be topic of the authors' further research.

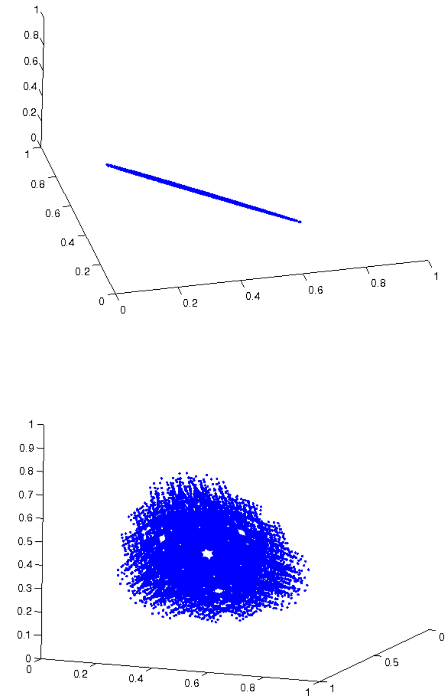

Fig. 1. Plot of stationary state distribution of possible HMMs which are equivalent to a given HMM (the two subplots give different views of the same plot).

IV. EQUIVALENCE FOR MOORE HMMS

In this section we investigate the set of equivalent Moore hidden Markov models. As was the case with Mealy models, one can always obtain a model equivalent to a given quasi or positive Moore model by permuting the states of the original model. In contrast to the Mealy case, we will show here that under certain conditions these permuted models are the only possible equivalent models of the same order than the original model.

The uniqueness of the (quasi) Moore model can be described by the following theorem

Theorem 3: Given a (quasi) Moore HMM $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X},q}, B_q, \pi_q(1), e_q)$ which is minimal as a quasi Mealy model[2] then the following holds: if all states of the Moore model have a different output distribution and if the state transition matrix $\Pi_{\mathbb{X},q}$ has full rank, then there does not exist any minimal (quasi) Moore model which is nontrivially equivalent to the given (quasi) Moore model[3].

Proof: Suppose that $(\tilde{\mathbb{X}}, \mathbb{Y}, \tilde{\Pi}_{\mathbb{X},q}, \tilde{B}_q, \tilde{\pi}_q(1), \tilde{e}_q)$ is an equivalent model to the given model $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X},q}, B_q, \pi_q(1), e_q)$ and has the same order as the given model. Then from Theorem 1, we know that there exists a transformation T such that

$$
\begin{aligned}
\tilde{\pi}_q(1) &= \pi_q(1) T^{-1}, \\
\operatorname{diag} \tilde{\beta}(\mathsf{y}) \tilde{\Pi}_{\mathbb{X},q} &= T \operatorname{diag} \beta(\mathsf{y}) \Pi_{\mathbb{X},q} T^{-1} \quad \forall \mathsf{y} \in \mathbb{Y}, \\
\tilde{e}_q &= T e_q.
\end{aligned} \tag{1}
$$

From the fact that $\Pi_{\mathbb{X},q}$ has full rank, it follows that $\tilde{\Pi}_{\mathbb{X},q}$ has full rank. So we conclude from (1) that there exist nonsingular matrices T and S such that

$$
\begin{aligned}
\operatorname{diag} \tilde{\beta}(\mathsf{y}) &= T \operatorname{diag} \beta(\mathsf{y}) S^{-1} \quad \forall \mathsf{y} \in \mathbb{Y}, \\
\tilde{\Pi}_{\mathbb{X},q} &= S \Pi_{\mathbb{X},q} T^{-1}.
\end{aligned}
$$

But from consistency of the HMM

[2]We say that a model is "minimal as a quasi Mealy model" if there does not exist any equivalent quasi Mealy model of lower order, i.e. if the order of the model is equal to the rank of the associated Hankel matrix [5].

[3]With a "trivial equivalent" model, we denote a model obtained by permuting the states of the original model.

$(\tilde{\mathbb{X}}, \mathbb{Y}, \tilde{\Pi}_{\mathbb{X},q}, \tilde{B}_q, \tilde{\pi}_q(1), \tilde{e}_q)$

$$\sum_{\mathsf{y}\in\mathbb{Y}} T \operatorname{diag} \beta(\mathsf{y}) S^{-1} = I$$

we find that

$$T = S.$$

Now we have that

$$\operatorname{diag} \tilde{\beta}(\mathsf{y}) = T \operatorname{diag} \beta(\mathsf{y}) T^{-1} \quad \forall \mathsf{y} \in \mathbb{Y},$$

and together with the fact that all states of the Moore model have a different output distribution, this allows us to conclude that T can only be equal to a permutation matrix. ■

If there do exist states with the same output distribution, but all the other conditions of Theorem 3 are fulfilled, then there exists a set of equivalent Moore models (apart from the models obtained by permuting the states). Suppose for instance that the output distribution of the first state equals the output distribution of the second state, in that case the transformation T of the proof of Theorem 3 is of the form

$$T = P \begin{bmatrix} T_{12} & 0 \\ 0 & I \end{bmatrix}, \tag{2}$$

where P is a permutation matrix, T_{12} a nonsingular matrix of size 2×2 with $T_{12}e = e$ and I is the unit matrix of appropriate dimensions. This gives a complete description of the equivalence set. Notice that all elements of the set of equivalent Moore models have the same output matrix B (up to a permutation of the states). They only differ in the state transition matrix $\Pi_{\mathbb{X}}$.

From Section III, we know that, under very general conditions, a (quasi) Mealy model with two outputs can be converted into an equivalent quasi Moore model with the same number of states. So every minimal quasi Moore model with two outputs is minimal as a quasi Mealy HMM. So from Theorem 3 we conclude that a minimal quasi Moore model with two outputs, with a full rank transition matrix and with a different output distribution for each state, has no equivalents (except trivial equivalents).

However, the theorem is also useful for hidden Markov models with more than two outputs. Consider for example the Moore model $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, B, \pi(1))$ with

$$\begin{aligned} \Pi_{\mathbb{X}} &= \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}, \\ B &= \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.1 & 0.1 & 0.8 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}, \\ \pi(1) &= \begin{bmatrix} 0.5405 & 0.1622 & 0.2973 \end{bmatrix}. \end{aligned}$$

One can show that the minimal order of an equivalent quasi Mealy model equals three. This allows us to conclude that the Moore model is minimal as a quasi Mealy model. In addition, the output distribution is different in every state and $\Pi_{\mathbb{X}}$ has full rank, so that we conclude from Theorem 3 that the only way to obtain a minimal Moore equivalent to the given model, is by permuting the states.

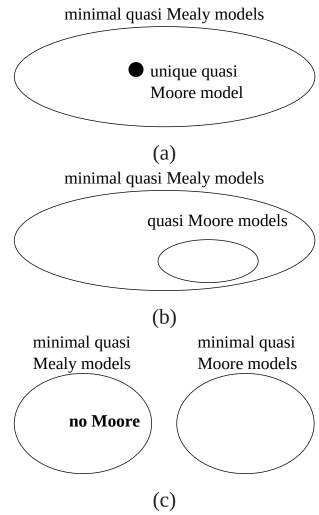

Fig. 2. The three main case concerning the equivalence classes of hidden Markov models.

V. SUMMARY OF THE RESULTS

In this section, we summarize the results so far, by showing the three different cases that can occur concerning the equivalence classes of Moore and Mealy models (see figure 2). Notice that in all cases, there is a whole set of equivalent minimal quasi Mealy models, as explained in Theorem 1. In case (a) (figure 2(a)), the quasi Moore model is minimal as a quasi Mealy model and every state has a different output distribution and the transition matrix has full rank so that we know from Theorem 3 that this minimal Moore model is unique. In case (b) (figure 2(b)), the quasi Moore model is again minimal as a quasi Mealy model, but the other conditions of Theorem 3 are not fulfilled, such that there exists a set of equivalent Moore models. In case (c) (figure 2(c)) there does not exist a Moore model of the same order as the minimal Mealy model, so any minimal Moore model is of higher order. In that case, nothing is proven about the class of equivalent minimal Moore models, but we expect that a whole set will exist.

VI. CONCLUSIONS

In this paper we considered the following problem for (quasi) hidden Markov models: given a minimal (quasi) hidden Markov model, what can be said about the set of all equivalent (quasi) hidden Markov models of the same order. For quasi hidden Markov models of Mealy type, a necessary and sufficient condition for two models to be equivalent was already proven in literature. In addition, we have proven a sufficient condition for two positive Mealy models to be equivalent. In a simulation example, we computed a set of equivalent positive Mealy models. We have also proven that, under certain conditions, the set of equivalent models for minimal quasi Moore models consists of only one element (up to a permutation of the states). It was shown that for Moore models with an output alphabet of two symbols, these conditions are always fulfilled. In future work, we will investigate the equivalence problem for positive HMMs in more detail.

ACKNOWLEDGMENTS

Bart Vanluyten is a research assistant with the Fund for Scientific Research Flanders (FWO-Vlaanderen). Katrien De Cock is a postdoctoral researcher, Jan Willems a professor and Bart De Moor a full professor with the Katholieke Universiteit Leuven. The SISTA research program is supported by: Research Council KUL: GOA AMBioRICS, CoE EF/05/006 Optimization in Engineering, several PhD/postdoc and fellow grants; Flemish Government: FWO: PhD/postdoc grants, projects, G.0407.02 (support vector machines), G.0197.02 (power islands), G.0141.03 (Identification and cryptography), G.0491.03 (control for intensive care glycemia), G.0120.03 (QIT), G.0452.04 (new quantum algorithms), G.0499.04 (Statistics), G.0211.05 (Nonlinear), G.0226.06 (cooperative systems and optimization), G.0321.06 (Tensors), G.0553.06 (VitamineD), research communities (ICCoS, ANMMM, MLDM); IWT: PhD Grants,GBOU (McKnow), Eureka-Flite2; Belgian Federal Science Policy Office: IUAP P5/22 ('Dynamical Systems and Control: Computation, Identification and Modelling', 2002-2006) ; PODO-II (CP/40: TMS and Sustainability); EU: FP5-Quprodis; ERNSI; Contract Research/agreements: ISMC/IPCOS, Data4s, TML, Elia, LMS, Mastercard

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