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An Algebraic ICA Algorithm for 3 Sources and 2 Sensors [1]

Lieven De Lathauwer, Bart De Moor and Joos Vandewalle[2]

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[2]K.U.Leuven, Dept. of Electrical Engineering (ESAT), Research group SISTA/COSIC, Kardinaal Mercierlaan 94, 3001 Leuven, Belgium, Tel. 32/16/32 18 05, Fax 32/16/32 19 70, WWW: *http://www.esat.kuleuven.ac.be/sista.* E-mail: *Lieven.DeLathauwer@esat.kuleuven.ac.be*, *Bart.DeMoor@esat.kuleuven.ac.be*, *Joos.Vandewalle@esat.kuleuven.ac.be.* This research was partially supported by the Flemish Government: (1) Research Council K.U.Leuven: Concerted Research Actions GOA-MIPS and GOA-MEFISTO-666, (2) the Fund for Scientific Research-Flanders (F.W.O.) projects G.0240.99 and G.0256.97, (3) the F.W.O. Research Communities ICCoS and ANMMM, and by the Belgian State, Prime Minister's Office - Federal Office for Scientific, Technical and Cultural Affairs: the Interuniversity Poles of Attraction Programmes IUAP P4-02 and IUAP P4-24. L. De Lathauwer is a post-doctoral researcher with the F.W.O. B. De Moor is a senior Research Associate with the F.W.O. and an Associate Professor with the K.U.Leuven. J. Vandewalle is a Full Professor with the K.U.Leuven. The scientific responsibility is assumed by the authors.

Abstract

In this paper we develop an algebraic algorithm for Independent Component Analysis with 3 complex-valued sources and 2 sensors. First we consider a generalization of an old theorem by Sylvester, which allows us to relate the problem with the approximation of a 4th-order tensor with Hermitean symmetry by a tensor of rank-1. We present an Alternating Least Squares algorithm for the computation of the result.

AN ALGEBRAIC ICA ALGORITHM FOR 3 SOURCES AND 2 SENSORS

L. De Lathauwer, B. De Moor and J. Vandewalle[*]
K.U.LEUVEN, E.E.Dept. (ESAT), SISTA/COSIC,
Kard. Mercierlaan 94, B-3001 Leuven, BELGIUM
Tel: +32 16 321805; fax: +32 16 321970
e-mail: Lieven.DeLathauwer@esat.kuleuven.ac.be

ABSTRACT

In this paper we develop an algebraic algorithm for Independent Component Analysis with 3 complex-valued sources and 2 sensors. First we consider a generalization of an old theorem by Sylvester, which allows us to relate the problem with the approximation of a 4th-order tensor with Hermitean symmetry by a tensor of rank-1. We present an Alternating Least Squares algorithm for the computation of the result.

1 INTRODUCTION

The basic statistical model for *Independent Component Analysis* (ICA), or *Blind Source Separation* (BSS), is in this paper denoted as

$$Y = \mathbf{M}X + N, \tag{1}$$

in which the observed vector Y, the source vector X and the noise vector N are zero-mean random vectors with values in $\mathbb{C}$. The components of X are mutually statistically independent, as well as statistically independent from the noise components. The goal of ICA consists of the estimation of the "mixing matrix" $\mathbf{M}$ and the corresponding realizations of X, given only realizations of Y.

In this paper we focus on the situation in which there are 3 sources but only 2 sensor elements. This set-up is is for instance interesting w.r.t. applications in mobile communications. The rectangular size of the mixing matrix makes it of course impossible to estimate the source values by mere matrix inversion, but the source values may nevertheless be reconstructed by exploiting prior knowledge on the source distributions, like e.g. the finite alphabet property [3].

Our approach is based on multilinear algebra. In this framework the ICA problem can be formulated as the decomposition of super-symmetric higher-order cumulant tensors in a minimal number of rank-1 terms, in which each rank-1 term consists of the contribution of one particular source signal to the overall linear combination. Higher-order tensors have the advantage, compared to matrices, that the number of rank-1 terms is not bounded by the dimension of the column nor the row space, which makes it a priori possible to address the ICA problem in a context where there are more sources than sensors.

The algorithm of this paper will be derived by exploring an old theorem proposed by Sylvester. First we will introduce some basic material in Section 2. Section 3 states Sylvester's theorem. The actual application to ICA is discussed in Section 4.

There is some overlap between this paper and [5]. In the latter paper the case of unsymmetric source distributions is explicitly considered; this is outside the scope of the current paper. On the other hand, our approach to case of symmetric source distributions is somewhat different from the approach followed in [5].

2 TENSORS AND QUANTICS

In this paper, a *higher-order tensor* can intuitively be imagined as a multi-indexed array of numerical values. First we touch on some basic definitions in multilinear algebra.

Definition 1 (Outer product) *The outer product $\mathcal{A} \circ \mathcal{B}$ of a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_P}$ and a tensor $\mathcal{B} \in \mathbb{C}^{J_1 \times J_2 \times \ldots \times J_Q}$, is defined by*

$$(\mathcal{A} \circ \mathcal{B})_{i_1 i_2 \ldots i_P j_1 j_2 \ldots j_Q} \stackrel{\text{def}}{=} a_{i_1 i_2 \ldots i_P} b_{j_1 j_2 \ldots j_Q},$$

for all values of the indices.

This product generalizes the construction $\mathbf{C} = U V^T$, in which U and V are vectors, in matrix algebra. As an example, an Nth-order tensor $\mathcal{A}$, equal to the outer

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product of N vectors $U^{(1)}$, $U^{(2)}$, ..., $U^{(N)}$, is given by the element-wise equation $a_{i_1 i_2 \ldots i_N} = u^{(1)}_{i_1} u^{(2)}_{i_2} \ldots u^{(N)}_{i_N}$. Such a tensor will be called a rank-1 tensor:

Definition 2 (Rank-1 tensor) *An Nth-order tensor $\mathcal{A}$ has rank 1 when it equals the outer product of N vectors $U^{(1)}$, $U^{(2)}$, ..., $U^{(N)}$:*

$$\mathcal{A} = U^{(1)} \circ U^{(2)} \circ \ldots \circ U^{(N)}.$$

Definition 3 (Rank) *The rank of an arbitrary Nth-order tensor $\mathcal{A}$, represented by $R = \mathtt{rank}(\mathcal{A})$, is the minimal number of rank-1 tensors that yield $\mathcal{A}$ in a linear combination.*

In multilinear algebra the concept of rank is not subject to the same limitations as in the matrix case. The basis of this paper is the fact that the rank of a $(K \times K \times \ldots \times K)$-tensor is not bounded by K in general.

Next, we provide straightforward generalizations of the definitions of scalar product and orthogonality.

Definition 4 (Scalar product) *The scalar product $\langle \mathcal{A}, \mathcal{B} \rangle$ of two tensors $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ is defined as*

$$\langle \mathcal{A}, \mathcal{B} \rangle \stackrel{\text{def}}{=} \sum_{i_1} \sum_{i_2} \ldots \sum_{i_N} b^*_{i_1 i_2 \ldots i_N} a_{i_1 i_2 \ldots i_N}.$$

Definition 5 (Orthogonality) *Tensors of which the scalar product equals 0, are mutually orthogonal.*

A *quantic* is an homogeneous polynomial. One can associate a quantic $f(x_1, x_2, \ldots, x_K)$ of degree N with a given Nth-order $(K \times K \times \ldots \times K)$-tensor $\mathcal{F}$ as follows:

$$f(x_1, x_2, \ldots, x_K) = \sum_{k_1 k_2 \ldots k_N = 1}^{K} f_{k_1 k_2 \ldots k_N} x_{k_1} x_{k_2} \ldots x_{k_N}. \tag{2}$$

Conversely, one can also associate a tensor with a given quantic. To make the representation unique we may assume that the tensor is invariant under arbitrary index permutations. For binary quantics, set $x_1 = x$ and $x_2 = y$ to simplify the notation. Then, the link amounts to

$$f(x,y) = \sum_{n=0}^{N} \binom{N}{n} f_{\mathbf{k}}\, x^{N-n} y^n, \tag{3}$$

in which the multi-index $\mathbf{k}$ contains $N-n$ times the value 1 and n times the value 2, and $\binom{N}{n} = \frac{N!}{(N-n)!\,n!}$.

We now define the apolar scalar product of quantics; we restrict ourselves to polynomials in two unknowns, which is the case of interest for this paper.

Definition 6 (Apolar scalar product) *The apolar scalar product $\langle f, g \rangle$ of two binary quantics of degree N is given by*

$$\langle f, g \rangle \stackrel{\text{def}}{=} \sum_{n=0}^{N} \binom{N}{n} f_{\mathbf{k}}\, g^*_{\mathbf{k}}.$$

This product simply corresponds to the scalar product of the associated tensors; the variational coefficient takes into account the number of times a given entry appears due to the symmetry.

For complex tensors with a specific Hermitean-type symmetry it may be more appropriate to consider the complex conjugate of a number of unknowns in the right-hand side of Eq. (2). Eq. (3) and the expression for the apolar scalar product of binary quantics then take a more complicated form.

3 A THEOREM BY SYLVESTER

The following theorem is an old result of multilinear algebra obtained by J.J. Sylvester [1]:

Theorem 1 *A binary quantic $f(x,y)$ of degree N,*

$$f(x,y) = \sum_{n=0}^{N} \phi_n \binom{N}{n} x^{N-n} y^n,$$

can be written as a sum of Nth powers of R distinct linear forms,

$$f(x,y) = \sum_{r=1}^{R} \lambda_r (\alpha_r x + \beta_r y)^N,$$

iff (i) there exists a vector $G \in \mathbb{C}^{R+1}$, with components g_r $(0 \leqslant r \leqslant R)$, such that

$$\begin{bmatrix} \phi_0 & \phi_1 & \cdots & \phi_R \\ \phi_1 & \phi_2 & \cdots & \phi_{R+1} \\ \vdots & \vdots & & \vdots \\ \phi_{N-R} & \phi_{N-R+1} & \cdots & \phi_N \end{bmatrix} G^* = 0, \tag{4}$$

and (ii) the polynomial $g(x,y) \stackrel{\text{def}}{=} \sum_{r=0}^{R} g_r x^{R-r} y^r$ admits R distinct roots, $f(x,y)$ and $g(x,y)$ being related in the sense that $g(x,y) = \prod_{r=1}^{R} (\beta^*_r x - \alpha^*_r y)$.

Proof. Call $X_r = (\alpha_r\, \beta_r)^T$ and $Y_r = (\beta^*_r\ -\alpha^*_r)^T$. The quantics $f(x,y)$ and $g(x,y)$ can be associated with the Nth order tensor $\mathcal{F} \stackrel{\text{def}}{=} \sum_{r=1}^{R} \lambda_r X_r \circ X_r \circ \cdots \circ X_r$ and the Rth order tensor $\mathcal{G} \stackrel{\text{def}}{=} Y_1 \circ Y_2 \circ \cdots \circ Y_R$ respectively. Eq. (4) is an apolar scalar product formulation of the fact that $\mathcal{G}$ is orthogonal to the $N - R + 1$ different Rth order tensors that can be obtained from $\mathcal{F}$ by keeping $N - R$ arbitrary indices fixed. ∎

Sylvester's theorem not only proves the existence of the R forms, but also gives a means to compute them. From the theorem it is clear that, for odd values of N, we have a generic rank of $R = \frac{N+1}{2}$, whereas for even values of N, $R = \frac{N}{2} + 1$, generically. It is then clear that when N is even, there are at least two vectors satisfying (4), because the Hankel matrix is of size $\frac{N}{2} \times (\frac{N}{2} + 2)$. To be more concrete, take as example $N = 4$. The first Hankel matrix having more columns than rows is of size 2×4, and obviously has generically 2 vectors in its null space [2]. On the other hand, when N is odd, there is generically a unique vector G satisfying (4).

4 APPLICATION IN ICA

Due to the multilinearity of cumulants, $\mathcal{C}^{(4,0)} \stackrel{\text{def}}{=} \mathtt{Cum}\{Y, Y, Y, Y\}$ can be expanded as

$$\mathcal{C}^{(4,0)} = \sum_{r=1}^{R} \kappa_r^{(4,0)} M_r \circ M_r \circ M_r \circ M_r, \tag{5}$$

in which $\kappa_r^{(4,0)} = \mathtt{Cum}\{X_r, X_r, X_r, X_r\}$ is the kurtosis of the rth source and M_r the rth column of the mixing matrix. Hence identifying the mixing matrix from $\mathcal{C}^{(4,0)}$ is equivalent to decomposing the quartic associated with it into a sum of fourth powers of R linear terms.

For $R = 2$, the problem consists of finding the kernel of the (3×3) Hankel matrix constructed as in Eq. (4). In the noiseless case, this matrix is rank deficient by definition. With noise, the set of equations is overdetermined and can be solved in a least-squares sense by computing the third right singular vector of the Hankel matrix.

For $R = 3$, the unknown G in Eq. (4) can only be restricted to a two-dimensional linear subspace. In order to fix this indeterminacy, we resort to the cumulant $\mathcal{C}^{(2,2)} \stackrel{\text{def}}{=} \mathtt{Cum}\{Y, Y, Y^*, Y^*\}$, which admits the related decomposition

$$\mathcal{C}^{(2,2)} = \sum_{r=1}^{R} \kappa_r^{(2,2)} M_r \circ M_r \circ M_r^* \circ M_r^*. \tag{6}$$

In [2] it was proposed to perform an exhaustive search for the free complex parameter in Eq. (4) that yields the factors that give the best match of $\mathcal{C}^{(2,2)}$ in Eq. (6). This was formulated as a search in $\mathbb{R}^2$ to obtain an associated 35-dimensional vector that is closest to an associated linear 3-dimensional subspace of the 35-dimensional space.

In Section 5 we will formulate a modification of Theor. 1 for tensors that have a symmetry like $\mathcal{C}^{(2,2)}$. This will be exploited in Section 6 to obtain an algorithm that works for symmetric source distributions at a lower cost than the scheme derived in [2]. This algorithm is iterative. On the other hand, for unsymmetric distributions non-iterative algorithms can be devised; we refer to [5].

5 THEOREM EXTENSION

Sylvester's theorem can be modified to take into account complex conjugates of the unknowns, as indicated in Section 2. For fourth-order tensors with the same symmetry as $\mathcal{C}^{(2,2)}$ we have the following theorem.

Theorem 2 *The binary quartic*[1]

$$f(x,y) = \sum_{m,n=0}^{2} \phi_{m,n} \binom{2}{m} \binom{2}{n} x^{2-m} (x^{2-n})^* y^m (y^n)^*$$

can be expanded as

$$f(x,y) = \sum_{r=1}^{3} \lambda_r \, |\alpha_r \, x + \beta_r \, y|^4,$$

iff (i) there exists a vector $G \in \mathbb{C}^6$ such that

$$\begin{bmatrix} \phi_{00} & \phi_{10} & \phi_{20} & \phi_{01} & \phi_{11} & \phi_{21} \\ \phi_{01} & \phi_{11} & \phi_{21} & \phi_{02} & \phi_{12} & \phi_{22} \end{bmatrix} G^* = 0, \tag{7}$$

with $G = G'^* \otimes G''$, in which $G' \in \mathbb{C}^2$ and $G'' \in \mathbb{C}^3$. Call the components of G' and G'' g'_r $(0 \leqslant r \leqslant 1)$ and g''_r $(0 \leqslant r \leqslant 2)$ respectively. (ii) The roots of $g(x,y) \stackrel{\text{def}}{=} g'(x,y)\, g''(x,y)$ are mutually different. $f(x,y)$ and $g(x,y)$ are related in the sense that $g(x,y) = \prod_{r=1}^{3} (\beta_r^* \, x - \alpha_r^* \, y)$.

Proof. Call $X_r = (\alpha_r \, \beta_r)^T$ and $Y_r = (\beta_r^* \; - \alpha_r^*)^T$. The quantic $f(x,y)$ can be associated with the 4th order tensor $\mathcal{F} \stackrel{\text{def}}{=} \sum_{r=1}^{R} \lambda_r X_r \circ X_r \circ X_r^* \circ X_r^*$. Eq. (7) expresses that $Y_1^* \circ Y_2 \circ Y_3$, $Y_2^* \circ Y_3 \circ Y_1$ and $Y_3^* \circ Y_1 \circ Y_2$ are orthogonal to the 2 third-order tensors that can be obtained from $\mathcal{F}$ by keeping the fourth index fixed. ∎

Here we considered only the case of a decomposition that involves three linear forms because this is the case required by our application; a more general formulation can be found in [5].

6 ALGORITHM

Let us have a closer look at Theor. 2. Let the matrix in Eq. (7) be partitioned as $(\mathbf{B}_1 \, \mathbf{B}_2)$, in which $\mathbf{B}_1, \mathbf{B}_2 \in \mathbb{C}^{2 \times 3}$. A vector G is then of the form required in Theor. 2 iff there exist complex values w_1 and w_2 and a vector $V \in \mathbb{C}^{3 \times 1}$ such that

$$(w_1^* \mathbf{B}_1 - w_2^* \mathbf{B}_2) \, V = 0 \tag{8}$$

in a non-trivial way. It is clear that for any value of w_1/w_2 an appropriate vector V can be determined, such that $\mathcal{C}^{(2,2)}$, like $\mathcal{C}^{(4,0)}$, leaves one free complex parameter in the solution set. However this algebraic variety is not a linear vector space, which makes $\mathcal{C}^{(2,2)}$ harder to handle here. The solutions we are interested in, consist of $W \stackrel{\text{def}}{=} (w_1 \; w_2)^T$ proportional to $(\beta_{\tilde{r}} \; - \alpha_{\tilde{r}})^T$, while $g''(x,y) = \prod_{r \neq \tilde{r}} (\beta_r \, x - \alpha_r \, y)$ $(1 \leqslant \tilde{r} \leqslant 3)$; the vector G'' is proportional to V.

As a matter of fact, we can develop a similar reasoning as the one in Theor. 2 for $\mathcal{C}^{(4,0)}$. The corresponding binary quartic can then be written as the sum

$$f(x,y) = \sum_{r=1}^{3} \lambda_r \, (\alpha_r \, x + \beta_r \, y)^4,$$

iff (i) there exists a vector $G \in \mathbb{C}^6$ such that

$$\begin{bmatrix} \phi_0 & \phi_1 & \phi_2 & \phi_1 & \phi_2 & \phi_3 \\ \phi_1 & \phi_2 & \phi_3 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix} G^* = 0, \tag{9}$$

[1] f is actually an homogeneous polynomial in four real variables. Even if it is not correct strictly speaking, it is sometimes referred to as a real "binary quartic" in complex variables.

with $G = G' \otimes G''$, in which $G' \in \mathbb{C}^2$ and $G'' \in \mathbb{C}^3$. Call the components of G' and G'' g'_r $(0 \leqslant r \leqslant 1)$ and g''_r $(0 \leqslant r \leqslant 2)$ respectively. (ii) The roots of $g(x,y) \stackrel{\text{def}}{=} g'(x,y)\, g''(x,y)$ are mutually different. $f(x,y)$ and $g(x,y)$ are related in the sense that $g(x,y) = \prod_{r=1}^{3}(\beta_r^*\, x - \alpha_r^*\, y)$.

Let now the matrix in Eq. (9) be partitioned as $(\mathbf{A}_1\ \mathbf{A}_2)$, in which $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{C}^{2\times 3}$. A vector G is then of the form required in Theor. 2 iff there exist complex values w_1 and w_2 and a vector $V \in \mathbb{C}^{3\times 1}$ such that

$$(w_1 \mathbf{A}_1 - w_2 \mathbf{A}_2)\, V = 0 \tag{10}$$

in a non-trivial way. Again, the solutions we are interested in, consist of W^T proportional to $(\beta_{\tilde{r}}\ -\alpha_{\tilde{r}})$, while $g''(x,y) = \prod_{r\neq\tilde{r}}(\beta_r\, x - \alpha_r\, y)$ $(1 \leqslant \tilde{r} \leqslant 3)$; the vector G'' is proportional to V.

The estimation of the mixing matrix now consists of the simultaneous solution of Eqs. (8) and (10). Let us define the third-order tensors $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{2\times 3\times 2}$ by

$$\begin{aligned} \mathcal{A}(:,:,1) &= \mathbf{A}_1, & \mathcal{A}(:,:,2) &= \mathbf{A}_2, \\ \mathcal{B}(:,:,1) &= \mathbf{B}_1, & \mathcal{B}(:,:,2) &= \mathbf{B}_2, \end{aligned}$$

in which we used MATLAB notation. It can be verified that solving (8)-(10) in a least-squares sense corresponds to the minimization of the following cost function

$$\sum_{ijkl} c_{ijkl} w_i v_j w_k^* v_l^*, \tag{11}$$

for unit-norm vectors W and V, in which the fourth-order tensor $\mathcal{C} \in \mathbb{C}^{2\times 3\times 2\times 3}$ is defined by the element-wise equation

$$c_{ijkl} = \sum_p a_{ijp} a_{klp}^* + b_{kip} b_{ilp}^*. \tag{12}$$

As explained in [6], this problem is related to the approximation of $\mathcal{C}$ by a rank-1 tensor of the form $W^* \circ V^* \circ W \circ V$.

An obvious solution strategy is to alternate between updates of W and V, which form each a linear least squares problem. Given W, V can be determined in an optimal way as the right singular vector of the matrix

$$\begin{pmatrix} w_1 \mathbf{A}_1 - w_2 \mathbf{A}_2 \\ w_1^* \mathbf{B}_1 - w_2^* \mathbf{B}_2 \end{pmatrix}$$

that corresponds to the smallest singular value. Given V, W can be determined in an optimal way as the second right singular vector of the matrix

$$\begin{pmatrix} \mathbf{A}_1 V & \mathbf{A}_2 V \\ \mathbf{B}_1^* V^* & \mathbf{B}_2^* V^* \end{pmatrix}.$$

Simulations show that this approach works fairly well in general, but sometimes convergence to local optima is observed. In this respect, reinitialization may be required.

Without exploiting prior knowledge on the source distributions, the mixing vectors can inevitably only be estimated up to multiplication with a scalar. Denoting the estimate of the rth mixing vector by $\hat{M}_r$, a perfect estimation corresponds to $M_r = \mu_r \hat{M}_r$, in which $\mu_r \in \mathbb{C}$ $(1 \leqslant r \leqslant 3)$. If the kurtosis values of the individual sources are known, then Eq. (5) leads to an overdetermined system of linear equations in $\{\mu_r^4\}$,

$$\mathcal{C}^{(4,0)} = \sum_{r=1}^{R} \kappa_r^{(4,0)}\, \mu_r^4\, \hat{M}_r \circ \hat{M}_r \circ \hat{M}_r \circ \hat{M}_r, \tag{13}$$

from which the unknown scaling factors can be determined. Note that the moduli $\{|\mu_r|\}$ can also be found by following a similar reasoning for $\mathcal{C}^{(2,2)}$. Source signals from a finite alphabet can now be estimated as those combinations of the possible values that minimize the least-squares error between the predicted and the true observations. For typical problem set-ups, involving e.g. QAM4 signals, this combinatorial problem is still feasible, taking into account that we consider only the case with 3 sources and 2 sensors.

7 CONCLUSION

We developed in this paper an algebraic ICA algorithm for 3 complex-valued sources and 2 sensors, based on an old theorem by Sylvester.

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