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New Algebraic Techniques for the Separation of DS-CDMA Signals [1]

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Abstract

This paper introduces new algebraic techniques for the separation of DS-CDMA signals received on multiple antennas. We show that the PARAFAC structure induced by the coding can be imposed by means of a simultaneous matrix diagonalization. The Constant Modulus property of the symbols can be imposed by means of a simultaneous matrix diagonalization involving the inverse transformation. We also investigate the algebraic structure of convolutive mixtures of DS-CDMA signals. The blind deconvolution problem leads us to the study of a generalized Canonical Decomposition in multilinear algebra.

NEW ALGEBRAIC TECHNIQUES FOR THE SEPARATION OF DS-CDMA SIGNALS

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This paper introduces new algebraic techniques for the separation of DS-CDMA signals received on multiple antennas. We show that the PARAFAC structure induced by the coding can be imposed by means of a simultaneous matrix diagonalization. The Constant Modulus property of the symbols can be imposed by means of a simultaneous matrix diagonalization involving the inverse transformation. We also investigate the algebraic structure of convolutive mixtures of DS-CDMA signals. The blind deconvolution problem leads us to the study of a generalized Canonical Decomposition in multilinear algebra.

1. INTRODUCTION

Let us start from the following noiseless / memoryless data model for multiuser DS-CDMA:

$$y_{ijk} = \sum_{r=1}^{R} a_{ir} c_{jr} s_{kr}. \tag{1}$$

y_{ijk} is the output of the ith antenna for chip j and symbol k ($1 \leqslant i \leqslant I$, $1 \leqslant j \leqslant J'$, $1 \leqslant k \leqslant K$, with I the number of antennas, J' the code length and K the number of transmitted symbols). a_{ir} is the fading / gain between user r and antenna element i, c_{jr} is the jth chip of the spreading sequence of user r and s_{kr} is the kth symbol transmitted by user r. If there is Inter-Chip-Interference (ICI) but no Inter-Symbol-Interference (ISI) (this can be realized by means of a “discard prefix” or “guard chips” strategy, as explained in [4]), we can keep the same model provided we replace c_{jr} by h_{jr}:

$$y_{ijk} = \sum_{r=1}^{R} a_{ir} h_{jr} s_{kr}. \tag{2}$$

h_{jr}, for varying j and fixed r, is the result of convolving the spreading sequence of user r with the impulse response of its propagation channel. Here we suppose that $1 \leqslant j \leqslant J$. Eq. (2) can be written in a tensor format as:

$$\mathcal{Y} = \sum_{r=1}^{R} A_r \circ H_r \circ S_r, \tag{3}$$

in which $\mathcal{Y} \in \mathbb{C}^{I \times J \times K}$, $A_r \in \mathbb{C}^I$, $H_r \in \mathbb{C}^J$ and $S_r \in \mathbb{C}^K$. Eq. (3) is a decomposition of $\mathcal{Y}$ in rank-1 terms. Such a decomposition is called a Parallel Factors Model (PARAFAC) or a Canonical Decomposition (CD). This multilinear point of view w.r.t. CDMA data was adopted for the first time in [4].

Define $\mathbf{A} = [A_1 \dots A_R]$, $\mathbf{H} = [H_1 \dots H_R]$, $\mathbf{S} = [S_1 \dots S_R]$. Eq. (3) has a number of inherent indeterminacies. First, the order of the rank-1 terms is arbitrary. Secondly, A_r, H_r, S_r may be rescaled ($1 \leqslant r \leqslant R$) provided the scaling factors compensate each other.

Now let us introduce the following variant of the "rank" of a matrix [2]:

Definition 1 *The k-rank $k(\mathbf{A})$ of a matrix $\mathbf{A}$ is the maximal number such that* any *set of k columns of $\mathbf{A}$ is linearly independent*

It was shown in [3, 4] that the CD (3) is unique, apart from the trivial determinacies mentioned in the previous paragraph, if

$$k(\mathbf{A}) + k(\mathbf{H}) + k(\mathbf{S}) \geqslant 2(R+1). \tag{4}$$

Assuming that $K \geqslant R$, this means that the decomposition (3) is generically unique, apart from the trivial determinacies mentioned earlier, if

$$R \leqslant \min(I, R) + \min(J, R) - 2. \tag{5}$$

(We call a property "generic" when it holds for all matrices, except for a set of Lebesgue measure 0.)

In Section 2 we will propose a weaker condition. Our proof is constructive. It allows to obtain the canonical components from a simultaneous diagonalization of a set of matrices. Section 3 shows that in this framework it is easy to impose the Constant Modulus property on the symbol estimates. Sections 4–6 form the second part of the paper. In Section 4 we generalize model (3) to convolutive mixtures (in which there is ISI). The generalized model is cast in a multilinear algebraic framework in Section 5. Section 6 discusses some issues that are specific for the problem in Section 4.

2. COMPUTATION OF THE CD BY MEANS OF A SIMULTANEOUS DIAGONALIZATION

In [1], we have proved that generically the CD (3), with $K \geqslant R$, is unique up to the trivial indeterminacies mentioned before, if $R(R-1) \leqslant I^2(J^2-J)/2$. The components may be obtained as follows:

1. Stack $\mathcal{Y}$ in an $(IJ \times K)$ matrix $\mathbf{Y}$.
2. R is the rank of $\mathbf{Y}$. Compute the R dominant singular values σ_r, left singular vectors U_r and right singular vectors V_r. $\mathbf{E}_r = \text{unvec}(\sigma_r U_r)$, with $\mathbf{E}_r \in \mathbb{C}^{I\times J}$. $\mathbf{U} = [U_1 \dots U_R]$, $\mathbf{V} = [V_1 \dots V_R]$, $\mathbf{\Sigma} = \text{diag}(\sigma_1, \dots, \sigma_R)$.
3. Construct $\{\mathbf{\Psi}_{rs} \in \mathbb{C}^{I\times J\times I\times J}\}_{1\leqslant r,s\leqslant R}$:
$$(\mathbf{\Psi}_{rs})_{ijkl} = (\mathbf{E}_r)_{ij}(\mathbf{E}_s)^*_{kl} - (\mathbf{E}_r)_{ik}(\mathbf{E}_s)^*_{jl}.$$
4. $G_{(r-1)R+s} = \text{vec}(\mathbf{\Psi}_{rs})$. $[G_1 \dots G_{R^2}] = \mathbf{G} \in \mathbb{C}^{I^2J^2\times R^2}$. Compute the R right singular vectors V_r corresponding to the smallest singular values λ_r. $\mathbf{B}_r = \text{unvec}(V_r)$ (possibly weighted inversely proportional to λ_r), with $\mathbf{B}_r \in \mathbb{C}^{R\times R}$.
5. Obtain $\mathbf{F} \in \mathbb{C}^{R\times R}$ from the simultaneous diagonalization [5]
$$\begin{aligned} \mathbf{B}_1 &= \mathbf{F}\cdot\mathbf{\Lambda}_1\cdot\mathbf{F}^H \\ &\vdots \\ \mathbf{B}_R &= \mathbf{F}\cdot\mathbf{\Lambda}_R\cdot\mathbf{F}^H. \end{aligned} \tag{6}$$
6. $\mathbf{U}\cdot\mathbf{\Sigma}\cdot\mathbf{F} = [P_1 \dots P_R] \in \mathbb{C}^{IJ\times R}$. $\mathbf{P}_r = \text{unvec}(P_r)$, with $\mathbf{P}_r \in \mathbb{C}^{I\times J}$.
7. Compute A_r and H_r (up to a scaling factor) as the dominant left and right singular vector of $\mathbf{P}_r$. $\mathbf{S} = \mathbf{V}^*\cdot\mathbf{F}^{-T}$.
8. [Optimize Eq. (3) in least-squares sense.]

In this algorithmic description, $\text{vec}(\mathbf{X})$ is a vector representation of a matrix $\mathbf{X}$, in which all the columns are stacked one after the other. More in general, we assume that, in $\text{vec}([x_{ijkl}])$, the index l varies faster than index k, which in turn varies faster than index j, etc. $\text{unvec}(\bullet)$ is the inverse operation of $\text{vec}(\bullet)$.

3. IMPOSING THE CONSTANT MODULUS PROPERTY

In the previous section, the symbol matrix $\mathbf{S}$ is estimated as $\mathbf{V}^* \cdot \mathbf{F}^{-T}$, in which $\mathbf{V}$ is given in the second step of the algorithm and $\mathbf{F}$ has to be determined from the simultaneous diagonalization (6). According to [5], determining a linear transformation ($\mathbf{F}^{-1}$) that makes a given data set ($\mathbf{V}^H$) constant modulus, also amounts to a simultaneous diagonalization. We have:

$$\begin{aligned} \mathbf{M}_1 &= \mathbf{F}^{-1} \cdot \mathbf{\Omega}_1 \cdot \mathbf{F}^{-H} \\ &\vdots \\ \mathbf{M}_R &= \mathbf{F}^{-1} \cdot \mathbf{\Omega}_R \cdot \mathbf{F}^{-H}, \end{aligned} \tag{7}$$

for a set of matrices $\{\mathbf{M}_r\} \in \mathbb{C}^{R \times R}$ that is computed from $\mathbf{V}$. Hence, one can impose both the PARAFAC structure and the CM property by solving Eqs. (6) and (7) simultaneously. One could bring (7) in the same form as (6) by considering $\{\mathbf{M}_r^{-H}\}$ instead of $\{\mathbf{M}_r\}$. (It is always possible to find a basis of R nonsingular $(R \times R)$ matrices for the matrix space spanned by $\{\mathbf{M}_r\}$.)

4. CONVOLUTIVE MIXTURES: PROBLEM DEFINITION

Let n denote a "super-index", combining indices j and k in Eq. (1). Let the sequence transmitted by user r ($c_{jr}s_{kr}$, $1 \leqslant j \leqslant J$, $1 \leqslant k \leqslant K$) be represented by d_{nr}, $1 \leqslant n \leqslant JK$. In the case of multipath propagation (leading to ISI), Eq. (1) generalizes to:

$$y_{in} = \sum_{r=1}^{R} a_{ir} \sum_{l'=0}^{L'-1} h'_{n-l',r} d_{l'r}, \tag{8}$$

in which L' is the channel length at the chip rate. This can be rewritten as

$$y_{ijk} = \sum_{r=1}^{R} a_{ir} \sum_{l=0}^{L-1} (\mathbf{H}_r)_{j,l} s_{k-l,r}, \tag{9}$$

with $\mathbf{H}_r \in \mathbb{C}^{J \times L}$; $L = \lceil \frac{L'}{J} \rceil$ is the channel length at the symbol rate. Eq. (9) can be written in a tensor format as

$$\mathcal{Y} = \sum_{r=1}^{R} A_r \circ (\mathbf{H}_r \cdot \mathbf{S}_r^T), \tag{10}$$

in which $\mathbf{S}_r^T \in \mathbb{C}^{L \times K}$ is a Toeplitz matrix of which the first row contains the subsequent symbols transmitted by user r. Eq. (10) is a Generalized CD (GCD). It is not a decomposition in rank-1 terms, but a decomposition in rank-$(1, L, L)$ terms, i.e., each term consists of the outer product of a vector and a rank-L matrix.

5. GENERALIZED CANONICAL DECOMPOSITION

In this section we discuss a new multilinear decomposition, of which the GCD in the previous section is a special case.

Let us introduce the following notation:

Definition 2 *The n-mode product of a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ by a matrix $\mathbf{U} \in \mathbb{C}^{J_n \times I_n}$, denoted by $\mathcal{A} \times_n \mathbf{U}$, is an $(I_1 \times I_2 \times \ldots \times I_{n-1} \times J_n \times I_{n+1} \ldots \times I_N)$-tensor of which the entries are given by*

$$(\mathcal{A} \times_n \mathbf{U})_{i_1 i_2 \ldots i_{n-1} j_n i_{n+1} \ldots i_N} \stackrel{\text{def}}{=} \sum_{i_n} a_{i_1 i_2 \ldots i_{n-1} i_n i_{n+1} \ldots i_N} u_{j_n i_n}.$$

Then we define:

Definition 3 *A Generalized Canonical Decomposition (GCD) or Generalized Parallel Factors Decomposition (GPARAFAC) of a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$ is a decomposition of $\mathcal{A}$ as a linear combination of a minimal number of rank-$(R_1, R_2, \ldots, R_N)$ terms:*

$$\mathcal{A} = \sum_r^R \mathcal{B}_r \times_1 \mathbf{U}_r^{(1)} \times_2 \mathbf{U}_r^{(2)} \ldots \times_N \mathbf{U}_r^{(N)}, \tag{11}$$

in which $\{\mathcal{B}_r\} \in \mathbb{C}^{R_1 \times R_2 \times \ldots \times R_N}$, and in which $\mathbf{U}_r^{(1)} \in \mathbb{C}^{I_1 \times R_1}$, $\mathbf{U}_r^{(2)} \in \mathbb{C}^{I_2 \times R_2}, \ldots,$ $\mathbf{U}_r^{(N)} \in \mathbb{C}^{I_N \times R_N}$ are nonsingular.

Like for the ordinary CD, the order of the different terms is arbitrary. A second indeterminacy is that any matrix $\mathbf{U}_r^{(n)}$ may be post-multiplied with an arbitrary nonsingular matrix $\mathbf{X} \in \mathbb{C}^{R_n \times R_n}$, provided the corresponding tensor $\mathcal{B}_r$ is replaced by$\mathcal{B}_r \times_n \mathbf{X}^{-1}$.

Define $\mathbf{U}^{(n)} = [\mathbf{U}_1^{(n)} \ldots \mathbf{U}_R^{(n)}]$ $(1 \leqslant n \leqslant N)$. Next, let us introduce the following generalization of the k-rank:

Definition 4 *The k'-rank $k'(\mathbf{A})$ of a partitioned matrix $\mathbf{A} = [\mathbf{A}_1 \ldots \mathbf{A}_R]$ is the maximal number such that the columns of any set of k' submatrices of $\mathbf{A}$ are linearly independent.*

We now have the following uniqueness theorem:

Theorem 1 *Consider the GCD in Eq. (11). This decomposition is generically unique if*

$$\sum_n k'(\mathbf{U}^{(n)}) \geqslant 2R + N - 1. \tag{12}$$

This result can be proved in a similar way as its CD-counterpart (4). However, whereas the proof of (4) is based on the Permutation Lemma in [3], the new proof is based on the following lemma:

Lemma 1 *Consider two partitioned matrices $\overline{\mathbf{A}} = [\overline{\mathbf{A}}_1 \ldots \overline{\mathbf{A}}_R]$ and $\mathbf{A} = [\mathbf{A}_1 \ldots \mathbf{A}_R]$, in which all submatrices are $(J \times L)$-dimensional. Let $w'(v^T\mathbf{A})$ be the number of submatrices of $\mathbf{A}$ of which v is not a left kernel vector. Suppose that $\mathbf{A}$ has no identically zero columns, and assume that the following implication holds:*

$$w'(v^T\overline{\mathbf{A}}) \leqslant L - k'(\mathbf{A}) + 1 \Rightarrow w'(v^T\mathbf{A}) \leqslant w'(v^T\overline{\mathbf{A}}). \tag{13}$$

Then we have that $\overline{\mathbf{A}} = \mathbf{A}(\Pi \otimes \mathbf{I})\Delta_{block}$, where Π is an $(R \times R)$ permutation matrix, $\mathbf{I} \in \mathbb{R}^{L\times L}$ the identity matrix and $\Delta_{block} \in \mathbb{C}^{RL\times RL}$ a block diagonal matrix containing full-rank $(L \times L)$ blocks.

In matrix algebra, the decomposition of a matrix in a sum of rank-R terms is by no means unique, and hence meaningless. The fact that the components do not have to be rank-1 (nor mutually orthogonal) to ensure uniqueness of the tensor decomposition, is a fundamental advantage of working in a multilinear framework. Because rank-$(R_1, R_2, \ldots)$ terms have a richer structure than rank-1 terms, we are convinced that the GCD is important for several signal processing problems.

Like the ordinary CD, the GCD can be computed by means of an Alternating Least Squares (ALS) algorithm. Let us illustrate this by means of the third-order case. Let $\mathbf{X}_{(I\times JK)}$ be an $(I \times JK)$ matrix in which the entries of an $(I \times J \times K)$ tensor $\mathcal{X}$ are stacked (the first index in a product varies slowlier than the second one). Then Eq. (11) can be written in a matrix format as follows:

$$\mathbf{A}_{(I_1\times I_2I_3)} = \mathbf{U}^{(1)} \cdot \mathrm{diag}\{\ldots (\mathbf{B}_r)_{(R_1\times R_2R_3)} \ldots\} \cdot [\ldots \mathbf{U}_r^{(2)} \otimes \mathbf{U}_r^{(3)} \ldots]^T, \tag{14}$$

or:

$$\mathbf{A}_{(I_1\times I_2 I_3)} \cdot \left([\ldots \mathbf{U}_r^{(2)} \otimes \mathbf{U}_r^{(3)} \ldots]^T\right)^\dagger = \mathbf{U}^{(1)} \cdot \mathrm{diag}\{\ldots (\mathbf{B}_r)_{(R_1\times R_2 R_3)} \ldots\}. \quad (15)$$

Hence, the estimate of $\mathbf{U}_r^{(1)} \cdot (\mathbf{B}_r)_{(R_1\times R_2R_3)}$ that is optimal in least-squares sense, given estimates of $\mathbf{U}^{(2)}$ and $\mathbf{U}^{(3)}$, can be computed as the best rank-R approximation of the corresponding submatrix in (15) ($1 \leqslant r \leqslant R$).

6. CONVOLUTIVE MIXTURES: PRACTICAL ASPECTS

We now discuss some practical consequences of the development in the previous section for the problem established in Section 4. Define $\mathbf{A} = [A_1 \ldots A_R]$, $\mathbf{H} = [\mathbf{H}_1 \ldots \mathbf{H}_R]$, and let $\mathbf{S} \in \mathbb{C}^{K\times R}$ contain the first column of each $\mathbf{S_r}$ ($1 \leqslant r \leqslant R$).

Theorem 1 implies that the solution is essentially unique if

$$k(\mathbf{A}) + k'(\mathbf{H}) + k(\mathbf{S}) \geqslant 2(R+1), \quad (16)$$

in which we have used that the k'-rank of the matrix in which the matrices $\{\mathbf{S}_r\}$ are stacked is generically equal to $k(\mathbf{S})$, due to the Toeplitz structure.

In this specific case, each step in the ALS procedure is a classical linear least squares problem. No best rank-R approximations, as in the general scheme, are required. For instance, given estimates of $\mathbf{A}$ and $\mathbf{H}$, Eq. (10) is simply a set of linear equations in the symbols.

7. CONCLUSION

In this paper we have formulated the PARAFAC structure and Constant Modulus constraints for the separation of DS-CDMA signals in terms of a joint simultaneous matrix diagonalization. Our approach has led to a better understanding of the way in which the number of users is bounded by the code length and the number of receive antennas. In the second part of the paper we have proposed a new multilinear algebraic decomposition, namely, the decomposition of a higher-order tensor in a sum of rank-$(R_1, R_2, \ldots)$ terms. A theorem establishing uniqueness of the decomposition under certain conditions and an algorithm for its computation have been presented. The results have been applied to the blind MIMO deconvolution of DS-CDMA signals.

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